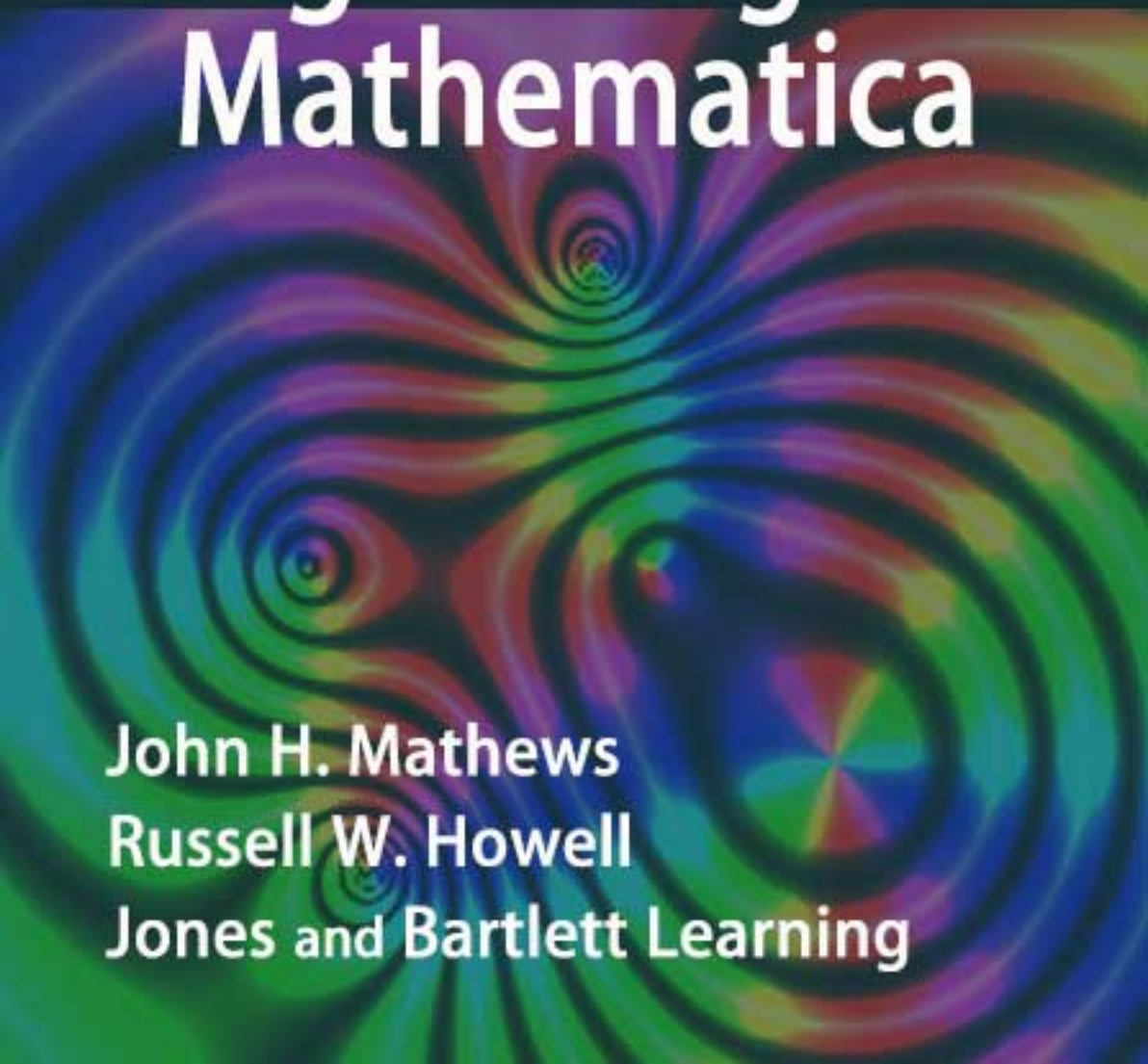
Complex Analysis for Mathematics and Engineering with Mathematica



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Complex Analysis for Mathematics and Engineering

by

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Complex Analysis *Mathematica* 8.0 Notebooks

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2. The Algebra of Complex Numbers
3. The Geometry of Complex Numbers
4. The Geometry of Complex Numbers, Continued
5. The Algebra of Complex Numbers, Revisited
6. The Topology of Complex Numbers

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Chapter 1 Complex Numbers

Overview

Get ready for a treat. You're about to begin studying some of the most beautiful ideas in mathematics. They are ideas with surprises. They evolved over several centuries, yet they greatly simplify extremely difficult computations, making some as easy as sliding a hot knife through butter. They also have applications in a variety of areas, ranging from fluid flow, to electric circuits, to the mysterious quantum world. Generally, they are described as belonging to the area of mathematics known as complex analysis.

Section 1.1 The Origin of Complex Numbers

Complex analysis can roughly be thought of as the subject that applies the theory of calculus to imaginary numbers. But what exactly are imaginary numbers? Usually, students learn about them in high school with introductory remarks from their teachers along the following lines: "We can't take the square root of a negative number. But let's pretend we can and begin by using the symbol $i = \sqrt{-1}$." Rules are then learned for doing arithmetic with these numbers. At some level the rules make sense. If $i = \sqrt{-1}$, it stands to reason that $i^2 = -1$. However, it is not uncommon for students to wonder whether they are really doing magic rather than mathematics.

If you ever felt that way, congratulate yourself! You're in the company of some of the great mathematicians from the sixteenth through the nineteenth centuries. They, too, were perplexed by the notion of roots of negative numbers. Our purpose in this section is to highlight some of the episodes in the very colorful history of how thinking about imaginary numbers developed. We intend to show you that, contrary to popular belief, there is really nothing imaginary about "imaginary numbers." They are just as real as "real numbers."

Our story begins in 1545. In that year the Italian mathematician Girolamo Cardano published Ars Magna (The Great Art), a 40-chapter masterpiece in which he gave for the first time an algebraic solution to the general cubic equation

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0.$$

Cardano did not have at his disposal the power of today's algebraic notation, and he tended to think of cubes or squares as geometric objects rather than algebraic quantities. Essentially, however, his solution began with the substiution $z = x - \frac{1}{3} a_2$. This move transforms $z^3 + a_2 z^2 + a_1 z + a_0 = 0$ into the cubic equation $x^3 + b x + c = 0$ without a squared term, which is called a depressed cubic and can be written as

$$x^3 + b x + c = 0.$$

You need not worry about the computational details, but the coefficients are $b = a_1 - \frac{1}{3} a_2^2$ and $c = a_0 - \frac{1}{3} a_1 a_2 + \frac{1}{27} \quad 2 a_2^3$.

Exploration.

To illustrate, begin with $z^3 + 9 z^2 + 24 z + 20 = 0$ and substitute $z = x - \frac{1}{3} a_2 = x - \frac{9}{3} = x - 3$. The equation then becomes $(x - 3)^3 + 9 (x - 3)^2 + 24 (x - 3) + 20 = 0$, which simplifies to $x^3 - 3 x + 2 = 0$.

Exploration.

If Cardano could get any value of x that solved a depressed cubic, he could easily get a corresponding solution to $z^3 + a_2 z^2 + a_1 z + a_0 = 0$ from the identity $z = x - \frac{1}{3} a_2$. Happily, Cardano knew how to solve a depressed cubic. The technique had been communicated to him by Niccolo Fontana who, unfortunately, came to be known as Tartaglia (the stammerer) due to a speaking disorder. The procedure was also independently discovered some 30 years earlier by Scipione del Ferro of Bologna. Ferro and Tartaglia showed that one of the solutions to the depressed cubic equation is

$$x = \sqrt[3]{\frac{-c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} + \sqrt[3]{-\frac{c}{2} - \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}.$$

Although Cardano would not have reasoned in the following way, today we can take this value for x and use it to factor the depressed cubic into a linear and quadratic term. The remaining roots can then be found with the quadratic formula.

For example, to solve $z^3 + 9 z^2 + 24 z + 20 = 0$, use the substitution $z = x - 3$ to get $x^3 - 3 x + 2 = 0$, which is a depressed cubic equation. Next, apply the "Ferro-Tartaglia" formula with $b = -3$ and $c = 2$ to get $\sqrt[3]{\frac{-2}{2} + \sqrt{\frac{2^2}{4} + \frac{(-3)^3}{27}}} + \sqrt[3]{-\frac{2}{2} - \sqrt{\frac{2^2}{4} + \frac{(-3)^3}{27}}} = (-1)^{1/3} + (-1)^{1/3} = -2$. Since $x = -2$ is a root, $x + 2$ must be a factor of $x^3 - 3 x + 2$. Dividing $x + 2$ into $x^3 - 3 x + 2$ gives $x^2 - 2 x + 1$, which yields the remaining (duplicate) roots of $x = 1$. The solutions to $z^3 + 9 z^2 + 24 z + 20 = 0$ are obtained by recalling $z = x - 3$, which yields the three roots $z_1 = -2 - 3 = -5$ and $z_2 = z_3 = 1 - 3 = -2$.

Exploration.

So, by using Tartaglia's work and a clever transformation technique, Cardano was able to crack what had seemed to be the impossible task of solving the general cubic equation. Surprisingly, this development played a significant role in helping to establish the legitimacy of imaginary numbers. Roots of negative numbers, of course, had come up earlier in the simplest of quadratic equations, such as $x^2 + 1 = 0$. The solutions we know today as $x = \pm\sqrt{-1}$, however, were easy for mathematicians to ignore. In Cardano's time, negative numbers were still being treated with some suspicion, as it was difficult to conceive of any physical reality corresponding to them. Taking square roots of such quantities was surely all the more ludicrous. Nevertheless, Cardano made some genuine attempts to deal with $\sqrt{-1}$. Unfortunately, his geometric thinking made it hard to make much headway. At one point he commented that the process of arithmetic that deals with quantities such as $\sqrt{-1}$ "involves mental tortures and is truly sophisticated." At another point he concluded that the process is "as refined as it is useless." Many mathematicians held this view, but finally there was a breakthrough.

In his 1572 treatise L'Algebra, Rafael Bombelli showed that roots of negative numbers have great utility indeed. Consider the depressed cubic $x^3 - 15x - 4 = 0$. Using $b = -15$ and $c = -4$ in the "Ferro-Tartaglia" formula for the depressed cubic, we compute
$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$, or in a somewhat different form,
$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$.

Simplifying this expression would have been very difficult if Bombelli had not come up with what he called a "wild thought." He suspected that if the original depressed cubic had real solutions, then the two parts of x in the preceding equation could be written as $u + v\sqrt{-1}$ and $u - v\sqrt{-1}$ for some real numbers u and v. That is, Bombeli believed
$u + v\sqrt{-1} = \sqrt[3]{2 + 11\sqrt{-1}}$ and $u - v\sqrt{-1} = \sqrt[3]{2 - 11\sqrt{-1}}$, which would mean $\left(u + v\sqrt{-1}\right)^3 = 2 + 11\sqrt{-1}$ and $\left(u - v\sqrt{-1}\right)^3 = 2 - 11\sqrt{-1}$. Then, using the well-known algebraic identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, and (letting $a = u$ and $b = v\sqrt{-1}$), and assuming that roots of negative numbers obey the rules of algebra, he obtained

$$\begin{aligned}\left(u + v\sqrt{-1}\right)^3 &= u^3 + 3u^2\left(v\sqrt{-1}\right) + 3(u)\left(v\sqrt{-1}\right)^2 + \left(v\sqrt{-1}\right)^3 \\ &= u^3 + 3(u)\left(v\sqrt{-1}\right)^2 + 3u^2\left(v\sqrt{-1}\right) + \left(v\sqrt{-1}\right)^3 \\ &= (u^3 - 3uv^2) + (3u^2v - v^3)\sqrt{-1} \\ &= u(u^2 - 3v^2) + v(3u^2 - v^2)\sqrt{-1} \\ &= 2 + 11\sqrt{-1}.\end{aligned}$$

By equating like parts, Bombelli reasoned that $u(u^2 - 3v^2) = 2$ and $v(3u^2 - v^2) = 11$. Perhaps thinking even more wildly, Bombelli then supposed that u and v were integers. The only integer factors of 2 are 2 and 1, so the equation $u(u^2 - 3v^2) = 2$ led Bombelli to conclude that $u = 2$ and $u^2 - 3v^2 = 1$. From this conclusion it follows that $v^2 = 1$, or $v = \pm 1$. Amazingly, $u = 2$ and $v = 1$ solve the second equation $v(3u^2 - v^2) = 11$, so Bombelli declared the values for u and v to be $u = 2$ and $v = 1$, respectively.

Since $\left(2 + \sqrt{-1}\right)^3 = 2 + 11\sqrt{-1}$, we clearly have $2 + \sqrt{-1} = \sqrt[3]{2 + 11\sqrt{-1}}$. Similarly, Bombelli showed that $2 - \sqrt{-1} = \sqrt[3]{2 - 11\sqrt{-1}}$. But this means that

$$\sqrt[3]{2+11\sqrt{-1}}+\sqrt[3]{2-11\sqrt{-1}}=\left(2+\sqrt{-1}\right)+\left(2-\sqrt{-1}\right)=4,$$

which was a proverbial bombshell. Prior to Bombelli, mathematicians could easily scoff at imaginary numbers when they arose as solutions to quadratic equations. With cubic equations, they no longer had this luxury. That $x = 4$ was a correct solution to the equation $x^3 - 15x - 4 = 0$ was indisputable, as it could be checked easily. However, to arrive at this very real solution, mathematicians had to take a detour through the uncharted territory of "imaginary numbers." Thus, whatever else might have been said about these numbers (which, today, we call complex numbers), their utility could no longer be ignored.

Exploration.

Prior to Bombelli, mathematicians could easily scoff at imaginary numbers when they arose as solutions to quadratic equations. With cubic equations, they no longer had this cavalier attitude. That $x = 4$ was a correct solution to the equation $x^3 - 15x - 4 = 0$ was indisputable. However, to arrive at this very real solution, mathematicians had to take a detour through the uncharted territory of "imaginary numbers." Thus, whatever else might have been said about these numbers (which, today, we call complex numbers), their utility could no longer be ignored. This was the dawning of a new era in mathematics, the "*Age of complex numbers*."

John Wallis leads the way to embed the real numbers in the plane.

Geometric Progress of John Wallis

As significant as Bombelli's work was his results left many issues unresolved. For example, his technique applied only to a few specialized cases. Could it be extended? Even if it could be extended a larger question remained: what possible physical representation could complex numbers have?

The last question remained unanswered for more than two centuries; University of New Hampshire professor Paul J. Nahin describes the progress as occurring in several stages (see ""An Imaginary tale: the Story of $\sqrt{-1}$", by Paul J Nahin, Princeton University Press, 2007, pages 48-55).

A preliminary step came in 1685 when the English mathematician John Wallis published "*A treatise of Algebra, both Historical and Practical*." Among the many contributions in that book two are particularly noteworthy for our purposes. They are displayed in Wallis' analysis of a problem from classical geometry that, at first glance, seems completely unrelated to complex numbers.

Problem 1.1. Construct a triangle determined by two sides and an angle *not* included between those sides.

We will get to Wallis' contributions in a moment. First, observe that Figure 1.1 illustrates the standard solution to Problem 1.1. Given side length a (represented by segment AB), angle α (determined by segments AB and BC), and side length b, draw an arc of a circle of radius b whose center is at point A. If the arc intersects segment BC at points E and F, then the resulting triangles ABE and ABF each satisfy the problem requirement.

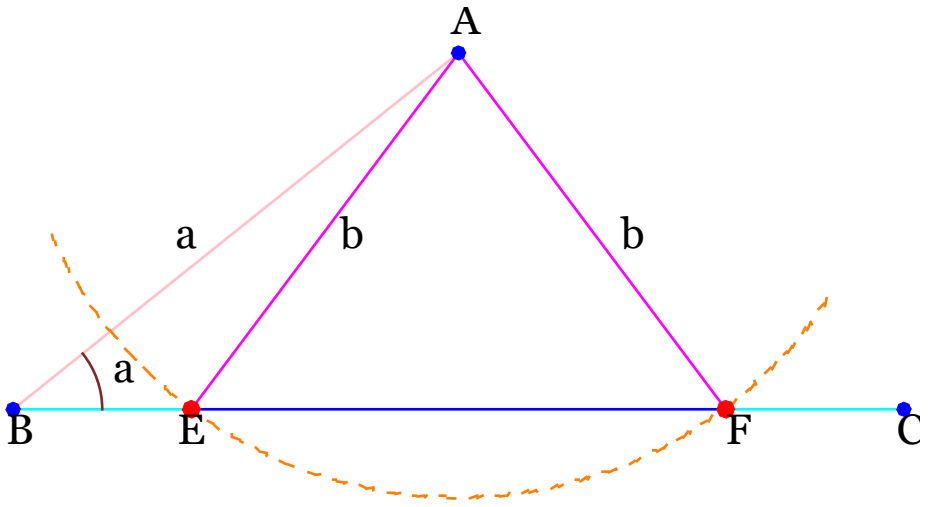

Figure 1.1 The Standard solution to Wallis' geometrical problem.

A Geometric Representation of Real Numbers

The representation of $\sqrt{b^2 - c^2}$ when $b > c$.

Wallis' first contribution allowed him to associate (real) numbers with the points E and F of Figure 1.1. The association came by way of a construct that may sound completely trivial to us, but that is only because we have been raised knowing about Wallis' idea: "*the real number line*." By choosing an arbitrary point to represent the number zero on a given line, Wallis declared that positive numbers could be viewed as corresponding distances to the *right* of zero, and negative numbers as corresponding (positive) distances to the *left* of zero.

To complete the association refer to Figure 1.2 (a) and think of segment BC as lying on a portion of the x-axis. Then draw a perpendicular segment AD to BC, and designate the origin to be at D. If the length of AD is c, then the Pythagorean theorem gives $\sqrt{b^2 - c^2}$ for the length of segments ED and DF. Combining this result with Wallis' number real line results in points E and F representing the real numbers:

$E = 0 - \sqrt{b^2 - c^2}$, and $F = 0 + \sqrt{b^2 - c^2}$.

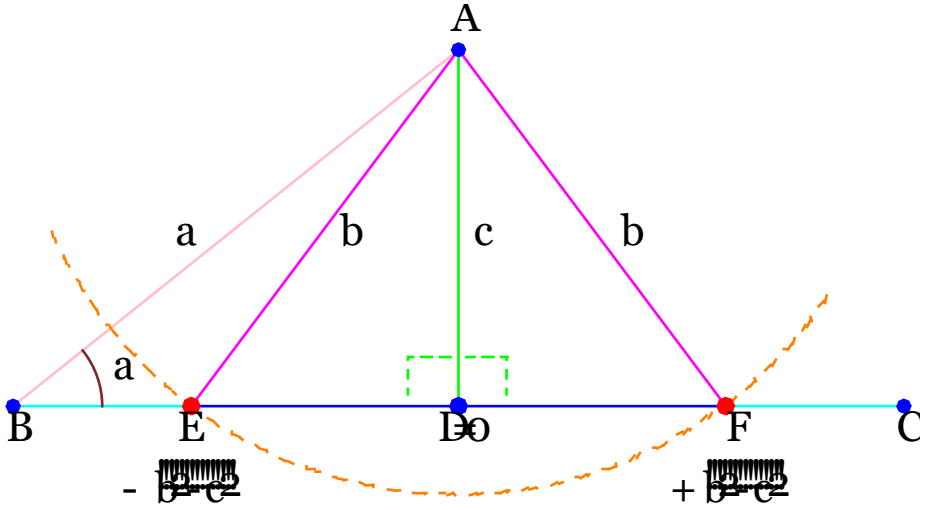

Figure 1.2 (a) Wallis' geometric depiction of "real numbers."

For example, if $b = 5$ and $c = 4$, the points E and F would represent -3 and $+3$, respectively, because

$$E = -\sqrt{5^2 - 4^2} = -3, \quad \text{and} \quad F = +\sqrt{5^2 - 4^2} = +3,$$

as shown below in the Figure 1.2 (b).

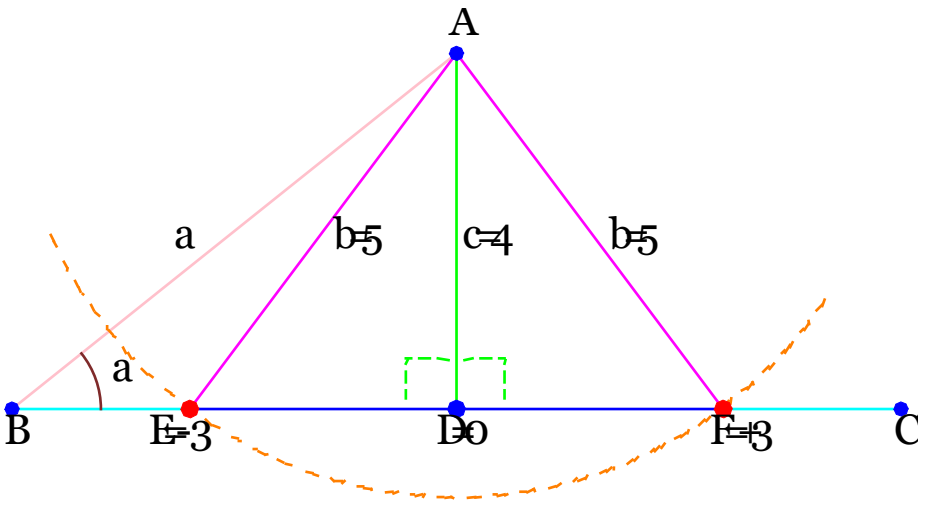

Figure 1.2 (b) Wallis' geometric depiction of the "real numbers," $E = -3$ and $F = 3$.

From both an algebraic and geometric viewpoint this procedure only makes sense if the stipulated length b is greater than or equal to c. If b were less than c then the algebraic expressions for points $E = -\sqrt{b^2 - c^2}$ and $F = +\sqrt{b^2 - c^2}$ would be meaningless, as the quantity $b^2 - c^2$ inside the square root would be negative. Viewed geometrically, if b were less than c then the arc of radius b that is centered at A would not be able to intersect the segment BC. In other words, if b were less than c Problem 1.1 would appear to have no solution.

A Geometric Representation of Complex Numbers

The representation of $\sqrt{b^2 - c^2}$ when $b < c$.

Appearances, of course, can be deceiving, and Wallis reinforced the truth of that ancient proverb when he came up with his second contribution. It was a solution to Problem 1.1 in the case when b is less than c. Figure 1.3 (a) illustrates how he did it. From the midpoint of AD Wallis drew a circle with diameter AD. Then, with A as a center he drew an arc of radius b. Because b is less than c the arc will intersect the circle at two points, say E and F, which now lie above the "*real number line*."

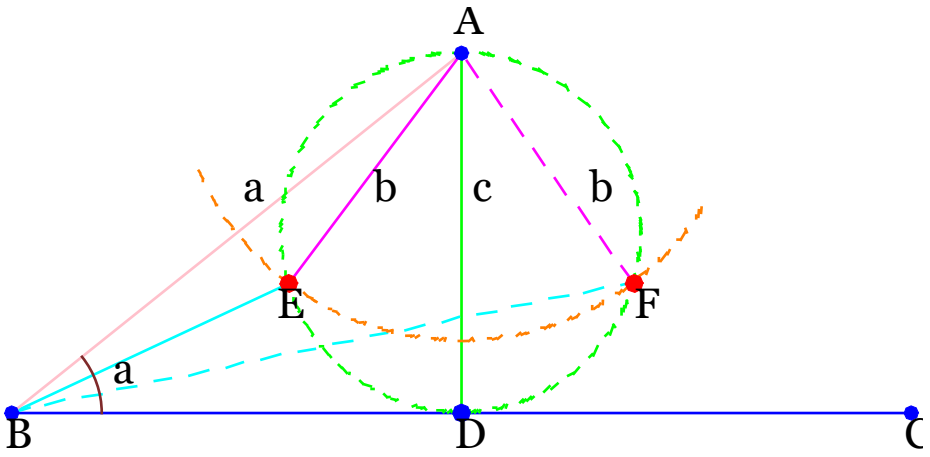

Figure 1.3 (a). Wallis' geometric depiction of the "complex numbers," E and F.

Again we get two triangles: ABE and ABF. Wallis claimed that these triangles each satisfy the requirement of Problem 1.1. You might object to this construction on the grounds that angle α is not part of either triangle. However, if you read the problem statement carefully, you will notice that it never states that the angle α had to be part of any triangle, only that it *determine* a triangle. From this perspective Wallis completely satisfied the requirement.

Notice that the points E and F are no longer on the x-axis as they were when b was greater than c, (and $\pm\sqrt{b^2 - c^2}$ are real numbers). They are now located somewhere above the x-axis, and it is not unreasonable to conclude that points E and F give, respectively, the geometric representations of the expressions $E = -\sqrt{b^2 - c^2}$ and $F = +\sqrt{b^2 - c^2}$ when b is less than c, (and $\pm\sqrt{b^2 - c^2}$ are complex numbers).

Although Wallis only hinted at such a conclusion, he nevertheless helped set the stage for thinking of real numbers as being embedded in a larger set of complex numbers, and that these numbers could be represented geometrically as vertical displacements from the x-axis. Unfortunately, if we tried to apply Wallis' method to construct complex numbers we would find it had some critical defects.

Wallis' depiction of $\pm\sqrt{-1}$

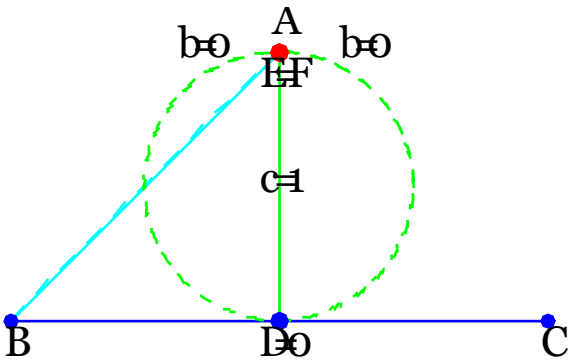

Figure 1.3 (b). An interpretation of Wallis' geometric depiction of "$E = -\sqrt{-1}$" and "$F = +\sqrt{-1}$".

Suppose that we choose b = 0 and c = 1, then Wallis' expression $\pm\sqrt{b^2 - c^2}$ becomes $\pm\sqrt{-1}$, and points E and F now coincide at point A, as shown above in Figure 1.3 (b). For certain, we can say that

the value "$F = +\sqrt{-1}$" is the proper choice we use today. So in some sense, Wallis was the first person to place the point $+\sqrt{-1}$ in it's proper location in the upper half-plane. But we surely would not want to equate the points "$E = -\sqrt{-1}$" and "$F = +\sqrt{-1}$." The reason for this anomaly was that the entire complex plane had not yet been invented. Thus, even with Wallis' work the jigsaw of getting the proper picture of complex numbers remained. It would be another century before someone put most of the pieces together.

Wallis's method has the undesirable consequence that $-\sqrt{-1}$ is represented by the same point as is $\sqrt{-1}$. Nevertheless, this interpretation helped set the stage for thinking of complex numbers as "points on the plane." By 1732, the great Swiss mathematician Leonhard Euler (pronounced "oiler") adopted this view concerning the n solutions to the equation $x^n - 1 = 0$. You will learn shortly that these solutions can be expressed as $\cos(\theta) + \sqrt{-1}\,\sin(\theta)$ for various values of θ; Euler thought of them as being located at the vertices of a regular polygon in the plane. Euler was also the first to use the symbol i for $\sqrt{-1}$. Today, this notation is still the most popular, although some electrical engineers prefer the symbol j instead so that they can use i to represent current.

Is it possible to modify slightly Wallis's picture of complex numbers so that it is consistent with the representation used today? To help you answer this question, refer to the article by Alec Norton and Benjamin Lotto, "Complex Roots Made Visible," The College Mathematics Journal, 15(3), June 1984, pp. 248--249, Jstor.

Two additional mathematicians deserve mention. The Frenchman Augustin-Louis Cauchy (1789-1857) formulated many of the classic theorems that are now part of the corpus of complex analysis. The German Carl Friedrich Gauss (1777--1855) reinforced the utility of complex numbers by using them in his several proofs of the fundamental theorem of algebra (see Chapter 6). In an 1831 paper, he produced a clear geometric representation of x+iy by identifying it with the point (x, y) in the coordinate plane. He also described how to perform arithmetic operations with these new numbers.

It would be a mistake, however, to conclude that in 1831 complex numbers were transformed into legitimacy. In that same year the prolific logician Augustus De Morgan commented in his book, On the Study and Difficulties of Mathematics, "We have shown the symbol $\sqrt{-a}$ to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility."

There are, indeed, genuine logical problems associated with complex numbers. For example, with real numbers $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ so long as both sides of the equation are defined. Applying this identity to complex numbers leads to
$1 = \sqrt{1} = \sqrt{(-1)\,(-1)} = \sqrt{(-1)}\,\sqrt{(-1)} = i * i = -1$. Plausible answers to these problems can be given, however, and you will learn how to resolve this apparent contradiction in Section 2.4. De Morgan's remark illustrates that many factors are needed to persuade mathematicians to adopt new theories. In this case, as always, a firm logical foundation was crucial, but so, too, was a willingness to modify some ideas concerning certain well-established properties of numbers.

As time passed, mathematicians gradually refined their thinking, and by the end of the nineteenth century complex numbers were firmly entrenched. Thus, as it is with many new mathematical or scientific innovations, the theory of complex numbers evolved by way of a very intricate process. But what is the theory that Tartaglia, Ferro, Cardano, Bombelli, Wallis, Euler, Cauchy, Gauss, and so many others helped produce? That is, how do we now think of complex numbers?

We explore this question in the remainder of this chapter.

The solution of the cubic equations. *Mathematica* can construct the solutions to the general cubic equation. However they are quite formidable.

Exploration.

Complex Analysis for Mathematics and Engineering

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1.2 The Algebra of Complex Numbers

We have seen that complex numbers came to be viewed as ordered pairs of real numbers. That is, a complex number z is defined to be

$$z = (x, y),$$

where x and y are both real numbers.

The reason we say ordered pair is because we are thinking of a point in the plane. The point (2, 3), for example, is not the same as (3, 2). The order in which we write x and y in the equation makes a difference. Clearly, then, two complex numbers are equal if and only if their x coordinates are and their y coordinates are equal. In other words,

$$(x, y) = (u, v) \quad \text{iff} \quad x = u \text{ and } y = v.$$

(Throughout this text, iff means if and only if.)

A meaningful number system requires a method for combining ordered pairs. The definition of algebraic operations must be consistent so that the sum, difference, product, and quotient of any two ordered pairs will again be an ordered pair. The key to defining how these numbers should be manipulated is to follow Gauss's lead and equate (x, y) with $x + iy$. Then, if $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are arbitrary complex numbers, we have

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$$

$$= (x_1 + i\, y_1) + (x_2 + i\, y_2)$$

$$= (x_1 + x_2) + i\, (y_1 + y_2)$$

$$= (x_1 + x_2, y_1 + y_2)$$

Thus, the following definitions should make sense.

Definition 1.1, (Addition)

Formula (1-8) $z_1 + z_2 = (x_1 + i\, y_1) + (x_2 + i\, y_2) = (x_1 + x_2, y_1 + y_2)$.

Derivation of Formula (1 – 8).

Definition 1.2, (Subtraction)

Formula (1-9) $z_1 - z_2 = (x_1 + i\, y_1) - (x_2 + i\, y_2) = (x_1 - x_2, y_1 - y_2)$.

Derivation of Formula (1 – 9).

Example 1.1. Given $z_1 = 3 + 7\, i$ and $z_2 = 5 - 6\, i$. **(a)** Find $z_1 + z_2$ and **(b)** $z_1 - z_2$.

$$z_1 + z_2 = (3, 7) + (5, -6) = (3 + 5, 7 - 6) = (8, 1) \text{ and}$$

$$z_1 + z_2 = (3, 7) - (5, -6) = (3 - 5, 7 + 6) = (-2, 13).$$

We can also use the notation $z_1 = 3 + 7\, i$ and $z_2 = 5 - 6\, i$:

$$z_1 + z_2 = 3 + 7\, i + 5 - 6\, i = 8 + i \text{ and}$$

$$z_1 - z_2 = 3 + 7\, i - (5 - 6\, i) = -2 + 13\, i.$$

Explore Solution 1.1. (a).

Explore Solution 1.1. (b).

Given the rationale we devised for addition and subtraction, it is tempting to define the product $z_1\, z_2$ as $z_1\, z_2 = (x_1\, x_2, y_1\, y_2)$. It turns out, however, that this is not a good definition, and we ask you in the exercises for this section to explain why. How, then, should products be defined? Again, if we equate (x, y) with $x + i\, y$ and assume, for the moment, that $i = \sqrt{-1}$ makes sense (so that $i^2 = -1$), we have

$$\begin{aligned} z_1\, z_2 &= (x_1, y_1)\,(x_2, y_2) \\ &= (x_1 + i\, y_1)\,(x_2 + i y_2) \\ &= x_1\, x_2 + i\, x_1\, y_2 + i\, x_2\, y_1 + i^2\, y_1\, y_2 \\ &= x_1\, x_2 - y_1\, y_2 + i\,(x_1\, y_2 + x_2\, y_1) \\ &= (x_1\, x_2 - y_1\, y_2, x_1\, y_2 + x_2\, y_1) \end{aligned}$$

Thus, it appears we are forced into the following definition.

Definition 1.3, (Complex Multiplication)

Formula (1-10) $z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$.

Derivation of Formula (1 – 10).

Example 1.2. Given $z_1 = 3 + 7\,i$ and $z_2 = 5 - 6\,i$. Find $z_1 z_2$.

$$z_1 z_2 = (3, 7)(5, -6)$$

$$= (3 * 5 - 7(-6), 3(-6) + 5 * 7)$$

$$= (15 + 42, -18 + 35)$$

$$= (57, 17)$$

We get the same answer by using the notation $z_1 = 3 + 7\,i$ and $z_2 = 5 - 6\,i$:

$$z_1 z_2 = (3, 7)(5, -6) = (3 + 7\,i)(5 - 6\,i)$$

$$= 15 - 18\,i + 35\,i - 42\,i^2$$

$$= 15 - 42(-1) + (-18 + 35)\,i$$

$$= 57 + 17\,i$$

$$= (57, 17)$$

Of course, it makes sense that the answer came out as we expected because we used the notation $x + i\,y$ as motivation for our definition in the first place.

Explore Solution 1.2.

To motivate our definition for division, we will proceed along the same lines as we did for multiplication, assuming $z_2 \neq 0$:

$$\frac{z_1}{z_2} = \frac{(x_1, y_1)}{(x_2, y_2)} = \frac{x_1 + i\,y_1}{x_2 + i\,y_2}$$

We need to figure out a way to write the preceding quantity in the form $x + i\,y$. To do this, we use a standard trick and multiply the numerator and denominator by $x_2 + i\,y_2$, which gives

$$\frac{z_1}{z_2} = \frac{(x_1+i\,y_1)\,(x_2-i\,y_2)}{(x_2+i\,y_2)\,(x_2-i\,y_2)}$$

$$= \frac{x_1\,x_2+y_1\,y_2+i\,(-x_1\,y_2+x_2\,y_1)}{x_2^2+y_2^2}$$

$$= \frac{x_1\,x_2+y_1\,y_2}{x_2^2+y_2^2} + i\,\frac{-x_1\,y_2+x_2\,y_1}{x_2^2+y_2^2}$$

$$= \left(\frac{x_1\,x_2+y_1\,y_2}{x_2^2+y_2^2}, \frac{-x_1\,y_2+x_2\,y_1}{x_2^2+y_2^2}\right)$$

Thus, we finally arrive at a rather odd definition.

Definition 1.4, (Complex Division)

Formula (1-11) $\frac{z_1}{z_2} = \frac{(x_1,\,y_1)}{(x_2,y_2)} = \left(\frac{x_1\,x_2+y_1\,y_2}{x_2^2+y_2^2}, \frac{-x_1\,y_2+x_2\,y_1}{x_2^2+y_2^2}\right)$, for $z_2 \neq 0$.

Derivation of Formula (1 – 11).

Example 1.3. Given $z_1 = 3 + 7\,i$ and $z_2 = 5 - 6\,i$. Find $\frac{z_1}{z_2}$.

$$\frac{z_1}{z_2} = \frac{(3,\,7)}{(5,-6)} = \left(\frac{15-42}{25+36}, \frac{18+35}{25+36}\right) = \left(-\frac{27}{61}, \frac{53}{61}\right)$$

As with the example for multiplication, we also get this answer if we use the notation $x + i\,y$:

$$\frac{z_1}{z_2} = \frac{(3,\,7)}{(5,-6)} = \frac{(3+7\,i)}{(5-6\,i)}$$

$$= \frac{(3+7\,i)}{(5-6\,i)}\,\frac{(5+6\,i)}{(5+6\,i)} = \frac{15+18\,i+35\,i+42\,i^2}{25+30\,i-30\,i-36\,i^2}$$

$$= \frac{15-42+(18+35)\,i}{(25+3)} = -\frac{27}{61} + \frac{53}{61}\,i$$

$$= \left(-\frac{27}{61}, \frac{53}{61}\right)$$

Explore Solution 1.3.

To perform operations on complex numbers, most mathematicians would use the notation $x + i\,y$ and engage in algebraic manipulations, as we did here, rather than apply the complicated-looking definitions we gave for those operations on ordered pairs. This procedure is valid because we used the $x + i\,y$ notation as a guide for defining the operations in the first place. Remember, though, that the $x + i\,y$ notation is nothing more than a convenient bookkeeping device for keeping track of how to manipulate ordered pairs. It is the ordered pair algebraic definitions that form the real foundation on which the complex number system is based. In fact, if you were to program a computer to do arithmetic on complex numbers, your program would perform calculations on ordered pairs, using exactly the definitions that we gave.

It turns out that our algebraic definitions give complex numbers all the properties we normally ascribe to the real number system. Taken together, they describe what algebraists call a field. In formal terms, a field is a set (in this case, the complex numbers) together with two binary opera-

tions (in this case, addition and multiplication) having the following properties.

(P1) Commutative Law for Addition.

$$z_1 + z_2 = z_2 + z_1.$$

Derivation of Property (P1).

(P2) Associative Law for Addition.

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$$

(P3) Additive Identity. There is a complex number ω such that

$$z + \omega = z \text{ for all complex numbers } z.$$

The number ω is obviously the ordered pair $(0, 0)$.

(P4) Additive Inverses. Given any complex number z, there is a complex number η (depending on z) with the property that

$$z + \eta = (0, 0).$$

Obviously, if $z = (x, y) = x + i\, y$, the number η will be $\eta = (-x, -y) = -x - i\, y = -z$.

(P5) Commutative Law for Multiplication.

$$z_1\, z_2 = z_2\, z_1.$$

(P6) Associative Law for Multiplication.

$$z_1\, (z_2\, z_3) = (z_1\, z_2)\, z_3.$$

(P7) Multiplicative Identity. There is a complex number ζ such that

$$\zeta z = z \text{ for all complex numbers } z.$$

As one might expect, it turns out that $(1, 0)$ is the unique complex number ζ with this property. We ask you to verify this identity in the exercises for this section.

(P8) Multiplicative Inverses. Given any number z other than the number (0, 0), there is a complex number (depending on z) which we shall denote by z^{-1} with the property that

$$z\, z^{-1} = (1, 0) = 1.$$

Based on our definition for division, it seems reasonable that the number z^{-1} would be

$$z^{-1} = \frac{(1,0)}{z} = \frac{1}{x+i\,y} \quad .$$

$$= \frac{x-i\,y}{(x+i\,y)\,(x-i\,y)} = \frac{x-i\,y}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$$

$$= \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$$

We ask you to confirm this result in the exercises for this section.

(P9) The Distributive Law.

$$z_1\,(z_2 + z_3) = z_1\,z_2 + z_1\,z_3.$$

Derivation of Property (P 9).

None of these properties is difficult to prove. Most of the proofs make use of corresponding facts in the real number system. To illustrate, we give a proof of property (P1).

Proof of the commutative law for addition: Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be arbitrary complex numbers. Then,

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$$

$$= (x_1 + x_2, y_1 + y_2) \quad \text{(by definition of addition of complex numbers)}$$

$$= (x_2 + x_1, y_2 + y_1) \quad \text{(by the commutative law for real numbers)}$$

$$= (x_2, y_2) + (x_1, y_1) \quad \text{(by definition of addition of complex numbers)}$$

$$= z_2 + z_1$$

Actually, you can think of the real number system as a subset of the complex number system. To see why, let's agree that, as any complex number of the form $(t, 0)$ is on the x axis, we can identify it with the real number t. With this correspondence, we can easily verify that our definitions for addition, subtraction, multiplication, and division of complex numbers are consistent with the corresponding operations on real numbers. For example, if x_1 and x_2 are real numbers, then

$$x_1\,x_2 = (x_1, 0)\,(x_2, 0) \quad \text{(by our agreed correspondence)}$$

$$= (x_1\,x_2 - 0, 0 + 0) \quad \text{(by definition of multiplication of complex numbers)}$$

$$= (x_1\,x_2, 0) \quad \text{(confirming the consistence of our correspondence)}$$

It is now time to show specifically how the symbol i relates to the quantity $\sqrt{-1}$. Note that

$$(0, 1)^2 = (0, 1)\,(0, 1)$$

$$= (0 - 1, 0 + 0) \quad \text{(by definition of multiplication of complex numbers)}$$

$$= (-1, 0)$$

$$= -1 \quad \text{(by our agreed correspondence)}$$

If we use the symbol $\mathbb{i}$ for the point $(0, 1)$, the preceding identity gives

$$\mathbb{i}^2 = (0, 1)^2 = -1,$$

which means $\mathbb{i} = (0, 1) = \sqrt{-1}$. So, the next time you are having a discussion with your friends and they scoff when you claim that $\sqrt{-1}$ is not imaginary, calmly put your pencil on the point $(0, 1)$ of the coordinate plane and ask them if there is anything imaginary about it. When they agree there isn't, you can tell them that this point, in fact, represents the mysterious $\sqrt{-1}$ in the same way that $(1, 0)$ represents 1.

We can also see more clearly now how the notation $x + \mathbb{i}\, y$ equates to (x, y). Using the preceding conventions (i.e., $x = (x, 0)$, etc.), we have

$$x + \mathbb{i}\, y = (x, 0) + (0, 1)\,(y, 0) \quad \text{(by our previously discussed conventions)}$$

$$= (x, 0) + (0, y) \quad \text{(by definition of multiplication of complex numbers)}$$

$$= (x, y) \quad \text{(by definition of addition of complex numbers)}$$

Thus we may move freely between the notations $x + \mathbb{i}\, y$ and (x, y), depending on which is more convenient for the context in which we are working. Students sometimes wonder whether it matters where the "$\mathbb{i}$" is located in writing a complex number. It does not. Generally, most texts place terms containing an "$\mathbb{i}$" at the end of an expression, and place the "$\mathbb{i}$" before a variable, but after a constant. Thus, we write $x + \mathbb{i}\, y$, $u + \mathbb{i}\, v$, etc., but $3 + 7\,\mathbb{i}$, $5 - 6\,\mathbb{i}$ and so forth. Because letters lower in the alphabet generally denote constants, you will usually (but not always) see the expression $a + b\,\mathbb{i}$, instead of $a + \mathbb{i}\, b$. Many authors write quantities like $1 + \mathbb{i}\,\sqrt{3}$ instead of $1 + \sqrt{3}\,\mathbb{i}$ to make sure the "$\mathbb{i}$" is not mistakenly thought to be inside the square root symbol. Additionally, if there is concern that the "$\mathbb{i}$" might be missed, it is sometimes placed before a lengthy expression, as in $2\cos\left(-\frac{5}{6}\pi + 2n\pi\right) + \mathbb{i}\, 2\sin\left(-\frac{5}{6}\pi + 2n\pi\right)$.

We close this section with three important definitions and a theorem involving them. We ask you for a proof of the theorem in the excercises.

Definition 1.5, (Real Part of z). The real part of $z = x + \mathbb{i}\, y$, denoted by $\mathrm{Re}(z)$, is the real number x.

Definition 1.6, (Imaginary Part of z). The imaginary part of $z = x + \mathbb{i}\, y$, denoted by $\mathrm{Im}(z)$, is the real number y.

Definition 1.7, (Conjugate of z). The conjugate of $z = x + \mathbb{i}\, y$, denoted by $\overline{z}$, is the complex number $(x, -y) = x - \mathbb{i}\, y$.

Example 1.4. Given $z_1 = -3 + 7\,i$ and $z_2 = 9 + 4\,i$.

1.4 (a) We have $\text{Re}[z_1] = \text{Re}[-3 + 7\,i] = -3$ and $\text{Re}[z_2] = \text{Re}[9 + 4\,i] = 9$.

1.4 (b) We have $\text{Im}[z_1] = \text{Im}[-3 + 7\,i] = 7$ and $\text{Im}[z_2] = \text{Im}[9 + 4\,i] = 4$.

1.4 (c) We have $\overline{z_1} = \overline{-3 + 7\,i} = -3 - 7\,i$ and $\overline{z_2} = \overline{9 + 4\,i} = 9 - 4\,i$.

Explore Solution 1.4 (a).

Explore Solution 1.4 (b).

Explore Solution 1.4 (c).

The following theorem gives some important facts relating to these operations. You will be asked for a proof in the exercises

Theorem 1.1. Suppose z_1, z_2 and z are arbitrary complex numbers. Then if $z_2 \neq 0$

(1 – 12) $\quad \overline{\overline{z}} = z$

(1 – 13) $\quad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(1 – 14) $\quad \overline{z_1\, z_2} = \overline{z_1}\ \overline{z_2}$

(1 – 15) $\quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$

(1 – 16) $\quad \text{Re}[z] = \frac{z + \overline{z}}{2}$

(1 – 17) $\quad \text{Im}[z] = \frac{z - \overline{z}}{2\,i}$

(1 – 18) $\quad \text{Re}[i\,z] = -\,\text{Im}[z]$

(1 – 19) $\quad \text{Im}[i\,z] = \text{Re}[z]$

Because of what it erroneously connotes, it is a shame that the term imaginary is used in Definition (1.6). It was coined by the brilliant mathematician and philosopher René Descartes (1596--1650) during an era when quantities such as $\sqrt{-1}$ were thought to be just that. Gauss, who was successful in getting mathematicians to adopt the phrase complex number rather than imaginary number, also suggested that they use lateral part of z in place of imaginary part of z. Unfortunately, that suggestion never caught on, and it appears we are stuck with what history has handed down to us.

Complex Analysis for Mathematics and Engineering

by

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1.3 The Geometry of Complex Numbers

Complex numbers are ordered pairs of real numbers, so they can be represented by points in the plane. In this section we show the effect that algebraic operations on complex numbers have on their geometric representations.

We can represent the number $z = x + iy = (x, y)$ by a position vector in the xy plane whose tail is at the origin and whose head is at the point (x, y). When the xy plane is used for displaying complex numbers, it is called the complex plane, or more simply, the z plane. Recall that $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$. Geometrically, $\mathrm{Re}(z)$ is the projection of $z = (x, y)$ onto the x axis, and $\mathrm{Im}(z)$ is the projection of z onto the y axis. It makes sense, then, to call the x axis the real axis and the y axis the imaginary axis, as Figure 1.3 illustrates.

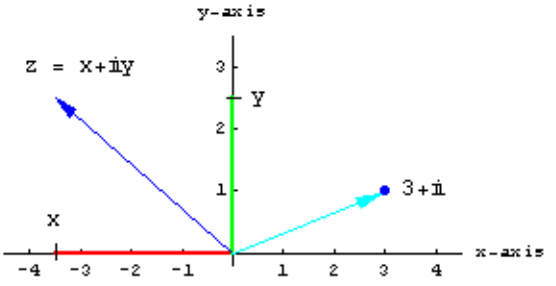

Figure 1.3 The complex plane.

Addition of complex numbers is analogous to addition of vectors in the plane. As we saw in Section 1.2, the sum of $z_1 = x_1 + iy_1 = (x_1, y_1)$ and $z_2 = x_2 + iy_2 = (x_2, y_2)$ is $(x_1 + x_2, y_1 + y_2)$. Hence, $z_1 + z_2$ can be obtained vectorially by using the "parallelogram law," where the vector sum is the vector represented by the diagonal of the parallelogram formed by the two original vectors. Figure 1.4 illustrates this notion.

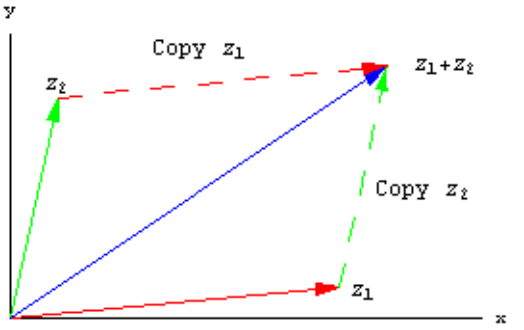

Figure 1.4 The sum $z_1 + z_2$.

The difference $z_1 - z_2$ can be represented by the displacement vector from the point $z_2 = (x_2, y_2)$ to the point $z_1 = (x_1, y_1)$, as Figure 1.5 shows.

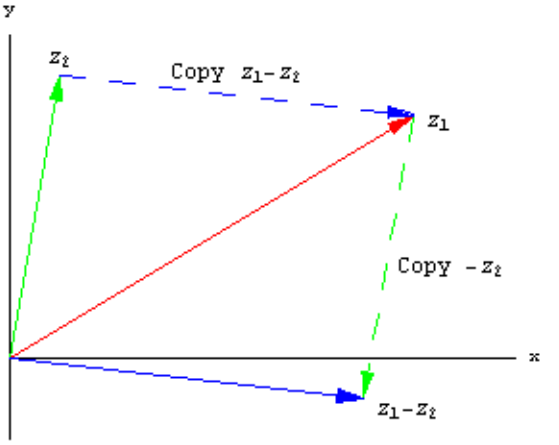

Figure 1.5 The difference $z_1 - z_2$.

Definition 1.8, (Modulus or Absolute Value). The modulus, or absolute value, of the complex number $z = x + i\,y$ is a nonnegative real number denoted by $|z|$ and is given by the equation

$$|z| = \sqrt{x^2 + y^2}.$$

The number $|z|$ is the distance between the origin and the point $z = (x, y)$. The only complex number with modulus zero is the number 0. The number $z = 3 + 4\,i$ has modulus $|4 + 3\,i| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$, and is depicted in Figure 1.6.

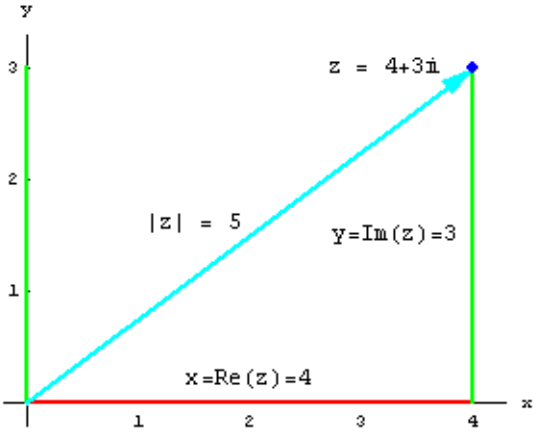

Figure 1.6 The real and imaginary parts of a complex number.

The numbers $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ and $|z|$ are the lengths of the sides of the right triangle

OPQ shown in Figure 1.7.

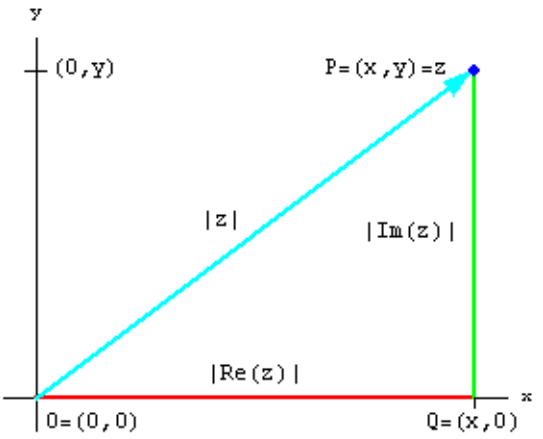

Figure 1.7 The moduli of z and its components.

The inequality $|z_1| < |z_2|$ means that the point z_1 is closer to the origin than the point z_2. Although obvious from Figure 1.7, it is still profitable to work out algebraically the standard results that

$$|x| = |\text{Re}(z)| \leq |z| \text{ and } |y| = |\text{Im}(z)| \leq |z|.$$

which we leave as an exercise for the reader.

The difference $z_1 - z_2$ represents the displacement vector from z_2 to z_1, so the distance between z_1 and z_2 is given by $|z_1 - z_2|$. We can obtain this distance by using Definitions (1.2) and (1-3) to obtain the familiar formula

$$\text{dist}(z_1, z_2) = |z_2 - z_1| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

If $z = (x, y) = x + i\,y$, then $-z = (-x, -y) = -x - i\,y$ is the reflection of z, through the origin, and $\overline{z} = (x, -y) = x - i\,y$ is the reflection of z through the x axis, as illustrated in Figure 1.8.

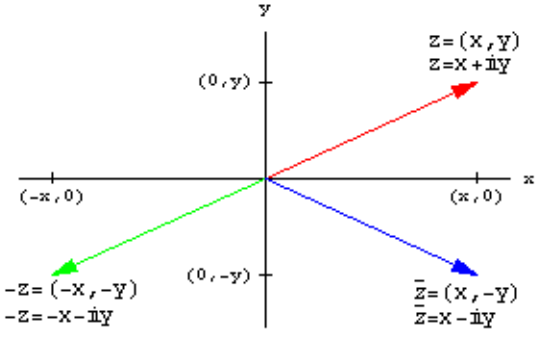

Figure 1.8 The geometry of negation and conjugation.

We can use an important algebraic relationship to establish properties of the absolute value that have geometric applications. Its proof is rather straightforward, and we ask you to give it in the exercises for this section.

$$|z|^2 = z\,\overline{z}.$$

A beautiful and important application of the above identity is its use in establishing the

triangle inequality, which states that the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side. Figure 1.9 illustrates this inequality.

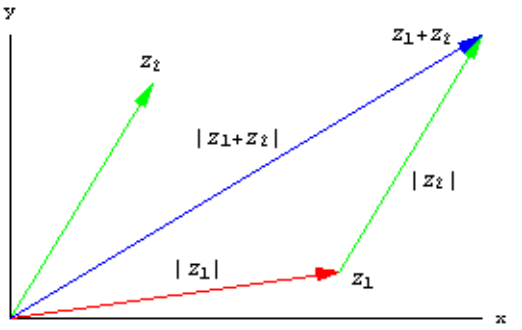

Figure 1.9 The triangle inequality.

Theorem 1.2, (Triangle Inequality). If z_1 and z_2 are arbitrary complex numbers, then

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof. We appeal to basic results:

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\,\overline{(z_1 + z_2)} && \text{(by Identity (1 – 22))}\\
&= (z_1 + z_2)\,(\overline{z_1} + \overline{z_2}) && \text{(by Identity (1 – 13))}\\
&= z_1\,\overline{z_1} + z_1\,\overline{z_2} + z_2\,\overline{z_1} + z_2\,\overline{z_2} && \text{(by the distributive law)}\\
&= |z_1|^2 + z_1\,\overline{z_2} + \overline{z_1}\,z_2 + |z_2|^2 && \text{(by the commutative law)}\\
&= |z_1|^2 + z_1\,\overline{z_2} + \overline{z_1\,\overline{z_2}} + |z_2|^2 && \text{(by (1 – 12))}\\
&= |z_1|^2 + 2\,\mathrm{Re}\,(z_1\,\overline{z_2}) + |z_2|^2 && \text{(by Identity (1 – 16))}\\
&\leq |z_1|^2 + 2\,|z_1\,\overline{z_2}| + |z_2|^2 && \text{(by Identity (1 – 22))}\\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

Taking square roots yields the desired inequality.

Example 1.5. To produce an example of which Figure 1.9 is a reasonable illustration, we let $z_1 = 7 + i$ and $z_2 = 3 + 5\,i$. Then $|z_1| = \sqrt{49 + 1} = \sqrt{50}$ and $|z_2| = \sqrt{9 + 25} = \sqrt{34}$. Clearly, $z_1 + z_2 = 10 + 6\,i$; hence $|z_1 + z_2| = \sqrt{100 + 36} = \sqrt{136}$. In this case, we can verify the triangle inequality without recourse to computation of square roots because

$$|z_1 + z_2| = \sqrt{100 + 36} = \sqrt{136} = 2\sqrt{34} = \sqrt{34} + \sqrt{34} < \sqrt{50} + \sqrt{34} = |z_1| + |z_2|,$$

thus

$$| z_1 + z_2 | \leq | z_1 | + | z_2 |.$$

Explore Solution 1.5.

We can also establish other important identities by means of the triangle inequality. Note that

$$\begin{aligned} | z_1 | &= | (z_1 + z_2) + (-z_2) | \\ &\leq | z_1 + z_2 | + | -z_2 | \\ &\leq | z_1 + z_2 | + | z_2 |. \end{aligned}$$

Subtracting $| z_2 |$ from the left and right sides of this string of inequalities gives an important relationship that will be used in determining lower bounds of sums of complex numbers:

$$| z_1 + z_2 | \geq | z_1 | - | z_2 |.$$

Using the identity $| z |^2 = z \overline{z}$ and the commutative and associative laws it follows that

$$\begin{aligned} | z_1 z_2 |^2 &= (z_1 z_2) \overline{(z_1 z_2)} \\ &= (z_1 \overline{z_1}) (z_2 \overline{z_2}) \\ &= | z_1 |^2 | z_2 |^2 \end{aligned}$$

Taking square roots of the terms on the left and right establishes another important identity

$$| z_1 z_2 | = | z_1 | | z_2 |.$$

As an exercise, we ask you to show

$$\left| \frac{z_1}{z_2} \right| = \frac{| z_1 |}{| z_2 |}, \text{ provided } z_2 \neq 0.$$

Example 1.6. Use the values $z_1 = 1 + 2\,i$ and $z_2 = 3 + 2\,i$, then $| z_1 | = \sqrt{1 + 4} = \sqrt{5}$ and $| z_2 | = \sqrt{9 + 4} = \sqrt{13}$. Also $z_1 z_2 = (1 + 2\,i)(3 + 2\,i) = -1 + 8\,i$; hence

$$| z_1 z_2 | = \sqrt{1 + 64} = \sqrt{65} = \sqrt{5}\,\sqrt{13} = | z_1 | | z_2 |,$$

thus

$$| z_1 z_2 | = | z_1 | | z_2 |.$$

Figure 1.10 illustrates the multiplication shown in Example1.6. The length of the $z_1 z_2$ vector apparently equals the product of the lengths of z_1 and z_2, confirming that $| z_1 z_2 | = | z_1 | | z_2 |$, but why is it located in the second quadrant when both z_1 and $z_2$are in the first quadrant? The answer to this question will become apparent to you in Section 1.4..

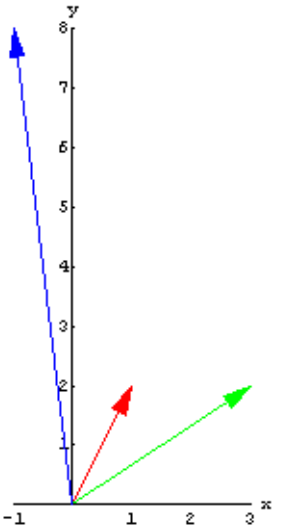

Figure 1.10 The geometry of multiplication.

Explore Solution 1.6.

Complex Analysis for Mathematics and Engineering

by

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1.4 The Geometry of Complex Numbers, Continued

In Secion 1.3 we saw that a complex number $z = x + iy$ could be viewed as a vector in the xy-plane whose tail is at the origin and whose head is at the point (x,y). A vector can be uniquely specified by giving its magnitude (i.e., its length) and direction (i.e., the angle it makes with the positive x-axis). In this section, we focus on these two geometric aspects of complex numbers.

Let r be the modulus of z (i.e., $r = |z|$), and let θ be the angle that the line from the origin to the complex number z makes with the positive x-axis. (Note: The number θ is undefined if $z=0$). We make the following definition.

Definition 1.9. (Polar Representation). The identity

$$z = (r\cos\theta, r\sin\theta) = r(\cos\theta + i\sin\theta)$$

is known as a polar representation of z, and the values r and θ are called polar coordinates of z, and is illustrated in Figure 1.11 (a).

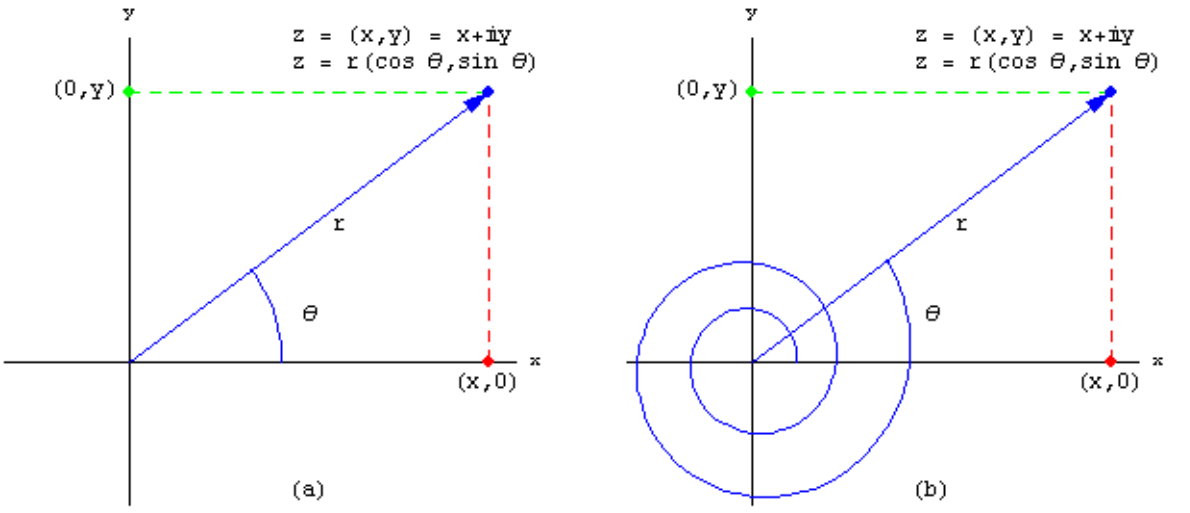

Figure 1.11 Polar representation of complex numbers.

As Figure 1.11(b) shows, the angle θ can be any value for which the identities $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$ hold true. Given $z \neq 0$, the collection of all values of θ for which $z = r(\cos\theta + i\sin\theta)$ is denoted by $\arg z$.

Example 1.7. If $z = 1 + i$, then $r = \sqrt{1+1} = \sqrt{2}$ and $z = \left(\sqrt{2}\cos\frac{\pi}{4}, \sqrt{2}\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ is a polar representation of z. The polar coordinates in this case are $r = \sqrt{2}$, and $\theta = \frac{\pi}{4}$.

Explore Solution 1.7.

Formally, we have the following definitions.

Definition 1.10, (Argument, arg z). If $z \neq 0$, we denote $\arg z$ by

$$\arg(z) = \{\theta : z = r(\cos\theta + i\sin\theta)\}.$$

If $\theta \in \arg z$, we say that θ is an argument of z.

Notice that we write $\theta \in \arg z$ as opposed to $\theta = \arg z$. This is because $\arg z$ is a set, and the designation $\theta \in \arg z$ indicates that θ belongs to that set. Notice also that, if $\theta_1 \in \arg z$ and $\theta_2 \in \arg z$, then there exists some integer n such that

$$\theta_2 = \theta_1 + 2n\pi.$$

Example 1.8. Since $1 + i = \sqrt{2}\left(\text{Cos}\left[\frac{\pi}{4}\right] + i\,\text{Sin}\left[\frac{\pi}{4}\right]\right)$, we have

$$\arg(1+i) = \left\{\frac{\pi}{4} + 2n\pi : n \text{ an integer}\right\}$$
$$= \left\{\ldots, \frac{-15\pi}{4}, \frac{-7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \frac{25\pi}{4}, \ldots\right\}.$$

Explore Solution 1.8.

Mathematicians have agreed to single out a special choice of $\theta \in \arg z$. It is that value of θ for which $-\pi < \theta \leq \pi$ as the following definition indicates.

Definition 1.11, (The principal value of argument, Arg z). Let $z \neq 0$, be a complex number. Then

$$\text{Arg}(z) = \{\theta : z = r(\cos\theta + i\sin\theta) \text{ and } -\pi < \theta \le \pi\}.$$

If $\theta = \text{Arg}(z)$, we call θ the argument of z.

Example 1.9. $\text{Arg}(1+i) = \frac{\pi}{4}$.

Explore Solution 1.9.

Remark. Clearly if $z = x + iy = r(\cos\theta + i\sin\theta)$, where $x \neq 0$, then

$$\arg(z) \subset \arctan\left(\frac{y}{x}\right).$$

where $\arctan\left(\frac{y}{x}\right) = \left\{\theta : \tan(\theta) = \frac{y}{x}\right\}$. Note that, as with $\arg(z)$, $\arctan\left(\frac{y}{x}\right)$ is a set (as opposed to Arctan, which is a number). We specifically identify $\arg(z)$ as a proper subset of $\arctan\left(\frac{y}{x}\right)$ because $\tan(\theta)$ has period π, whereas $\cos(\theta)$ and $\sin(\theta)$ have period 2π. In selecting the proper values for $\arg(z)$, we must be careful in specifying the choices of $\arctan\left(\frac{y}{x}\right)$ so that the point z associated with r and θ lies in the appropriate quadrant.

Example 1.10. If $z = -\sqrt{3} - i = r(\cos\theta + i\sin\theta)$, then

$$r = |z| = \left|-\sqrt{3} - i\right| = \sqrt{3+1} = 2 \quad \text{and}$$

$$\theta \in \arctan\left(\frac{y}{x}\right) = \arctan\frac{-1}{-\sqrt{3}} = \left\{\frac{\pi}{6} + n\pi : n \text{ is an integer}\right\}.$$

It would be a mistake to use $\frac{\pi}{6}$ as an acceptable value for θ, as the point z associated with $r = 2$ and $\theta = \frac{\pi}{6}$ is in the first quadrant, whereas $z = -\sqrt{3} - i$ is in the third quadrant. A correct choice for θ is $\theta = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$. Thus

$$\sqrt{3} - i = 2\,\text{Cos}\left[-\frac{5\pi}{6}\right] + i\,2\,\text{Sin}\left[-\frac{5\pi}{6}\right], \text{ and}$$

$$-\sqrt{3} - i = 2\,\text{Cos}\left[-\frac{5\pi}{6} + 2n\pi\right] + i\,2\,\text{Sin}\left[-\frac{5\pi}{6} + 2n\pi\right],$$

where n is any integer. In this case,

$$\text{Arg}\left(-\sqrt{3} - i\right) = -\frac{5\pi}{6}, \text{ and}$$

$$\arg\left(-\sqrt{3} - i\right) = \left\{-\frac{5\pi}{6} + 2n\pi : n \text{ is an integer}\right\}.$$

Remark. Note that $\arg\left(-\sqrt{3} - i\right) = \left\{-\frac{5\pi}{6} + 2n\pi : n \text{ is an integer}\right\}$ is indeed a proper subset of $\arctan\frac{-1}{-\sqrt{3}} = \left\{\frac{\pi}{6} + n\pi : n \text{ is an integer}\right\}$.

Explore Solution 1.10.

If $z \neq 0$ is on the y-axis, it would be a mistake to attempt to find $\text{Arg}\, z$ by looking at $\arctan \frac{y}{x}$, as $x = 0$, so $\frac{y}{x}$ is undefined. We emphasize these special cases:

$$\text{If } y > 0, \text{ then } \text{Arg}\,(0 + i\,y) = \frac{\pi}{2}, \text{ and}$$

$$\text{If } y < 0, \text{ then } \text{Arg}\,(0 + i\,y) = -\frac{\pi}{2}.$$

Example 1.11. If $z = x + i\,y = 0 + 4\,i$, it would be a mistake to attempt to find $\text{arg}\, z$ by looking at $\arctan \frac{y}{x}$, as $x = 0$, so $\frac{y}{x} = \frac{4}{0}$ is undefined. In this case

$$\text{Arg}\,(4\,i) = \text{Arg}\,(0 + 4\,i) = \frac{\pi}{2}, \text{ and}$$

$$\text{arg}\,(4\,i) = \left\{\frac{\pi}{2} + 2\,n\,\pi : n \text{ is an integer}\right\}.$$

As we shall see in Chapter 2, $\text{Arg}\, z$ is a discontinuous function of z because it "jumps" by an amount of $2\,\pi$ as z crosses the negative real axis.

In Chapter 5 we define e^z for any complex number z. You will see that this complex exponential has all the properties of real exponentials that you studied in earlier mathematics courses. That is, $e^{z_1}\,e^{z_2} = e^{z_1+z_2}$, and so forth. You will also see, amazingly, that if $z = x + i\,y$, then

$$e^z = e^{x+i\,y} = e^x\,(\cos y + i \sin y).$$

We will establish this result rigorously in Chapter 5, but there is a plausible explanation we can give now.

If e^z has the normal properties of an exponential, it must be that $e^{x+i\,y} = e^x\,e^{i\,y}$. Now, recall from Calculus the values of three infinite series:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}\,x^k, \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k)!}\,x^{2\,k}, \text{ and } \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\,k+1)!}\,x^{2\,k+1}.$$

Substituting $i\,y$ for x in the infinite series for e^x gives $e^{i\,y} = \sum_{k=0}^{\infty} \frac{1}{k!}\,(i\,y)^k + \sum_{k=0}^{\infty} \frac{1}{k!}\,i^k\,y^k$. At this point, our argument loses rigor because we have not talked about infinite series of complex numbers, let alone whether such series converge. Nevertheless, if we merely take the last series as a formal expression and split it into two series according to whether the index k is even ($k=2n$) or odd ($k=2n+1$), we get

$$\begin{aligned}
e^{i\,y} &= \sum_{k \text{ is even}}^{\infty} \frac{1}{k!}\,i^k\,y^k + \sum_{k \text{ is odd}}^{\infty} \frac{1}{k!}\,i^k\,y^k \\
&= \sum_{n=0}^{\infty} \frac{1}{(2\,n)!}\,i^{2\,n}\,y^{2\,n} + \sum_{n=0}^{\infty} \frac{1}{(2\,n+1)!}\,i^{2\,n+1}\,y^{2\,n+1} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2\,n)!}\,(i^2)^n\,y^{2\,n} + \sum_{n=0}^{\infty} \frac{1}{(2\,n+1)!}\,(i^2)^n\,i\,y^{2\,n+1} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2\,n)!}\,(-1)^n\,y^{2\,n} + i \sum_{n=0}^{\infty} \frac{1}{(2\,n+1)!}\,(-1)^n\,y^{2\,n+1} \\
&= \cos y + i \sin y
\end{aligned}$$

Exploration

Thus, it seems the only possible value for e^z is $e^z = e^x (\cos y + i \sin y)$. We will use this result freely from now on, and, as stated, supply a rigorous proof in Chapter 5.

If we set $x=0$ and let θ take the role of y in the above equation, we get a famous result known as Euler Formula:

$$e^{i\theta} = \cos\theta + i \sin\theta = (\cos\theta, \sin\theta).$$

If θ is a real number, $e^{i\theta}$ will be located somewhere on the circle with radius 1 centered at the origin. This assertion is easy to verify because

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1.$$

Figure 1.12 illustrates the location of the points $e^{i\theta}$ for various values of θ.

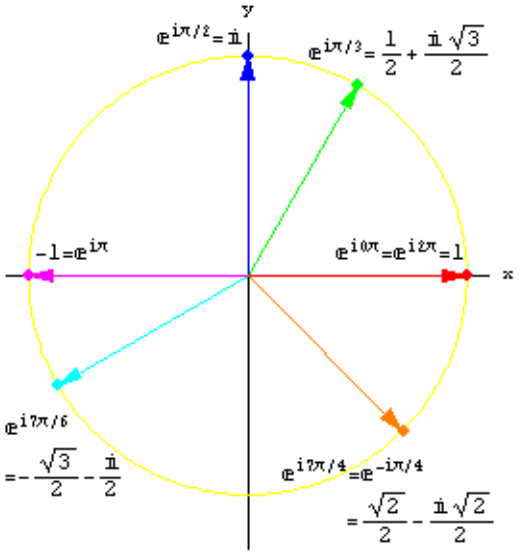

Figure 1.12 The location of $e^{i\theta}$ for various values of θ.

Notice that, when $\theta = \pi$, we get $e^{i\pi} = (\cos\pi, \sin\pi) = (-1, 0) = -1$, so

$$e^{i\pi} + 1 = 0.$$

Euler was the first to discover this relationship; it is referred to as Euler's identity. It has been labeled by many mathematicians as the most amazing relation in analysis---and with good reason. Symbols with a rich history are miraculously woven together---the constant used by Hippocrates as early as Hippocrates as early as 400 B.C.; e the base of the natural logarithms; the basic concepts of addition (+) and equality (=); the foundational whole numbers 0 and 1; and i, the number that is the central focus of this book.

Euler's formula is of tremendous use in establishing important algebraic and geometric properties of complex numbers. You will see shortly that it enables you to multiply complex numbers with great ease. It also allows you to express a polar form of the complex number z in a more compact way. Recall that if $r = |z|$ and $\theta \in \arg z$, then $z = r(\cos\theta + i \sin\theta)$. Using Euler's formula we can now write z in its exponential form:

$$z = r e^{i\theta}.$$

Example 1.12. In example 1.10 we investigated the polar form of $z = -\sqrt{3} - i$, and saw that $r = 2$ and $\theta = -\frac{5\pi}{6}$. Now we have

$$z = -\sqrt{3} - i = 2\, e^{i\,(-5\pi/6)} = 2\, e^{-i\,\frac{5\pi}{6}}.$$

Explore Solution 1.12.

Together with the rules for exponentiation that we will verify in Chapter 5, the exponential form has interesting applications. If $z_1 = r_1\, e^{i\theta_1}$ and $z_2 = r_2\, e^{i\theta_2}$, then

$$z_1\, z_2 = r_1\, e^{i\theta_1}\, r_2\, e^{i\theta_2} = r_1\, r_2\, e^{i\,(\theta_1+\theta_2)}$$

$$= r_1\, r_2\, (\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2))$$

Figure 1.13 illustrates the geometric significance of this equation.

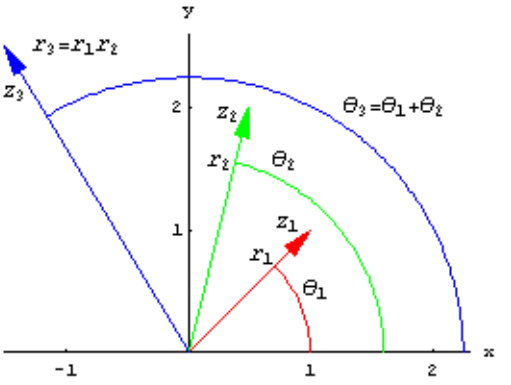

Figure 1.13 The product of two complex numbers $z_3 = z_1\, z_2$

We have already seen that the modulus of the product is the product of the moduli; that is, $|z_1\, z_2| = |z_1|\,|z_2|$. The above identity establishes that an argument of $z_1\, z_2$ is an argument of z_1 plus an argument of z_2. It also answers the question posed at the end of Section 1.3 regarding why the product $z_1\, z_2$ was in a different quadrant than either z_1 or z_2. It further offers an interesting explanation as to why the product of two negative real numbers is a positive real number. The negative numbers, each of which has an angular displacement of π radians, combine to produce a product that is rotated to a point with an argument of $\pi + \pi = 2\pi$ radians, coinciding with the positive real axis.

Using exponential form, if $z \neq 0$, we can write $\arg z$, a little more compactly as

$$\arg z = \{\theta : z = r\, e^{i\theta}\}.$$

Doing so enables us to see a nice relationship between the sets $\arg(z_1\, z_2)$, $\arg(z_1)$, and $\arg(z_2)$.

Theorem 1.3. If $z_1 = r_1\, e^{i\theta_1} \neq 0$ and $z_2 = r_2\, e^{i\theta_2} \neq 0$, then as sets,

$$\arg(z_1\, z_2) = \arg(z_1) + \arg(z_2).$$

Before proceeding with the proof, we recall two important facts about sets. First, to establish the equality of two sets, we must show that each is a subset of the other. Second, the sum of two

sets is the sum of all combinations of elements from the first and second sets, respectively. In this case, $\arg(z_1) + \arg(z_2) = \{\theta_1 + \theta_2 : \theta_1 \in \arg(z_1) \text{ and } \theta_2 \in \arg(z_2)\}$.

Proof Let $\theta \in \arg(z_1 z_2)$. Because $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$, it follows that $\theta_1 + \theta_2 \in \arg(z_1 z_2)$. Hence, there is some integer n such that $\theta = \theta_1 + \theta_2 + 2n\pi$. Further, as $z_1 = r_1 e^{i\theta_1}$ we have $\theta_1 \in \arg(z_1)$. Likewise, $z_2 = r_2 e^{i\theta_2}$ gives $\theta_2 \in \arg(z_2)$. But if $\theta_2 \in \arg(z_2)$, then $\theta_2 + 2n\pi \in \arg(z_2)$. This result shows that $\theta = \theta_1 + (\theta_2 + 2n\pi) \in \arg(z_1) + \arg(z_1)$. Thus, $\arg(z_1 z_2) \subset \arg(z_1) + \arg(z_2)$. The proof that $\arg(z_1) + \arg(z_2) \subset \arg(z_1 z_2)$ is left as an exercise.

Using the exponential form, $z = r e^{i\theta}$, we see that $z^{-1} = \frac{1}{z} = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta}$. In other words,

$$z^{-1} = \frac{1}{r}[\cos(-\theta) + i \sin(-\theta)]$$

$$= \frac{1}{r} e^{-i\theta}$$

Recalling that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, we also have

$$\overline{z} = r(\cos\theta - i \sin\theta) = r[\cos(-\theta) + i \sin(-\theta)]$$

$$= r e^{-i\theta}$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

$$= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

If z is in the first quadrant, the positions of the numbers z, $\overline{z}$, and z^{-1} are as shown in Figure 1.14 when $|z| < 1$. Figure 1.15 depicts the situation when $|z| > 1$.

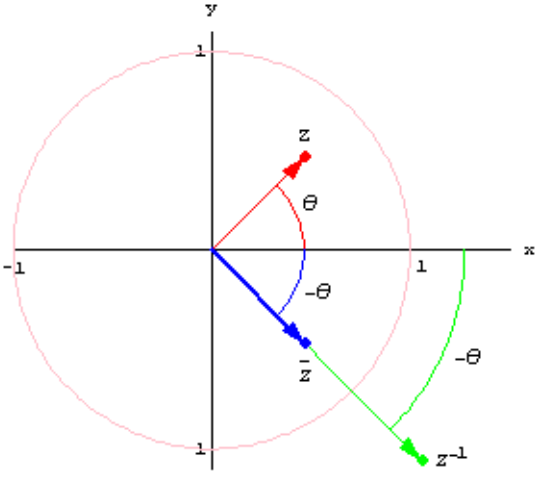

Figure 1.14 Relative positions z, $\overline{z}$, and z^{-1}, when $|z| < 1$.

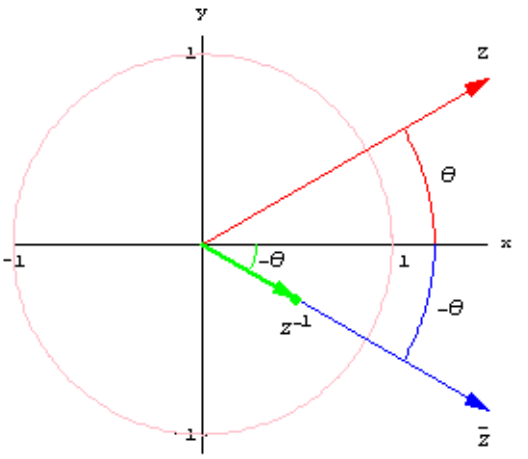

Figure 1.15 Relative positions of z, $\bar{z}$, and z^{-1}, when $|z| > 1$.

Example 1.13. If $z = 1 + i$, then $r = |1 + i| = \sqrt{1+1} = \sqrt{2}$ and $\theta = \text{Arg}(z) = \text{Arg}(1 + i) = \frac{\pi}{4}$. Therefore

$$z^{-1} = \frac{1}{r} e^{-i\pi/4}$$

$$= \frac{1}{\sqrt{2}}\left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{2} - \frac{i}{2}$$

the modulus is $|z^{-1}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = \frac{1}{|z|}$, and the argument is $\text{Arg}(z^{-1}) = -\frac{\pi}{4} = -\text{Arg}(z)$.

Explore Solution 1.13.

Example 1.14. Given $z_1 = 8\,i$ and $z_2 = 1 + i\sqrt{3}$, compute $\frac{z_1}{z_2}$ using polar computations.

If $z_1 = 8\,i$ and $z_2 = 1 + i\sqrt{3}$, then representative polar forms for these numbers are $z_1 = 8\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$ and $z_2 = 2\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$. Hence

$$\frac{z_1}{z_2} = \frac{8}{2} e^{i\left(\frac{\pi}{2} - \frac{\pi}{3}\right)}$$

$$= \frac{8}{2}\left[\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)\right]$$

$$= 4\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]$$

$$= 2\sqrt{3} + 2\,i$$

the modulus is $\left|\frac{z_1}{z_2}\right| = \frac{8}{2} = \frac{|z_1|}{|z_2|}$, and the argument is

$\text{Arg}\left(\frac{z_1}{z_2}\right) = \frac{\pi}{6} = \frac{\pi}{2} - \frac{\pi}{3} = \text{Arg}(z_1) - \text{Arg}(z_2)$.

Caveat. The formula $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2)$ does not hold for all complex numbers z_1 and z_2. Whereas, the formula $\text{arg}\left(\frac{z_1}{z_2}\right) = \text{arg}(z_1) - \text{arg}(z_2)$ does hold for all complex numbers z_1 and z_2, and is left for the reader to verify.

Explore Solution 1.14.

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1.5 The Algebra of Complex Numbers, Revisited

The real numbers are deficient in the sense that not all algebraic operations on them produce real numbers. Thus, for $\sqrt{-1}$ to make sense, we must consider the domain of complex numbers. Do complex numbers have this same deficiency? That is, if we are to make sense of expressions such as $\sqrt{1+i}$, must we appeal to yet another new number system? The answer to this question is no. In other words, any reasonable algebraic operation performed on complex numbers gives complex numbers. Later we show how to evaluate intriguing expressions such as $(-1)^i$. For now we only look at integral powers and roots of complex numbers.

The important players in this regard are the exponential and polar forms of a non-zero complex number $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. By the laws of exponents (which, you recall, we have promised to prove in Chapter 5) we have

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)],$$

and

$$z^{-n} = (re^{i\theta})^{-n} = r^{-n}e^{-in\theta} = r^{-n}[\cos(-n\theta) + i\sin(-n\theta)].$$

Example 1.15. Show that $\left(-\sqrt{3} - i\right)^3 = -8i$ in two ways.

Solution. (Method 1): The binomial formula (Exercise 14 of Section 1.2) gives

$$\left(-\sqrt{3} - i\right)^3 = 1\left(-\sqrt{3}\right)^3 + 3\left(-\sqrt{3}\right)^2(-i) + 3\left(-\sqrt{3}\right)(-i)^2 + (-i)^3$$

$$= -3\sqrt{3} + -9i + 3\sqrt{3} + i$$

$$= -8i$$

(Method 2): Using iIdentity stated above and Example 1.12 yields

$$\left(-\sqrt{3}-i\right)^3 = \left(2\,e^{i\,(-5\pi/6)}\right)^3$$

$$= (2)^3\left(e^{i\,(-5\pi/6)}\right)^3$$

$$= 8\left(e^{i\,(-15\pi/6)}\right)$$

$$= 8\left(\text{Cos}\left[\frac{-15\pi}{6}\right] + i\,\text{Sin}\left[\frac{-15\pi}{6}\right]\right)$$

$$= -8\,i$$

Explore Solution 1.15.

Which method would you use if you were asked to compute $\left(-\sqrt{3}-i\right)^{30}$?

Example 1.16. Evaluate $\left(-\sqrt{3}-i\right)^{30}$.

Solution. $\left(-\sqrt{3}-i\right)^{30} = \left(2\,e^{i\,(-5\pi/6)}\right)^{30}$

$$= 2^{30}\left(e^{i\,(-5\pi/6)}\right)^{30}$$

$$= 2^{30}\left(e^{i\,(-150\pi/6)}\right)$$

$$= 2^{30}\left(\text{Cos}\left[\frac{-150\pi}{6}\right] + i\,\text{Sin}\left[\frac{-150\pi}{6}\right]\right)$$

$$= 2^{30}\left(\text{Cos}[-25\pi] + i\,\text{Sin}[-25\pi]\right)$$

$$= -2^{30}$$

Explore Solution 1.16.

Extra Example. Evaluate $\left(-\sqrt{3}-i\right)^{-6} = \frac{-1}{64}$ in two ways.

Explore Extra Solution.

An interesting application of the laws of exponents comes from putting the equation $\left(e^{i\theta}\right)^n = e^{i\,n\theta}$ into its polar form. Doing so gives

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta),$$

which is known as De Moivre's Formula, in honor of the French mathematician Abraham de Moivre (1667--1754).

Example 1.17. Use DeMoivre's formula to show that
$\cos(5\theta) = \cos^5(\theta) - 10\cos^3(\theta)\sin^2(\theta) + 5\cos[\theta](\theta)\sin^4(\theta)$.

Solution. If we let $n=5$ and use the binomial formula to expand the left side of De Moivre's formula, we obtain

$$(\cos\theta + i \sin\theta)^5 = \cos^5\theta + i\, 5\cos^4\theta \sin\theta - 10\cos^3\theta \sin^2\theta - 10\, i \cos^2\theta \sin^3\theta + 5\cos\theta \sin^4\theta + i \sin^5\theta$$

$$(\cos\theta + i \sin\theta)^5 = \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta + i\left(5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta\right)$$

$$\cos(5\theta) + i \sin(5\theta) = \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta + i\left(5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta\right)$$

The real part of right side of this expression is $\cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$. Equating it to the real part of $\cos(5\theta) + i \sin(5\theta)$ establishes the desired result. Furthermore, it can be shown that $\sin(5\theta) = 5\cos^4\theta \sin\theta - 10\cos^2\theta \sin^3\theta + \sin^5\theta$.

Explore Solution 1.17.

A key aid in determining roots of complex numbers is a corollary to the Fundamental Theorem of Algebra. We prove this theorem in Chapter 6. Our proofs must be independent of the conclusions we derive here because we are going to make use of the corollary now.

Theorem 1.4 (Fundamental Theorem of Algebra). If $P(z)$ is a polynomial of degree n ($n > 0$) with complex coefficients, then the equation $P(z) = 0$ has precisely n (not necessarily distinct) solutions.

Proof. Refer to Chapter 6.

Example 1.18. Let $P(z) = z^3 + (2 - 2i) z^2 + (-1 - 4i) z - 2$. and find its zeros. This polynomial of degree 3 can be written as $P(z) = (z - i)^2 (z + 2)$. Hence the equation $P(z) = 0$ has solutions $z_1 = z_2 = i$ and $z_3 = -2$. Thus, in accordance with the Fundamental Theorem of Algebra, we have three solutions, with z_1 and z_2 being repeated roots.

Explore Solution 1.18.

The corollary implies that if we can find *n distinct* solutions to the equation $z^n = c$ (or $z^n - c = 0$) , we will have found *all* the solutions. We begin our search for these solutions by looking at the simpler equation $z^n = 1$. Solving this equation will enable us to handle the more general one quite easily.

To solve $z^n = 1$ we first note an important condition that determines when two nonzero complex numbers are equal. If we let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 = z_2 \text{ (i.e., } r_1 e^{i\theta_1} = r_2 e^{i\theta_2}) \text{ iff } r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2\pi k,$$

where k is an integer. That is, two complex numbers are equal iff their moduli agree and an argument of one equals an argument of the other to within an integral multiple of 2π.

We now find all solutions to $z^n = 1$ in two stages, with each stage corresponding to one direction in the iff part of the above relation. First, we show that if we have a solution to $z^n = 1$, then

the solution must have a certain form. Second, we show that any quantity with that form is indeed a solution.

For the first stage, suppose that $z = r e^{i\theta}$ is a solution to $z^n = 1$. Putting the latter equation in exponential form gives $r^n e^{i n\theta} = 1 e^{0 i}$, so we must have $r^n = 1$ and $n\theta = 0 + 2\pi k$. In other words, $r = 1$ and $\theta = \frac{2\pi k}{n}$, where k is an integer.

So, if $z = r e^{i\theta}$ is a solution to $z^n = 1$, then $r = 1$ and $\theta = \frac{2\pi k}{n}$ must be true. This observation completes the first stage of our solution strategy. For the second stage, we note that if $r = 1$, and $\theta = \frac{2\pi k}{n}$, then $z = r e^{i\theta} = e^{i 2\pi k/n}$ is indeed a solution to $z^n = 1$ because $z^n = (e^{i 2\pi k/n})^n = e^{i 2\pi k} = 1$. For example, if n=7 and k=3, then $z = e^{i 6\pi/7}$ is a solution to $z^7 = 1$ because $(e^{i 6\pi/7})^7 = e^{i 6\pi} = 1$.

Furthermore, it is easy to verify that we get n distinct solutions to $z^n = 1$ (and, therefore, all solutions) by setting $k = 0, 1, 2, \ldots, n-1$. The solutions for $k = n, n+1, \ldots$ merely repeat those for $k = 0, 1, 2, \ldots, n-1$, because the arguments so generated agree to within an integral multiple of 2π. As we stated in Section 1.1, the n solutions can be expressed as

$$z_k = e^{i 2k\pi/n} = \cos\frac{2\pi k}{n} + i \sin\frac{2\pi k}{n} \quad \text{for} \quad k = 0, 1, \cdots, n-1.$$

They are called the *n*th roots of unity.

When $k=0$ in the above equation, we get $z_0 = e^{i 2*0*\pi/n} = 1$, which is a rather trivial result. The first interesting root of unity occurs when $k=1$, giving $z_1 = e^{i 2\pi/n}$. This particular value shows up so often that mathematicians have given it a special symbol.

Definition 1.9 (Primitive nth Root of Unity). For any natural number n, the value ω_n given by

$$\omega_n = e^{i 2\pi/n} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$$

is called the primitive *n*th root of unity.

By De Moivre's formula, the *n*th roots of unity can be expressed as

$$1, \omega_n, \omega_n^2, \omega_n^3, \ldots, \omega_n^{n-1}.$$

Geometrically, the *n*th roots of unity are equally spaced points that lie on the unit circle $C_1(0) = \{z : |z| = 1\}$ and form the vertices of a regular polygon with n sides.

Example 1.19. The solutions to the equation $z^8 = 1$ are given by the eight values

$$z_k = e^{i\frac{2\pi k}{8}} = \cos\frac{2\pi k}{8} + i \sin\frac{2\pi k}{8} \quad \text{for} \quad k = 0, 1, 2, \ldots, 7.$$

In Cartesian form, these solutions are $\pm 1, \pm i, \pm\frac{\sqrt{2}+i\sqrt{2}}{2}, \pm\frac{\sqrt{2}-i\sqrt{2}}{2}$. The primitive 8th root of unity is

$$\omega_n = e^{i\frac{2\pi}{8}} = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{\sqrt{2}+i\sqrt{2}}{2}.$$

Figure 1.18 illustrates this result.

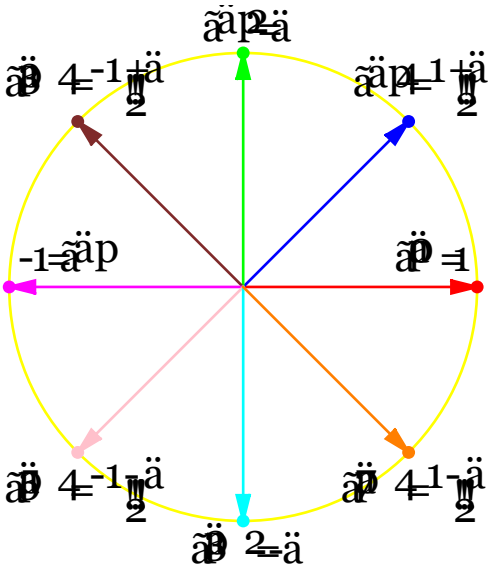

Figure 1.18 The eight eighth roots of unity.

Explore Solution 1.19.

The procedure for solving $z^n = 1$ is easy to generalize in solving $z^n = c$ for any nonzero complex number c. If $c = \rho\, e^{i\phi} = \rho\,(\cos\phi + i \sin\phi)$ and $z = r\, e^{i\theta}$, then $z^n = c$ iff $r^n\, e^{i n\theta} = \rho\, e^{i\phi}$. But this last equation is satisfied iff

$$r^n = \rho, \quad \text{and} \quad n\theta = \phi + 2k\pi, \quad \text{where } k \text{ is an integer.}$$

As before, we get n distinct solutions given by

$$z_k = \rho^{\frac{1}{n}}\, e^{i\,\frac{\phi + 2\pi k}{n}} = \rho^{\frac{1}{n}} \left(\cos\frac{\phi + 2\pi k}{n} + i \sin\frac{\phi + 2\pi k}{n}\right) \quad \text{for}$$

$k = 0, 1, \cdots, n-1$.

Each of the above solutions can be considered an n^{th} root of c. Geometrically, the n^{th} roots of c are equally spaced points that lie on the circle $C_{\rho^{\frac{1}{n}}} = \left\{z : |z| = \rho^{\frac{1}{n}}\right\}$ and form the vertices of a regular polygon with n sides. Figure 1.19 illustrates the case for $n=5$.

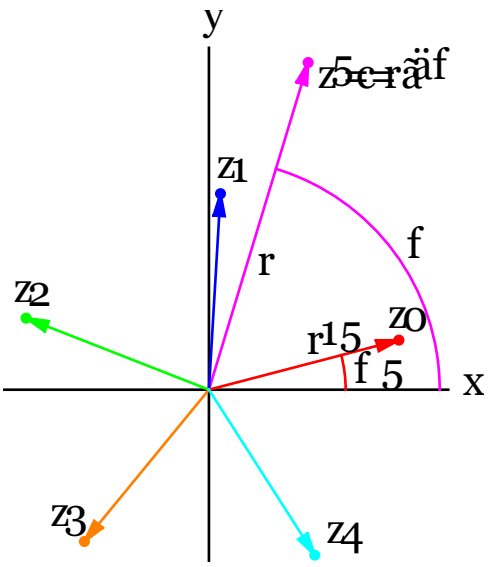

Figure 1.19 The five solutions to the equation $z^5 = c$.

It is interesting to note that if ζ is any particular solution to the equation $z^n = c$, then *all* solutions can be generated by multiplying by the various n^{th} roots of unity. That is, the solution set is

$$\{\zeta\, \omega_n^j\}_{j=0}^{n-1} = \{\zeta, \zeta\, \omega_n, \zeta\, \omega_n^2, \ldots, \zeta\, \omega_n^{n-1}\}.$$

The reason for this is that if $\zeta^n = c$, then for any $j = 0, 1, \cdots, n-1$ we have

$$(\zeta\, \omega_n^j)^n = \zeta^n\, (\omega_n^j)^n = \zeta^n\, (\omega_n^n)^j = \zeta^n\, (1) = c,$$

and that multiplying a number by $\omega_n = e^{i\frac{2\pi}{n}}$ increases an argument of that number by $\frac{2\pi}{n}$, so that $\{\zeta\, \omega_n^j\}_{j=0}^{n-1}$ contain n distinct values.

Example 1.20. Find all the cube roots of $8i = 8\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$, i.e. find all solutions to the equation $z^3 = 8i$.

Solution.

$$z_k = 8^{\frac{1}{3}} e^{i\left(\frac{\pi}{2} + 2\pi k\right)/3}$$

$$= 2\left(\cos\frac{\frac{\pi}{2} + 2\pi k}{3} + i \sin\frac{\frac{\pi}{2} + 2\pi k}{3}\right)$$

$$= 2\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i \sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$$

for $k = 0, 1, 2, \ldots, n-1$. The Cartesian forms of the solutions (shown in Figure 1.20) are

$$z_0 = \sqrt{3} + i, \quad z_1 = -\sqrt{3} + i, \quad \text{and} \quad z_2 = -2i.$$

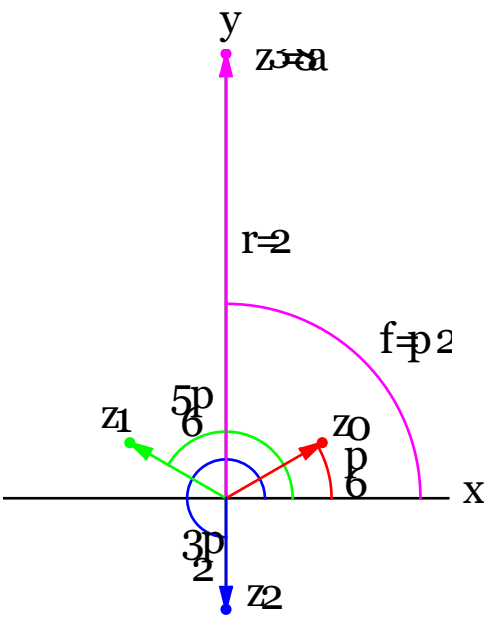

Figure 1.20 The point $z = 8\,i$ and its three cube roots $z_0, z_1,$ and z_2.

Explore Solution 1.20.

Is the quadratic formula valid in the complex domain? The answer is yes, but we will delay it's presentation until Section 2.2

Complex Analysis for Mathematics and Engineering

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1.6 The Topology of Complex Numbers

In this section we investigate some basic ideas concerning sets of points in the plane. The first concept is that of a curve. Intuitively, we think of a curve as a piece of string placed on a flat surface in some type of meandering pattern. More formally, we define a curve to be the range of a continuous complex-valued function $z(t)$ defined on the interval $[a, b]$. That is, a curve C is the range of a function given by $z(t) = (x(t), y(t)) = x(t) + i\,y(t)$, for $a \le t \le b$, where both $x(t)$ and $y(t)$ are continuous real-valued functions. If both $x(t)$ and $y(t)$ are differentiable, we say that the curve is smooth. A curve for which $x(t)$ and $y(t)$ are differentiable except for a finite number of points is called piecewise smooth.

Definition (Curve).

We specify a curve C as

$$C: z(t) = x(t) + i\,y(t) \quad \text{for} \quad a \le t \le b,$$

and say that $z(t)$ is a parametrization for the curve C. Notice that with this parametrization, we are specifying a direction of the curve C, and we say that C is a curve that goes from the initial point $z(a) = (x(a), y(a)) = x(a) + i\,y(a)$ to the terminal point $z(b) = (x(b), y(b)) = x(b) + i\,y(b)$. If we had another function whose range was the same set of points as z(t) but whose initial and final points were reversed, we would indicate the curve this function defines by $-C$.

Example 1.21. Find parametrizations for C and $-C$, where C is the straight line segment beginning at $z_0 = x_0 + i\,y_0$ and ending at $z_1 = x_1 + i\,y_1$.

Solution. Refer to Figure 1.21. The vector form of a line shows that the direction of C is $z_1 - z_0$. As z_0 is a point on C, its vector equation is

(1 – 48) $C: z(t) = z_0 + (z_1 - z_0)\, t$ for $0 \le t \le 1$, or

$$C: z(t) = (x_0 + (x_1 - x_0)\, t) + i\, (y_0 + (y_1 - y_0)\, t) \text{ for } 0 \le t \le 1.$$

Clearly one parametrization for $-C$ is

$$-C: \gamma(t) = z_1 + (z_0 - z_1)\, t \text{ for } 0 \le t \le 1, \text{ or}$$

Remark. Note that $\gamma(t) = z(1 - t)$, which illustrates a general principle: If C is a curve parametrized by $z(t)$ for $0 \le t \le 1$, then one parametrization for $-C$ will be $\gamma(t) = z(1 - t)$, for $0 \le t \le 1$.

Explore Solution 1.21.

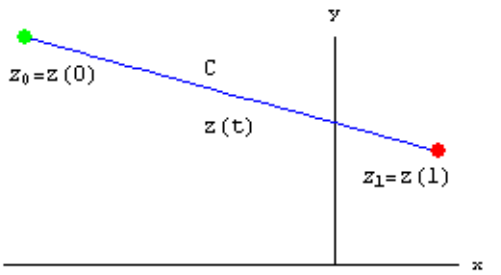

Figure 1.21 The straight-line segment C joining z_0 to z_1.

Extra Example 1. Find the equation of the line segment with the initial point $z_0 = -3 + 2\, i$ and the terminal point $z_1 = 1 + i$.

Explore Extra Solution 1.

A curve C having the property that $z(a) = z(b)$ is said to be a closed curve. The line segment (1-48) is not a closed curve. The range of $z(t) = x(t) + i\, y(t)$, where $x(t) = \sin 2\, t \cos t$, and $y(t) = \sin 2\, t \sin t$ for $0 \le t \le 2\, \pi$ is a closed curve because $z(0) = (0, 0) = z(2\, \pi)$. The range of $z(t)$ is the four-leaved rose shown in Figure 1.22. Note that, as t goes from 0 to $\frac{\pi}{2}$, the point is on leaf 1; from $\frac{\pi}{2}$ to π to , it is on leaf 2; between π and $\frac{3\, \pi}{2}$, it is on leaf 3; and finally, for t between $\frac{3\, \pi}{2}$ and $2\, \pi$, it is on leaf 4.

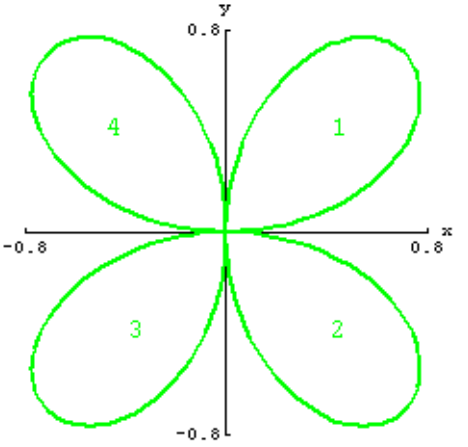

Figure 1.22 The curve $x(t) = \sin 2\, t \cos t$, and $y(t) = \sin 2\, t \sin t$ for

$0 \le t \le 2\pi$, which forms a four-leaved rose.

Note further, in Figure 1.22, that at the point $(0, 0)$, the curve has crossed over itself (at points other than those corresponding with $t = 0$ and $t = 2\pi$); we want to be able to distinguish when a curve does not cross over itself in this way. The curve C is called simple if it does not cross over itself, except possibly at its initial and terminal points. In other words, the curve $C : z(t)$, for $a \le t \le b$, is simple provided that $z(t_1) \ne z(t_2)$ whenever $t_1 \ne t_2$, except possibly when $t_1 = a$ and $t_2 = b$.

Extra Example 2. The curve $x(t) = \sin 2t \cos t$, and $y(t) = \sin 2t \sin t$ for $0 \le t \le 2\pi$.
Remark. The curve looks like a "four leafed rose".

Explore Extra Solution 2.

Example 1.22. Show that the circle C with center $z_0 = x_0 + i\, y_0$ and radius R can be parameterized to form a simple closed curve.

Solution. Note that the required parametrization is

$$C : z(t) = z_0 + R\, e^{i t}$$

$$C : z(t) = x_0 + i\, y_0 + R\, e^{i t}$$

$$C : z(t) = (x_0 + R \cos t) + i\, (y_0 + R \sin t),$$

$$\text{for } 0 \le t \le 2\pi.$$

Figure 1.23 shows that, as t varies from 0 to 2π, the circle is traversed counterclockwise. If you were traveling around the circle in this manner, its interior would be on your left. When a simple closed curve is parametrized in this fashion, we say that the curve has a positive orientation. We will have more to say about this idea shortly.

Explore Solution 1.22.

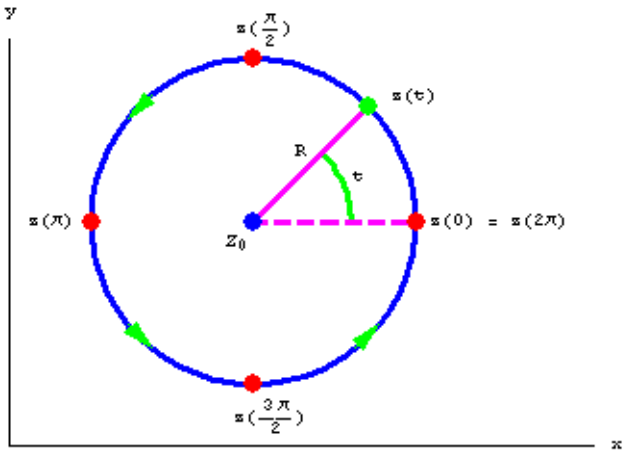

Figure 1.23 The simple closed curve $z(t) = z_0 + R\, e^{i t}$, for $0 \le t \le 2\pi$.

We need to develop some vocabulary that will help describe sets of points in the plane. One fundamental idea is that of an $\in$ neighborhood of the point z_0. It is the open disk of radius $\in > 0$

about z_0 shown in Figure 1.24. Formally, it is the set of all points satisfying the inequality $|z - z_0| < \epsilon$ and is denoted $D_\epsilon(z_0)$. That is,

(1-49) $$D_\epsilon(z_0) = \{z : |z - z_0| < \epsilon\}.$$

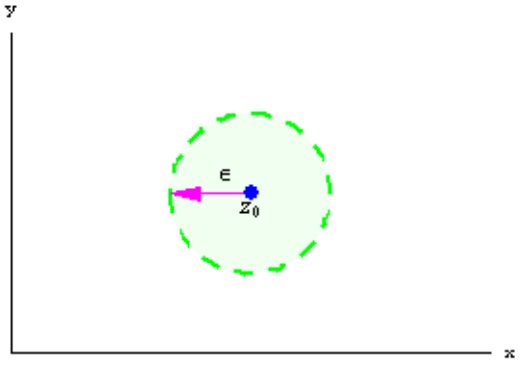

Figure 1.24 An ϵ neighborhood of the point z_0.

Example 1.23. The solution sets of the inequalities $|z| < 1$, $|z - i| < 2$, and $|z + 1 + 2i| < 3$ are neighborhoods of the points 0, i, and $-1 - 2i$, with radii 1, 2, and 3, respectively. They can also be expressed as $D_1(0)$, $D_2(i)$, and $D_3(-1 - 2i)$.

We also define $\overline{D_\epsilon}(z_0)$, the closed disk of radius ϵ centered at z_0, and $D_\epsilon^*(z_0)$, the punctured disk of radius ϵ centered at $z_0$as

(1-50) $$\overline{D_\epsilon}(z_0) = \{z : |z - z_0| \le \epsilon\}, \text{ and}$$

(1-51) $$D_\epsilon^*(z_0) = \{z : 0 < |z - z_0| < \epsilon\}.$$

Definitions (Interior Point, Exterior Point, Boundary Point).

The point z_0 is said to be an interior point of the set S provided that there exists an ϵ neighborhood of z_0 that contains only points of S; z_0 is called an exterior point of the set S if there exists an ϵ neighborhood of z_0 that contains no points of S. If z_0 is neither an interior point nor an exterior point of S, then it is called a boundary point of S and has the property that each ϵ neighborhood of z_0 contains both points in S and points not in S. Figure 1.25 illustrates this situation.

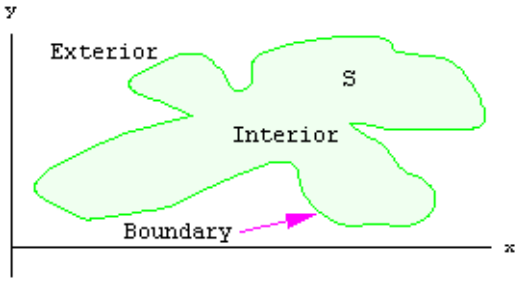

Figure 1.25 The interior, exterior, and boundary of a set S.

The boundary of $D_R(z_0)$ is the circle depicted in Figure 1.23. We denote this circle $C_R(z_0)$ and refer to it as the circle of radius R centered at z_0. Thus

(1-52) $$C_R(z_0) = \{z : |z - z_0| = R\}.$$

We use the notation $C_R^+(z_0)$ to indicate that the parametrization we chose for this simple closed curve resulted in a positive orientation; $C_R^-(z_0)$ denotes the same circle, but with a negative

orientation. (In both cases, counterclockwise denotes the positive direction.) Using notation that we have already introduced, we get $C_R^-(z_0) = -C_R^+(z_0)$.

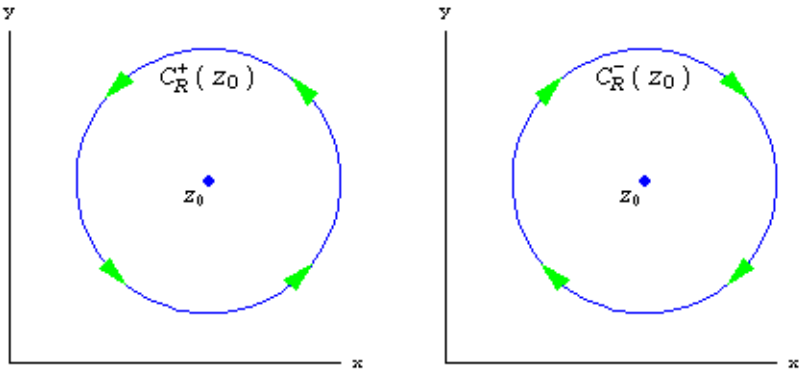

Figure 1.C The positively oriented curve $C_R^+(z_0)$ and its opposite $C_R^-(z_0) = -C_R^+(z_0)$.

Example 1.24. Let $S = \{z : |z| < 1\}$. **(a)** Find the interior of S. **(b)** Find the exterior of S. **(c)** Find boundary of S.

Solution. We show that every point of S is an interior point of S. Let z_0 be a point of S. Then $|z_0| < 1$, and we can choose $\epsilon = 1 - |z_0| > 0$. We claim that $D_\epsilon(z_0) \subset S$. If $z \in D_\epsilon(z_0)$, then

$$|z| = |z - z_0 + z_0| \le |z - z_0| + |z_0| < \epsilon + |z_0| = 1 - |z_0| + |z_0| = 1$$

Hence the ϵ neighborhood of z_0 is contained in S, which shows that z_0 is an interior point of S. It follows that the interior of S is the set S itself.

Similarly, it can be shown that the exterior of S is the set $\{z : |z| > 1\}$. The boundary of S is the unit circle $C_1(0) = \{z : |z| = 1\}$. This condition is true because if $z_0 = e^{i\theta_0}$ is any point on the circle, then any ϵ-neighborhood of z_0 will contain the point $\left(1 - \frac{\epsilon}{2}\right) e^{i\theta_0}$, which belongs to S, and $\left(1 + \frac{\epsilon}{2}\right) e^{i\theta_0}$, which does not belong to S. We leave the details as an exercise.

Explore Solution 1.24 (a). Find the interior of S.

Explore Solution 1.24 (b). Find the exterior of S.

Explore Solution 1.24 (c). Find the boundary of S.

The point z_0 is called an accumulation point of the set S if, for each ϵ, the punctured disk $D_\epsilon^*(z_0)$ contains at least one point of S. We ask you to show in the exercises that the set of accumulation points of $D_1(0)$ is $\overline{D_1}(0)$, and that there is only one accumulation point of $S = \left\{\frac{i}{n} : n = 1, 2, \ldots\right\}$, namely, the point 0.
We also ask you to prove that a set is closed if and only if it contains all of its accumulation points.

A set S is called an open set if every point of S is an interior point of S. Thus, Example 1.25 shows that $D_1(0)$ is open. A set S s called a closed set if it contains all its boundary points. A set S is said to be a connected set if every pair of points z_1 and z_2 contained in S can be joined by a curve that lies entirely in S. Roughly speaking, a connected set consists of a "single piece." The unit disk

$D_1(0) = \{z : |z| < 1\}$ is a connected open set. We ask you to verify in the exercises that, if z_1 and z_2 lie in $D_1(0)$, then the straight-line segment joining them lies entirely in $D_1(0)$. The annulus $A = \{z : 1 < |z| < 2\}$ is a connected open set because any two points in A can be joined by a curve C that lies entirely in A, as shown in Figure 1.26. The set $B = \{z : |z+2| < 1 \text{ or } |z-2| < 1\}$ consists of two disjoint disks. We leave it as an exercise for you to show that the set is not connected, as shown in Figure 1.27.

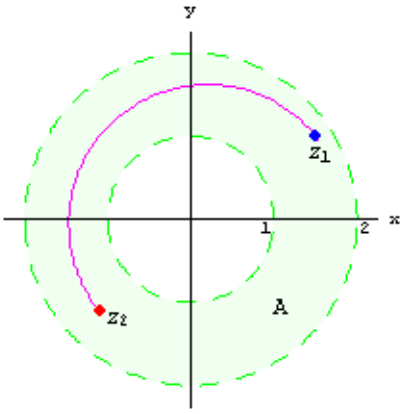

Figure 1.26 The annulus $A = \{z : 1 < |z| < 2\}$ is a connected set.

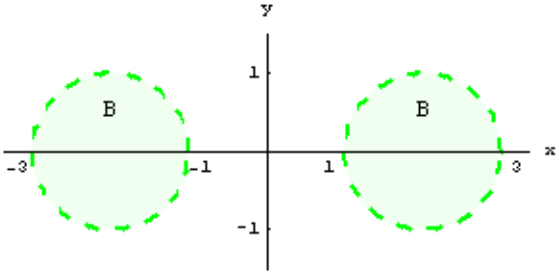

Figure 1.27 The set $B = \{z : |z+2| < 1 \text{ or } |z-2| < 1\}$ is not a connected set.

We call a connected open set a domain. In the exercises we ask you to show that the open unit disk $D_1(0) = \{z : |z| < 1\}$ is a domain and that the closed unit disk $\overline{D_1}(0) = \{z : |z| \le 1\}$ is not a domain. The term domain is a noun and is a type of set. In Chapter 2 we note that it also refers to the set of points on which a function is defined. In the latter context, it does not necessarily mean a connected open set.

Example 1.25. Show that the right half-plane $H = \{z : \text{Re}(z) > 0\}$ is a domain.

Solution. First we show that H is connected. Let z_0 and z_1 be any two points in H. We claim the obvious, that the straight-line segment $C : z(t) = z_0 + (z_1 - z_0)\,t$, for $0 \le t \le 1$, lies entirely within H. To prove this claim, we let $z(t^*) = z_0 + (z_1 - z_0)\,t^*$, for some $t^* \in [0, 1]$, be an arbitrary point on C. We must show that $\text{Re}[z(t^*)] > 0$. Now,

(1 – 53)
$$\begin{aligned} \mathrm{Re}[z(t^*)] &= \mathrm{Re}[z_0 + (z_1 - z_0)\,t^*] \\ &= \mathrm{Re}[(1 - t^*)\,z_0 + t^*\,z_1] \\ &= \mathrm{Re}[(1 - t^*)\,z_0] + \mathrm{Re}[t^*\,z_1] \\ &= (1 - t^*)\,\mathrm{Re}[z_0] + t^*\,\mathrm{Re}[z_1] \end{aligned}$$

If $t = 0$, the last expression becomes $\mathrm{Re}[z_0]$, which is greater than zero because $z_0 \in H$. Likewise, if $t = 1$, then Equation (1-53) becomes $\mathrm{Re}[z_1]$, which also is positive. Finally, if $0 \le t \le 1$, then each term in Equation (1-53) is positive, so in this case we also have $\mathrm{Re}[z(t^*)] > 0$.

To show that H is open, we suppose without loss of generality that $\mathrm{Re}[z_0] < \mathrm{Re}[z_1]$. We claim that $D_\epsilon(z_0) \subset H$, where $\epsilon = \mathrm{Re}[z_0]$. We leave the proof of this claim as an exercise.

A domain, together with some, none, or all its boundary points, is called a region. For example, the horizontal strip $\{z : 1 < \mathrm{Im}(z)\}$ is a region. A set formed by taking the union of a domain and its boundary is called a closed region; thus $\{z : 1 \le \mathrm{Im}(z) \le 2\}$ is a closed region. A set S is said to be a bounded set if it can be completely contained in some closed disk, that is, if there exists an $R > 0$ such that for each z in S we have $|z| \le R$. The rectangle given by $\{z : |x| \le 4 \text{ and } |y| \le 3\}$ is bounded because it is contained inside the disk $\overline{D_5}(0) = \{z : |z| \le 5\}$. A set that cannot be enclosed by any closed disk is called an unbounded set.

We mentioned earlier that a simple closed curve is positively oriented if its interior is on the left when the curve is traversed. How do we know, though, that any given simple closed curve will have an interior and exterior? Theorem 1.6 guarantees that this is indeed the case. It is due in part to the work of the French mathematician Marie Ennemond Camille Jordan (1838--1922).

Theorem 1.5 (Jordan Curve Theorem). The compliment of any simple closed curve C can be partitioned into two mutually exclusive domains I and E in such a way that I is bounded, E is unbounded, and C is the boundary for both I and E. In addition $I \cup E \cup C$ is the entire complex plane. The domain I is called the interior of C, and the domain E is called the exterior of C.

The Jordan curve theorem is a classic example of a result in mathematics that seems obvious but is very hard to demonstrate, and its proof is beyond the scope of this book. Jordan's original argument, in fact, was inadequate, and not until 1905 was a correct version finally given by the American topologist Oswald Veblen (1880-1960). The difficulty lies in describing the interior and exterior of a simple closed curve analytically, and in showing that they are connected sets. For example, in which domain (interior or exterior) do the two points depicted in Figure 1.28 lie? If they are in the same domain, how, specifically, can they be connected with a curve? If you appreciated the subtleties involved in showing that the right half-plane of Example 1.26 is connected, you can begin to appreciate the obstacles that Veblen had to navigate.

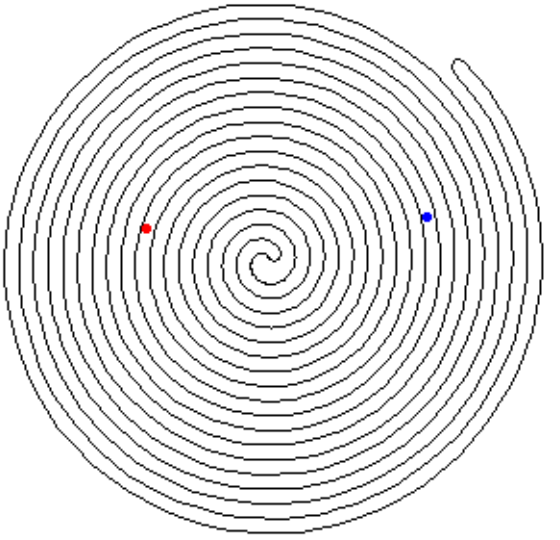

Figure 1.28 Are the points z_1 (blue), and z_2 (red) in the interior or exterior of this simple closed curve?

Exploration

Although an introductory treatment of complex analysis can be given without using this theorem, we think it is important for the well-informed student at least to be aware of it.

Complex Analysis for Mathematics and Engineering

by

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Chapter 2 Complex Functions

Overview

The last chapter developed a basic theory of complex numbers. For the next few chapters we turn our attention to functions of complex numbers. They are defined in a similar way to functions of real numbers that you studied in calculus; the only difference is that they operate on complex numbers rather than real numbers. This chapter focuses primarily on very basic functions, their representations, and properties associated with functions such as limits and continuity. You will learn some interesting applications as well as some exciting new ideas.

2.1 Functions and Linear Mappings

A complex-valued function f of the complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We write $w = f(z)$ and call w the image of z under f. A simple example of a complex-valued function is given by the formula $w = f(z) = z^4$. The set D is called the domain of f, and the set of all images $\{w = f(z) : z \in D\}$ is called the range of f. When the context is obvious, we omit the phrase complex-valued, and simply refer to a function f, or to a complex function f.

We can define the domain to be any set that makes sense for a given rule, so for $w = f(z) = z^4$, we could have the entire complex plane for the domain D, or we might artificially restrict the domain to some set such as $D = D_1(0) = \{z : |z| < 1\}$. Determining the range for a function defined by a formula is not always easy, but we will see plenty of examples later on. In some contexts functions are referred to as mappings or transformations.

In Section 1.6, we used the term domain to indicate a connected open set. When speaking about the domain of a function, however, we mean only the set of points on which the function is defined. This distinction is worth noting, and context will make clear the use intended.

Just as z can be expressed by its real and imaginary parts, $z = x + iy$, we write $f(z) = w = u + iv$, where u and v are the real and imaginary parts of w, respectively. Doing so gives us the representation

$$w = f(z) = f(x, y) = f(x + i\,y) = u + i\,v.$$

Because u and v depend on x and y, they can be considered to be real-valued functions of the real variables x and y; that is,

$$u = u(x, y) \text{ and } v = v(x, y).$$

Combining these ideas, we often write a complex function f in the form

$$f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y).$$

Figure 2.1 illustrates the notion of a function (mapping) using these symbols.

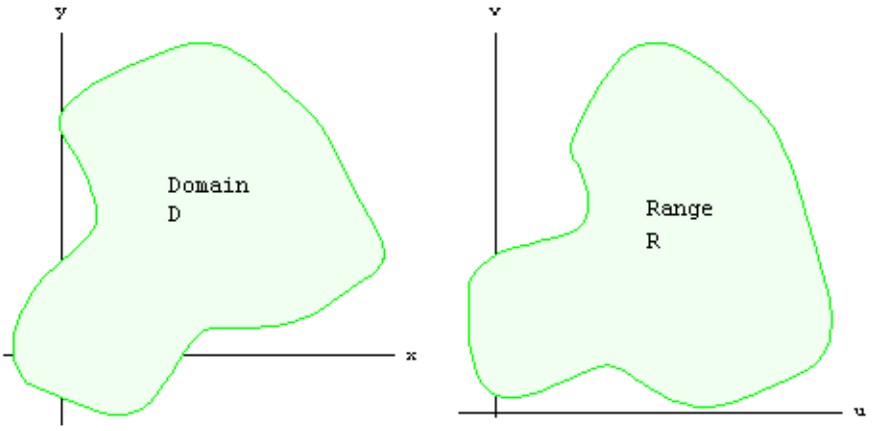

Figure 2.1 The mapping $w = f(z) = u(x, y) + i\,v(x, y)$.

There are two methods for defining a complex function in *Mathematica*.

Exploration.

We now give several examples that illustrate how to express a complex function.

Example 2.1. Write $f(z) = z^4$ in the for $f(z) = u(x, y) + i\,v(x, y)$.

Solution. Using the binomial formula, we obtain

$$\begin{aligned} f(z) &= f(x + i\,y) = (x + i\,y)^4 \\ &= x^4 + 4\,x^3\,(i\,y) + 6\,x^2\,(i\,y)^2 + 4\,x\,(i\,y)^3 + (i\,y)^4 \\ &= x^4 + 4\,i\,x^3\,y - 6\,x^2\,y^2 - 4\,i\,x\,y^3 + y^4 \\ &= x^4 - 6\,x^2\,y^2 + y^4 + i\,(4\,x^3\,y - 4\,x\,y^3) \\ &= u(x, y) + i\,v(x, y) \end{aligned}$$

so that $u(x, y) = x^4 - 6\,x^2\,y^2 + y^4$ and $v(x, y) = 4\,x^3\,y - 4\,x\,y^3$.

Explore Solution 2.1.

Example 2.2. Express the function $f(z) = \overline{z}\,\mathrm{Re}[z] + z^2 + \mathrm{Im}[z]$ in the form $f(z) = u(x, y) + i\,v(x, y)$.

Solution. Using the elementary properties of complex numbers, it follows that

$$f(z) = (x - i\,y)\,x + (x + i\,y)^2 + y = (2\,x^2 - y^2 + y) + i\,(x\,y)$$

so that $u(x, y) = 2\,x^2 - y^2 + y$ and $v(x, y) = x\,y$.

Explore Solution 2.2.

Examples 2.1 and 2.2 show how to find $u(x, y)$ and $v(x, y)$ when a rule for computing f is given. Conversely, if $u(x, y)$ and $v(x, y)$ are two real-valued functions of the real variables x and y, they determine a complex-valued function $f(z) = u(x, y) + i\,v(x, y)$, and we can use the formulas

$$x = \frac{z + \overline{z}}{2} \text{ and } y = \frac{z - \overline{z}}{2\,i}$$

to find a formula for f involving the variables z and $\overline{z}$.

Example 2.3. Express $f(z) = 4\,x^2 + i\,4\,y^2$ by a formula involving the variables z and $\overline{z}$.

Solution. Calculation reveals that

$$f(z) = 4\left(\frac{z + \overline{z}}{2}\right)^2 + i\,4\left(\frac{z - \overline{z}}{2\,i}\right)^2$$

$$= z^2 + 2\,z\,\overline{z} + \overline{z}^2 - i\left(z^2 - 2\,z\,\overline{z} + \overline{z}^2\right)$$

$$= (1 - i)\,z^2 + (2 + 2\,i)\,z\,\overline{z} + (1 - i)\,\overline{z}^2$$

Explore Solution 2.3.

Using $z = r\,e^{i\,\theta}$ in the expression of a complex function f may be convenient. It gives us the polar representation

$$f(z) = f(r\,e^{i\,\theta}) = U(r, \theta) + i\,V(r, \theta),$$

where U and V are real functions of the real variables r and θ.

Remark. For a given function f, the functions u and v defined above are different from those used previously in $f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$ which used Cartesian coordinates instead of polar coordinates.

Example 2.4. Express $f(z) = z^2$ in both Cartesian and polar form.

Solution. For the Cartesian form, a simple calculation gives

$$f(z) = f(x + i\,y) = (x + i\,y)^2$$

$$= x^2 - y^2 + 2\,i\,x\,y$$

$$= u(x, y) + i\,v(x, y)$$

so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2\,x\,y$.

For the polar form, we get v

$$f(z) = f(r\,e^{i\theta}) = (r\,e^{i\theta})^2 = r^2\,e^{i\,2\theta}$$

$$= r^2\,(\cos 2\theta + i \sin 2\theta)$$

$$= r^2 \cos 2\theta + i\,r^2 \sin 2\theta$$

$$= U(r, \theta) + i\,V(r, \theta)$$

so that $U(r, \theta) = r^2 \cos 2\theta$ and $V(r, \theta) = r^2 \sin 2\theta$.

Explore Solution 2.4.

Remark. Once we have defined u and v for a function f in Cartesian form, we must use different symbols if we want to express f in polar form. As is clear here, the functions u and U are quite different, as are v and V. Of course, if we are working only in one context, we can use any symbols we choose.

For a given function f, the functions u and v defined here are different from those defined by equation (2-1), because equation (2-1) involves Cartesian coordinates and equation (2-2) involves polar coordinates.

Example 2.5. Express $f(z) = z^5 + 4\,z^2 - 6$ in polar form.

Solution. We obtain

$$f(z) = f(r\,e^{i\theta}) = (r\,e^{i\theta})^5 + 4\,(r\,e^{i\theta})^2 - 6$$

$$= r^5\,e^{i\,5\theta} + 4\,r^2\,e^{i\,2\theta} - 6$$

$$= r^5 \cos 5\theta + 4\,r^2 \cos 2\theta - 6 + i\,(r^5 \sin 5\theta + 4\,r^2 \sin 2\theta)$$

$$= U(r, \theta) + i\,V(r, \theta)$$

so that $U(r, \theta) = r^5 \cos 5\theta + 4\,r^2 \cos 2\theta - 6$ and $V(r, \theta) = r^5 \sin 5\theta + 4\,r^2 \sin 2\theta$.

Explore Solution 2.5.

We now look at the geometric interpretation of a complex function. If D is the domain of real-valued functions u(x,y) and v(x,y), the equations

$$u = u(x, y) \quad \text{and} \quad v = v(x, y)$$

describe a transformation (or mapping) from D in the xy plane into the uv plane, also called the w plane. Therefore, we can also consider the function

$$w = f(z) = u(x, y) + i\, v(x, y)$$

to be a transformation (or mapping) from the set D in the z plane onto the range R in the w plane. This idea was illustrated in Figure 2.1. In the following paragraphs we present some additional key ideas. They are staples for any kind of function, and you should memorize all the terms in bold.

If A is a subset of the domain D of f, the set $B = \{w = f(z) : z \in A\}$ is called the image of the set A, and f is said to map A onto B. The image of a single point is a single point, and the image of the entire domain, D, is the range, R. The mapping $w = f(z)$ is said to be from A into S if the image of A is contained in S. Mathematicians use the notation $f : A \to S$ to indicate that a function maps A into S. Figure 2.2 illustrates a function f whose domain is D and whose range is R. The shaded areas depict that the function maps A onto B. The function also maps A into R, and, of course, it maps D onto R.

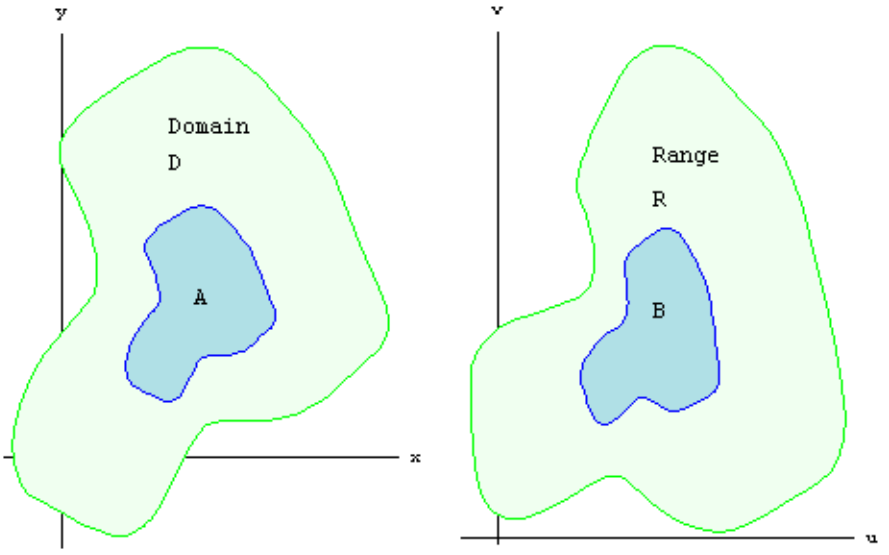

Figure 2.2 $w = f(z)$ maps A onto B; $w = f(z)$ maps A into R.

The inverse image of a point w is the set of all points z in D such that $w = f(z)$. The inverse image of a point may be one point, several points, or nothing at all. If the last case occurs then the point w is not in the range of f. For example, if $w = f(z) = i\, z$, the inverse image of the point -1 is the single point i, because $w = f(i) = i\,(i) = -1$, and i is the only point that maps to -1. In the case of $w = f(z) = z^2$, the inverse image of the point -1 is the set $\{i, -i\}$. You will learn in Chapter 5 that, if $w = f(z) = e^z$, the inverse image of the point 0 is the empty set---there is no complex number z such that $e^z = 0$.

The inverse image of a set of points, S, is the collection of all points in the domain that map into S. If f maps D onto R it is possible for the inverse image of R to be function as well, but the original function must have a special property: a function f is said to be one-to-one if it maps distinct points $z_1 \neq z_2$ onto distinct points $f(z_1) \neq f(z_2)$. Many times an easy way to prove that a function f is one-to-one is to suppose $f(z_1) = f(z_2)$, and from this assumption deduce that z_1 must equal z_2. Thus, $f(z) = i z$ is one-to-one because if $f(z_1) = f(z_2)$, then $i z_1 = i z_2$. Dividing both sides of the last equation by i gives $z_1 = z_2$. Figure 2.3 illustrates the idea of a one-to-one function: distinct points get mapped to distinct points.

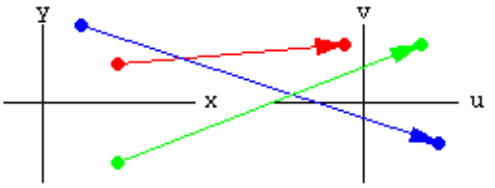

Figure 2.3 A function $w = f(z)$ that is one-to-one.

The function $f(z) = z^2$ is not one-to-one because $-i \neq i$, but $f(i) = f(-i) = -1$. Figure 2.4 depicts this situation: at least two different points get mapped to the same point.

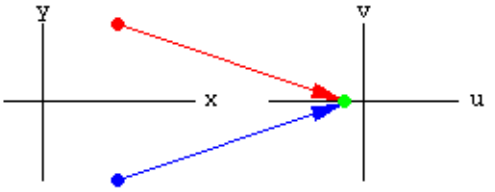

Figure 2.4 A function that is not one-to-one.

In the exercises we ask you to demonstrate that one-to-one functions give rise to inverses that are functions. Loosely speaking, if $w = f(z)$ maps the set A one-to-one and onto the set B, then for each w in B there exists exactly one point z in A A such that $w = f(z)$. For any such value of z we can take the equation $w = f(z)$ and "solve" for z as a function of w. Doing so produces an inverse function $z = g(w)$ where the following equations hold:

$$g(f(z)) = z \quad \text{for all } z \in A, \quad \text{and}$$

$$f(g(w)) = w \quad \text{for all } z \in B$$

Conversely, if $w = f(z)$ and $z = g(w)$ are functions that map A into B and B into A, respectively, and the above hold, then f maps the set A one-to-one and onto the set B.

Further, if f is a one-to-one mapping from D onto T and if A is a subset of D, then f is a one-to-one mapping from A onto its image B. We can also show that, if $\zeta = f(z)$ is a one-to-one mapping from A onto B and $w = g(\zeta)$ is a one-to-one mapping from B onto S, then the composite mapping $w = g(f(z))$ is a one-to-one mapping from A onto S.

We usually indicate the inverse of f by the symbol f^{-1}. If the domains of f and f^{-1} are A and B respectively, then we write

$$f^{-1}(f(z)) = z \quad \text{for all } z \in A, \text{ and}$$

$$f(f^{-1}(w)) = w \quad \text{for all } w \in B.$$

Also, for $z_0 \in A$ and $w_0 \in B$.

$$w_0 = f(z_0) \quad \text{iff} \quad f^{-1}(w_0) = z_0, \quad \text{and}$$

$$z_0 = f^{-1}(w_0) \quad \text{iff} \quad f(z_0) = w_0.$$

Example 2.6. If $w = f(z) = i\,z$ for any complex number z, find $f^{-1}(w)$.

Solution. We can easily show f is one-to-one and onto the entire complex plane. We solve for z, given $w = f(z) = i$, to get $z = \frac{w}{i} = -i\,w$. This result implies that $f^{-1}(w) = -i\,w$ for all complex numbers w.

Remark. Once we have specified $f^{-1}(w) = -i\,w$ for all complex numbers w, we note that there is nothing magical about the symbol w. We could just as easily write $f^{-1}(z) = -i\,z$ for all complex numbers z.

Explore Solution 2.6.

We now show how to find the image B of a specified set A under a given mapping $u + i\,v = w = f(z)$. The set A is usually described with an equation or inequality involving x and y. Using inverse functions, we can construct a chain of equivalent statements leading to a description of the set B in terms of an equation or an inequality involving u and v.

Example 2.7. Show that the function $f[z] = i\,z$ maps the line $y = x + 1$ in the xy plane onto the line $v = -u - 1$ in the w plane.

Solution. Method 1: With $A = \{(x, y) : y = x + 1\}$, we want to describe $B = f(A)$. We let $z = x + i\,y \in A$ and get

$$\begin{aligned}
&u + i\,v = w = f(z) \in B \\
&\Longleftrightarrow f^{-1}(w) = z = x + i\,y \in A \\
&\Longleftrightarrow i\,w \in A \\
&\Longleftrightarrow v - i\,u \in A \\
&\Longleftrightarrow (v, -u) \in A \\
&\Longleftrightarrow u = v + 1 \\
&\Longleftrightarrow v = -u - 1
\end{aligned}$$

where $\Longleftrightarrow$ is the notation for "if and only if." Note what this result says: $u + i\,v = w \in B \Longleftrightarrow v = -u - 1$. The image of A under f, therefore, is the set $B = \{(u, v) : v = -u - 1\}$.

Method 2: We write $u + i\,v = w = f(z) = i\,(x + i\,y) = -y + i\,x$ and note that the transformation can be given by the equations $u = -y$ and $v = x$. Because A is described by $A = \{x + i\,y : y = x + 1\}$, we can substitute $u = -y$ and $v = x$ into the equation $y = x + 1$ to obtain $-u = v + 1$, which we can rewrite as $v = -u - 1$. If you use this method, be sure to pay careful attention to domains and ranges.

Explore Solution 2.7.

We now look at some elementary mappings. If we let $B = a + ib$ denote a fixed complex constant, the transformation

$$w = T(z) = z + B = x + a + i(y + b)$$

is a one-to-one mapping of the z-plane onto the w-plane and is called a translation. This transformation can be visualized as a rigid translation whereby the point z is displaced through the vector $B = a + ib$ to its new position $w = T(z)$. The inverse mapping is given by

$$z = T^{-1}(w) = w - B = u - a + i(v - b)$$

and shows that T is a one-to-one mapping from the z-plane onto the w-plane. The effect of a translation is depicted in Figure 2.5.

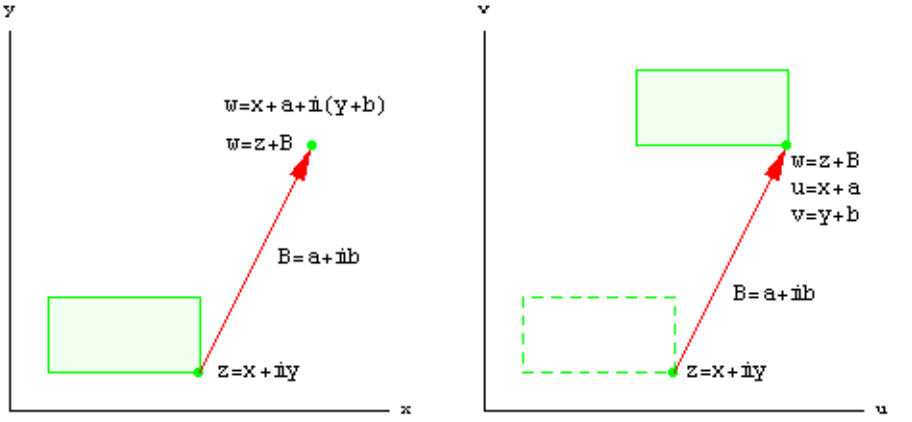

Figure 2.5 The translation $w = T(z) = z + B = x + a + i(y + b)$.

If we let α be a fixed real number, then for $z = r e^{i\theta}$, the transformation

$$w = R(z) = z e^{i\alpha} = r e^{i\theta} e^{i\alpha} = r e^{i(\theta+\alpha)}$$

is a one-to-one mapping of the z-plane onto the w-plane and is called a rotation. It can be visualized as a rigid rotation whereby the point z is rotated about the origin through an angle α to its new position $w = R(z)$. If we use polar coordinates and designate $w = \rho e^{i\phi}$ in the w-plane, then the inverse mapping is

$$z = R^{-1}(w) = w e^{-i\alpha} = \rho e^{i\phi} e^{-i\alpha} = \rho e^{i(\phi-\alpha)}.$$

This analysis shows that R is a one-to-one mapping from the z-plane onto the w-plane. The effect of rotation is depicted in Figure 2.6.

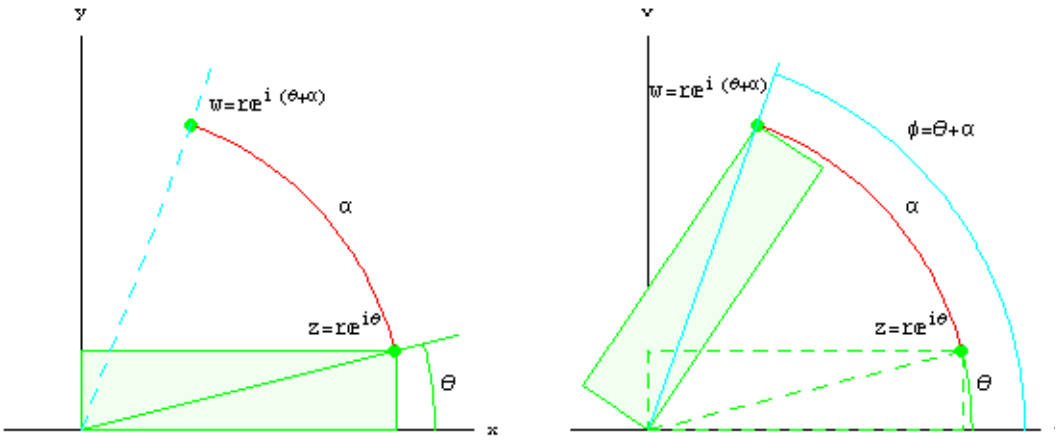

Figure 2.6 The rotation $w = R(z) = r\,e^{i(\theta+\alpha)}$.

Example 2.8. The ellipse centered at the origin with a horizontal major axis of 4 units and vertical minor axis of 2 units can be represented by the parametric equation

$$s(t) = 2\cos t + i \sin t = (2\cos t, \sin t), \text{ for } 0 \le t \le 2\pi.$$

Suppose we wanted to rotate the ellipse by an angle of $\frac{\pi}{6}$ radians and shift the center of the ellipse 2 units to the right and 1 unit up. Using complex arithmetic, we can easily generate a parametric equation $r(t)$ that does so:

$$r[t] = s[t]\, e^{i\pi/6} + (2 + i)$$

$$= (2\cos t + i \sin t)\left(\cos\tfrac{\pi}{6} + i \sin\tfrac{\pi}{6}\right) + (2 + i)$$

$$= \left(2\cos t\cos\tfrac{\pi}{6} - \sin t \sin\tfrac{\pi}{6}\right) + i\left(2\cos t \sin\tfrac{\pi}{6} + \sin t\cos\tfrac{\pi}{6}\right) + (2 + i)$$

$$= \left(\sqrt{3}\cos t - \tfrac{1}{2}\sin t + 2\right) + i\left(\cos t + \tfrac{\sqrt{3}}{2}\sin t + 1\right)$$

$$= \left(\sqrt{3}\cos t - \tfrac{1}{2}\sin t + 2,\ \cos t + \tfrac{\sqrt{3}}{2}\sin t + 1\right)$$

for $0 \le t \le 2\pi$. Figure 2.7 shows parametric plots of these ellipses.

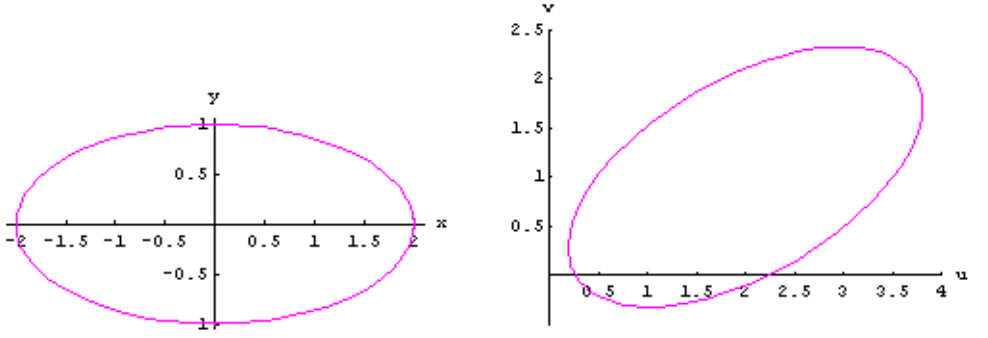

Figure 2.7 **(a)** Plot of the original ellipse $s(t) = 2\cos t + i \sin t$ **(b)** Plot of the rotated ellipse $r(t) = s(t)\, e^{i\pi/6} + (2 + i)$

Explore Solution 2.8.

If we let $K > 0$ be a fixed positive real number, then the transformation

$$w = S(z) = K z = K x + i K y$$

is a one-to-one mapping of the z-plane onto the w-plane and is called a magnification. If $K > 1$, it has the effect of stretching the distance between points by the factor K. If $K < 1$, then it reduces the distance between points by the factor K. The inverse transformation is given by

$$z = S^{-1}(w) = \frac{1}{K} w = \frac{1}{K} u + i \frac{1}{K} v$$

and shows that S is a one-to-one mapping from the z-plane onto the w-plane. The effect of magnification is shown in Figure 2.8.

Figure 2.8 The magnification $w = S(z) = K z = K x + i K y$.

Finally, if we let $A = K e^{i\alpha}$ and $B = a + i b$, where $K > 0$ is a positive real number, then the transformation

$$w = L(z) = A z + B$$

is a one-to-one mapping of the z-plane onto the w-plane and is called a linear transformation. It can be considered as the composition of a rotation, a magnification, and a translation. It has the effect of rotating the plane though an angle given by $\alpha = \text{Arg } A$, followed by a magnification by the factor $K = |A|$, followed by a translation by the vector $B = a + ib$. The inverse mapping is given by $z = L^{-1}(w) = \frac{1}{A} w - \frac{B}{A}$ and shows that L is a one-to-one mapping from the z-plane onto the w-plane.

Example 2.9. Show that the linear transformation $w = f(z) = i z + i$ maps the right half plane $\text{Re}(z) > 1$ onto the upper half plane $\text{Im}(w) > 2$.

Solution. Method 1: Let $A = \{(x, y) : x \geq 1\}$. To describe $B = f(A)$, we solve $w = i z + i$ for z to get $z = \frac{w-i}{i} = -i w - 1 = f^{-1}(w)$. We have the following

$$\begin{aligned}
& u + iv = w = f(z) \in B \\
\Longleftrightarrow\ & f^{-1}(w) = z \in A \\
\Longleftrightarrow\ & -i w - 1 \in A \\
\Longleftrightarrow\ & v - 1 - i u \in A \\
\Longleftrightarrow\ & (v - 1, -u) \in A \\
\Longleftrightarrow\ & v - 1 \geq 1 \\
\Longleftrightarrow\ & v \geq 2
\end{aligned}$$

Thus $B = \{w = u + iv : v \geq 2\}$, which is the same as saying $\text{Im}(w) > 2$.

Method 2: When we write $w = f(z)$ in Cartesian form as

$$w = u + iv = i(x + i y) + i = -y + i(x + 1),$$

we see that the transformation can be given by the equations $u = -y$ and $v = x + 1$. Substituting $x = v - 1$ in the inequality $\mathrm{Re}(z) = x \geq 1$ gives $v - 1 \geq 1$, or $v \geq 2$, which is the upper half-plane $\mathrm{Im}(w) > 2$.

Method 3: The effect of the transformation $w = f(z)$ is a rotation of the plane through the angle $\alpha = \frac{\pi}{2}$ (when z is multiplied by i) followed by a translation by the vector $B = i$. The first operation yields the set $\mathrm{Im}(w) \geq 1$. The second shifts this set up 1 unit, resulting in the set $\mathrm{Im}(w) > 2$. We illustrate this result in Figure 2.9.

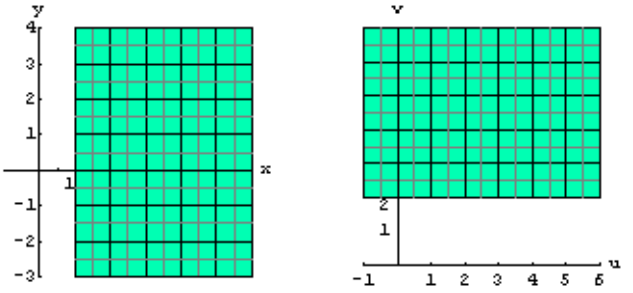

Figure 2.9 The linear transformation $w = f(z) = i z + i$.

Explore Solution 2.9.

Translations and rotations preserve angles. First, magnifications rescale distance by a factor K, so it follows that triangles are mapped onto similar triangles, preserving angles. Then, because a linear transformation can be considered to be a composition of a rotation, a magnification, and a translation, it follows that linear transformations preserve angles. Consequently, any geometric object is mapped onto an object that is similar to the original object; hence linear transformations can be called similarity mappings.

Note. The usage of the phrase "linear transformation" in a "complex analysis course" is different than that the usage in "linear algebra courses".

Example 2.10. Show that the image of the open disk $D_1(-1-i) = \{z : |z + 1 + i| < 1\}$ under the linear transformation $w = f(z) = (3 - 4i) z + 6 + 2i$ is the open disk $D_5(-1+3i) = \{w : |w + 1 - 3i| < 5\}$.

Solution. The inverse transformation is $z = \frac{w-6-2i}{3-4i}$, so if we designate the range of f as B, then

$$\begin{aligned}
&w = f(z) \in B \Leftrightarrow f^{-1}(w) = z \in D_1(-1-i)\\
\Longleftrightarrow\ &\frac{w-6-2i}{3-4i} \in D_1(-1-i)\\
\Longleftrightarrow\ &\left|\frac{w-6-2i}{3-4i} + 1 + i\right| < 1\\
\Longleftrightarrow\ &\left|\frac{w-6-2i}{3-4i} + 1 + i\right| |3 - 4i| < 1\,|3 - 4i|\\
\Longleftrightarrow\ &|w - 6 - 2i + (1 + i)(3 - 4i)| < 5\\
\Longleftrightarrow\ &|w + 1 - 3i| < 5
\end{aligned}$$

Hence the disk with center $-1-i$ and radius 1 is mapped one-to-one and onto the disk with center $-1+3i$ and radius 5 as shown in Figure 2.10.

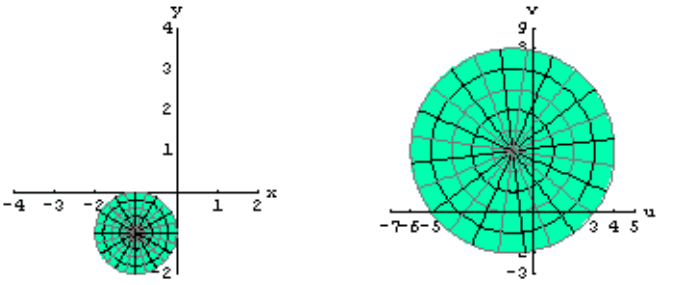

Figure 2.10 The mapping $w = f(z) = (3 - 4\,i)\,z + 6 + 2\,i$.

Explore Solution 2.10.

Example 2.11. Show that the image of the right half plane $\mathrm{Re}(z) = x > 1$ under the linear transformation $w = (-1 + i)\,z - 2 + 3\,i$ is the half plane $v > u + 7$.

Solution. The inverse transformation is given by

$$z = \frac{w+2-3\,i}{-1+i} = \frac{u+2+i\,(v-3)}{-1+i},$$

which we write as

$$x + i\,y = \frac{-u+v-5}{2} + i\,\frac{-u-v+1}{2}.$$

Substituting $x = \frac{-u+v-5}{2}$ into e $\mathrm{Re}(z) = x > 1$ gives $\frac{-u+v-5}{2} > 1$, which simplifies $v > u + 7$. Figure 2.11 illustrates the mapping.

Explore Solution 2.11.

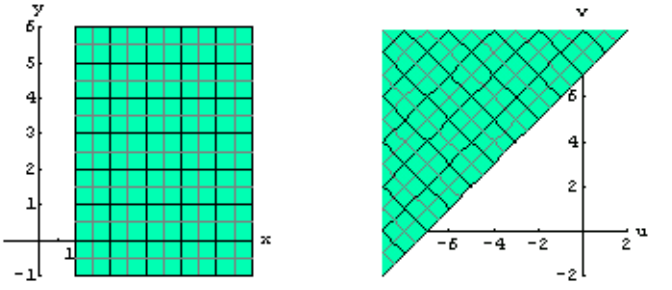

Figure 2.11 The the linear transformation $w = (-1 + i)\,z - 2 + 3\,i$.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

by

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2.2 The Mappings $w = z^n$ and $w = z^{1/n}$

In this section we turn our attention to power functions. For $z = r\,e^{i\theta}$, we can express the function $w = f(z) = z^2$ in polar coordinates by

$$w = f(z) = z^2 = r^2\,e^{i2\theta}.$$

If we also use polar coordinates for $w = \rho\,e^{i\phi}$ in the w -plane, we can express this mapping by the system of equations

$$\rho = r^2 \text{ and } \phi = 2\theta.$$

Because an argument of the product $(z)(z)$ is twice an argument of z, we say that $f(z)$ doubles angles at the origin. Points that lie on the ray $r > 0$, $\theta = \alpha$ are mapped onto points that lie on the ray $\rho > 0$, $\phi = 2\alpha$. If we now restrict the domain of $w = f(z) = z^2$ to the region

$$A = \left\{ r\,e^{i\theta} : r > 0 \text{ and } \frac{-\pi}{2} < \theta \le \frac{\pi}{2} \right\},$$

then the image of A under the mapping $w = z^2$ can be described by the set

$$B = \{\rho\,e^{i\phi} : \rho > 0 \text{ and } -\pi < \phi \le \pi\},$$

which consists of all points in the w-plane except the point $w=0$.

The inverse mapping of f, which we shall denote by $g(z)$, is then

$$z = g(w) = w^{1/2} = \rho^{1/2}\,e^{i\phi/2},$$

where $w \in B$. That is

$$z = g(w) = w^{1/2} = |w|^{1/2} e^{i \operatorname{Arg}(w)/2},$$

where $w \neq 0$. The function g is so important that we call special attention to it with a formal definition.

Definition 2.1 (Principal Square Root Function). The function

$$z = g(w) = w^{1/2} = |w|^{1/2} e^{i \operatorname{Arg}(w)/2}, \quad \text{for} \quad w \neq 0.$$

is called the principal square root function.

It is left as an exercise to show that f and g satisfy equations $g(f(z)) = z$ and $f(g(w)) = w$, and thus are inverses of each other that map the set A one-to-one and onto the set B and the set B one-to-one and onto the set A, respectively. Figure 2.12. illustrates this relationship.

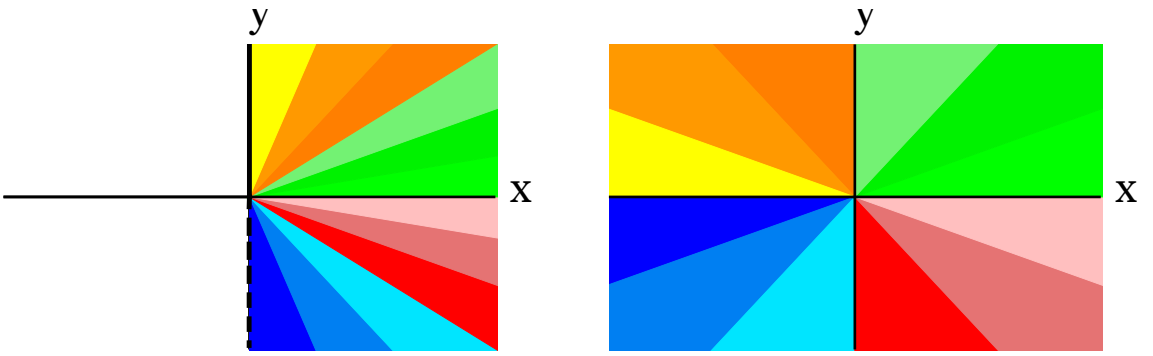

Figure 2.12 The mapping $w = f(z) = z^2$ (and $z = g(w) = w^{1/2}$).

What are the images of rectangles under the mapping $w = z^2$? To find out, we make use of the Cartesian form

$$w = u + i\, v = f(z) = z^2 = x^2 - y^2 + i\, 2\, x\, y = (x^2 - y^2, 2\, x\, y) = (u, v)$$

and the resulting system of equations

$$u = x^2 - y^2, \text{ and}$$

$$v = 2\, x\, y\,.$$

Example 2.12. Show that the transformation $w = f(z) = z^2$, usually maps vertical and horizontal lines onto parabolas and use this fact to find the image of the rectangle $R = \{(x, y) : 0 < x < a, 0 < y < b\}$. **(a)** Find the image of the vertical line $x = a$. **(b)** Find the image of the horizontal line $y = b$.

Solution. Using the above equations, we determine that the vertical line $x = a$ is mapped onto the set of points given by the equations $u = a^2 - y^2$ and $v = 2\, a\, y$. If $a \neq 0$, then $y = \frac{v}{2\, a}$ and

$$u = a^2 - \frac{v^2}{4\, a^2},$$

this equation represents a parabola with vertex at $(a^2, 0)$, oriented horizontally, and opening to

the left. If $a > 0$, the set $\{(u, v) : u = a^2 - y^2, v = 2\,a\,y\}$ has $v > 0$ precisely when $y > 0$, so the part of the line $x = a$ lying above the x axis is mapped to the top half of the parabola.

The horizontal line $y = b$ is mapped onto the parabola given by the equations $u = x^2 - b^2$ and $v = 2\,x\,b$. If $b \neq 0$, then as before we get

$$u = -b^2 + \frac{v^2}{4\,b^2},$$

this equation represents a parabola with vertex at $-b^2$, oriented horizontally and opening to the right. If $b > 0$, the part of the line $y = b$ to the right of the y axis is mapped to the top half of the parabola because the set $\{(u, v) : u = x^2 - b^2, v = 2\,b\,x\}$ has $v > 0$ precisely when $x > 0$.

Quadrant I is mapped onto quadrants I and II by $w = z^2$, so the rectangle $R = \{(x, y) : 0 < x < a, 0 < y < b\}$ is mapped onto the region bounded by the top halves of the parabolas given by $u = a^2 - \frac{v^2}{4\,a^2}$ and $v = -b^2 + \frac{v^2}{4\,b^2}$ and the u axis. The vertices $0, a^2, a^2 + 2\,i\,a\,b - b^2,$ and $-b^2$ of the rectangle are mapped onto the four points $0, a^2, a^2 - b^2 + 2\,i\,a\,b,$ and $-b^2$, respectively, as indicated in Figure 2.13.

Finally, we can easily verify that the vertical line $\{(0, y) : -\infty < y < \infty\}$ is mapped to the set $\{(-y^2, 0) : -\infty < y < \infty\}$. This is simply the set of non-positive real numbers. Similarly, the horizontal line $\{(x, 0) : -\infty < x < \infty\}$ is mapped to the set $\{(x^2, 0) : -\infty < x < \infty\}$, which is the set of non-negative real numbers.

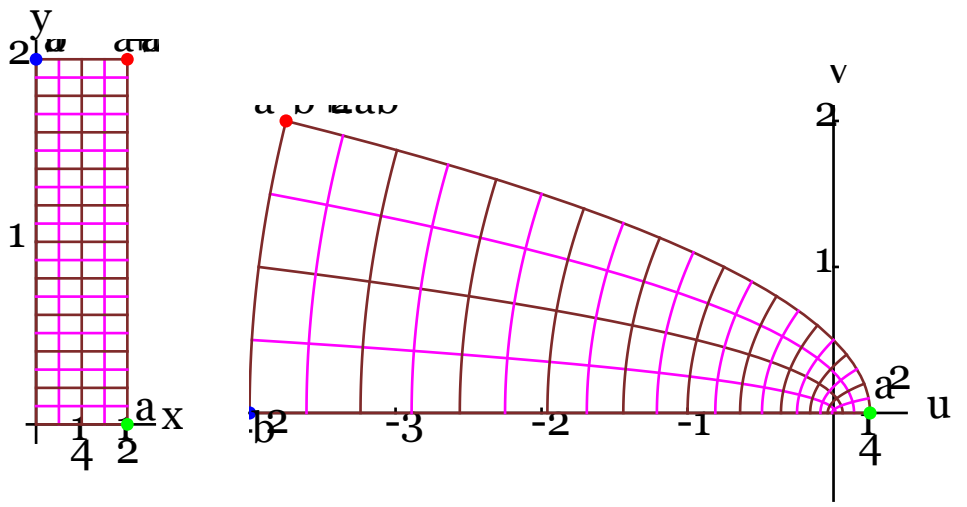

Figure 2.13 The transformation $w = f(z) = z^2$.

Explore Solution 2.12.

Extra Example 1. Consider the mapping $w = z^2$. Find the image of the ray $r > 0, \Theta = \alpha$ and circle $r = c$.

Explore Extra Solution 1.

We can use knowledge of the inverse mapping $z = w^2$ to get further insight into how the mapping $w = z^{1/2}$ acts on rectangles. If we let $z = x + i\,y \neq 0$, then

$$z = x + i\,y = w^2 = u^2 - v^2 + i\,2\,u\,v,$$

and we note that the point $z = x + i\,y$ in the z plane is related to the point $w = u + i\,v = z^{1/2}$ in the w-plane by the system of equations

$$x = u^2 - v^2, \quad \text{and}$$

$$y = 2\,u\,v.$$

Example 2.13. The transformation $w = f(z) = z^{1/2}$ usually maps vertical and horizontal lines onto portions of hyperbolas.
(a) Find the image of the vertical line $x = a$. **(b)** Find the image of the horizontal line $y = b$.

Solution. Let $a > 0$. The above equations map the right half-plane given by $\text{Re}(z) > a$ (i.e., $x > a$) onto the region in the right half-plane satisfying $u^2 - v^2 > a$ and lying to the right of the hyperbola $u^2 - v^2 = a$. If $b > 0$, the equations $x = u^2 - v^2$ and $y = 2\,u\,v$ map the upper half-plane $\text{Im}(z) > b$ (i.e., $y > b$) onto the region in quadrant I satisfying $2\,u\,v > b$ and lying above the hyperbola $2\,u\,v = b$. This situation is illustrated in Figure 2.14. We leave as an exercise the investigation of what happens when $a = 0$ or $b = 0$.

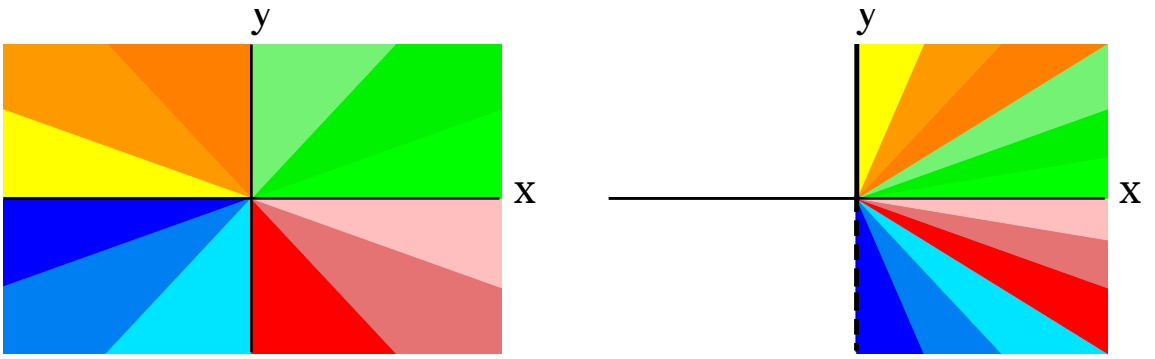

Figure 2.14 The mapping $w = f(z) = z^{1/2}$ and $(z = f^{-1}(w) = w^2)$.

Explore Solution 2.13.

What happens to images of regions under the mapping

$$w = f(z) = z^{1/2} = |z|^{1/2}\, e^{i\,\text{Arg}(z)/2} = r^{1/2}\, e^{i\theta/2} \quad \text{for } r\, e^{i\theta} \neq 0,$$

where $-\pi < \theta \leq \pi$? If we use polar coordinates for $w = \rho\, e^{i\phi}$ in the w plane, we can represent this mapping by the system

$$\rho = r^{1/2} \text{ and } \phi = \frac{\theta}{2}.$$

This Equation indicate that the argument of $f(z)$ is half the argument of z and that the modulus of $f(z)$ is the square root of the modulus of z. Points that lie on the ray $r > 0, \theta = \alpha$ are mapped onto the ray $\rho > 0, \phi = \frac{\alpha}{2}$. The image of the z plane (with the point $z = 0$ deleted) consists of the right half-plane $\text{Re}(w) > 0$ together with the positive v axis. The mapping is shown in Figure 2.15.

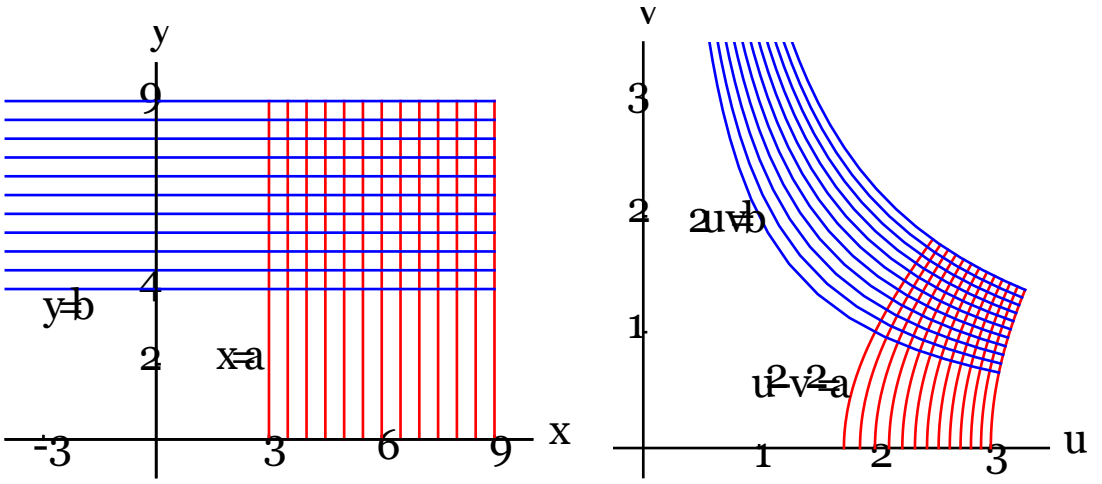

Figure 2.15 The mapping $w = f(z) = z^{1/2}$ (and $z = f^{-1}(w) = w^2$).

Extra Example 2. Consider the mapping $w = z^{1/2}$. Find the image of the ray $r > 0, \theta = \alpha$ and circle $r = c$.

Explore Extra Solution 2.

We can easily extend what we've done to integer powers greater than 2. We begin by letting n be a positive integer, considering the function $w = f(z) = z^n$, for $z = r e^{i\theta} \neq 0$, and then expressing it in the polar coordinate form

$$w = f(z) = z^n = r^n e^{in\theta}.$$

If we use polar coordinates for $w = \rho e^{i\phi}$ in the w -plane, this mapping can be given by the system of equations

$$\rho = r^n \quad \text{and}$$

$$\phi = n\theta.$$

The image of the ray $r > 0, \theta = \alpha$ is the ray $\rho > 0$, $\phi = n\alpha$ and the angles at the origin are increased by the factor n. The functions $\cos n\theta$ and $\sin n\theta$ are periodic with period $\frac{2\pi}{n}$, so f is in general an n-to-one function; that is, n points in the z-plane are mapped onto each non-zero point in the w-plane.

If we now restrict the domain of $w = f(z) = z^n$ to the region

$$E = \left\{ r e^{i\theta} : r > 0 \text{ and } \frac{-\pi}{n} < \theta \leq \frac{\pi}{n} \right\},$$

then the image of E under the mapping $w = z^n$ can be described by the set

$$F = \{ \rho e^{i\phi} : \rho > 0 \text{ and } -\pi < \phi \leq \pi \},$$

which consists of all points in the w-plane except the point w=0. The inverse mapping of f, which we shall denote by g, is then

$$z = g(w) = w^{1/n} = \rho^{1/n}\, e^{i\phi/n},$$

where $w \in F$. That is

$$z = g(w) = w^{1/n} = |w|^{1/n}\, e^{i\,\mathrm{Arg}(w)/n},$$

where $w \neq 0$. As with the principle square root function, we make an analogous definition for n^{th} roots.

Definition 2.2 (Principal n^{th} Root Function). The function

$$z = g(w) = w^{1/n} = |w|^{1/n}\, e^{i\,\mathrm{Arg}(w)/n}, \text{ for } w \neq 0$$

is called the principal n^{th} root function.

We leave as an exercise to show that f and g are inverses of each other that map the set E one-to-one and onto the set F and the set F one-to-one and onto the set E, respectively. Figure 2.16 illustrates this relationship.

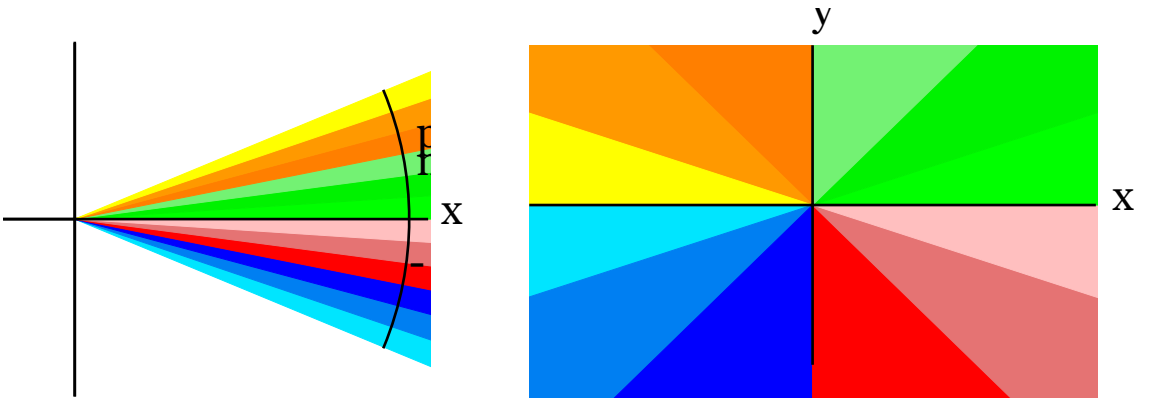

Figure 2.16 The mapping $w = f(z) = z^n$ (and $z = g(w) = w^{1/n}$).

Extra Example 3. Explore the mapping $w = z^3$.

Explore Extra Solution 3.

Extra Example 4. Explore the mapping $w = z^4$.

Explore Extra Solution 4.

Extra Example 5. Explore the mapping $w = z^{1/3}$.

Explore Extra Solution 5.

The Quadratic Formula

We are now able to present a familiar result. It's proof, which is left as an exercise, depends on the ideas in this section, and in Section 1.5.

Theorem 2.1 (Quadratic Formula). If $a z^2 + b z + c = 0$, then the solution set for z is

$$\left\{ \frac{-b + \left(b^2 - 4\,a\,c\right)^{\frac{1}{2}}}{2\,a} \right\},$$

where by $(b^2 - 4\,a\,c)^{\frac{1}{2}}$ we mean all distinct square roots of the number inside the parenthesis.

Proof . The proof is left as an exercise for the reader.

Example 1.21. Find all solutions to the equation $z^2 + (1 + i)\,z + 5\,i = 0$.

Solution. The quadratic formula gives $\frac{-(1+i) + \left((1+i)^2 - 4\,(1)\,(5\,i)\right)^{\frac{1}{2}}}{2\,(1)} = \frac{-(1+i) + (-18\,i)^{\frac{1}{2}}}{2}$. As $-18\,i = 18\,e^{i\left(-\frac{\pi}{2}\right)}$, we compute $(-18\,i)^{\frac{1}{2}} = (18)^{\frac{1}{2}}\,e^{i\left(-\frac{\pi}{2} + 2\,k\,\pi\right)/2}$ for k = 0 and k = 1. In Cartesian form, this expression reduces to

$$(18)^{\frac{1}{2}}\,e^{i\left(-\frac{\pi}{4}\right)} = 3\sqrt{2}\,\left(\cos\frac{-\pi}{4} + i\,\sin\frac{-\pi}{4}\right) = 3\sqrt{2}\,\left(\frac{1-i}{\sqrt{2}}\right) = 3 - 3\,i,$$

and

$$(18)^{\frac{1}{2}}\,e^{i\left(\frac{3\pi}{4}\right)} = 3\sqrt{2}\,\left(\cos\frac{3\pi}{4} + i\,\sin\frac{3\pi}{4}\right) = 3\sqrt{2}\,\left(\frac{-1+i}{\sqrt{2}}\right) = -3 + 3\,i.$$

Thus, our solution set is $\left\{\frac{-(1+i) + (3-3\,i)}{2}, \frac{-(1+i) + (-3+3\,i)}{2}\right\} = \{1 - 2\,i, -2 + i\}$.

Explore Solution 1.21.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

```
Needs["Graphics`ComplexMap`"];
```

Complex Analysis for Mathematics and Engineering

by

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2.3 Limits and Continuity

Let $u = u(x,y)$ be a real-valued function of the two real variables x and y. Recall that u has the limit u_0 as (x, y) approaches (x_0, y_0) provided that the value of $u(x, y)$ can be made to get as close as we please to the value u_0 by taking (x, y) to be sufficiently close to (x_0, y_0). When this happens we write

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0.$$

In more technical language, u has the limit u_0 as (x,y) approaches (x_0, y_0) iff $|u(x, y) - u_0|$ can be made arbitrarily small by making both $|x - x_0|$ and $|y - y_0|$ small. This condition is like the definition of a limit for functions of one variable. The point (x, y) is in the xy plane, and the distance between (x, y) and (x_0, y_0) is $\sqrt{(x - x_0)^2 + (y - y_0)^2}$. With this perspective we can now give a precise definition of a limit.

Definition 2.3 (limit of u(x,y)). The expression $\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0$ means that for each number $\in > 0$, there corresponds a number $\delta > 0$ such that

(2-15) $$|u(x, y) - u_0| < \in \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

Example 2.14. Show, if $u(x, y) = \frac{2x^3}{x^2 + y^2}$, then $\lim_{(x,y)\to(0,0)} u(x, y) = 0$.

Solution. If $x = r\cos\theta$ and $y = r\sin\theta$, and $z = x + i\,y \neq 0$ then

$$u(x, y) = \frac{2r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = 2r\cos^3\theta.$$

Because $\sqrt{(x - 0)^2 + (y - 0)^2} = r$ and because $|\cos^3\theta| < 1$, we have

$$|u(r, \theta)| = 2r\,|\cos^3\theta| < \in \quad \text{whenever} \quad \sqrt{(x - 0)^2 + (y - 0)^2} = r < \frac{\in}{2}.$$

Hence, for any $\epsilon > 0$, Inequality (2-15) is satisfied for $\delta = \frac{\epsilon}{2}$; that is, $u(x, y)$ has the limit $u_0 = 0$ as (x, y) approaches $(0, 0)$.

Explore Solution 2.14.

The value u_0 of the limit must not depend on how (x, y) approaches (x_0, y_0), so $u(x, y)$ must approach the value u_0 when (x, y) approaches (x_0, y_0) along any curve that ends at the point (x_0, y_0). Conversely, if we can find two curves C_1 and C_2 that end at (x_0, y_0) along which $u(x, y)$ approaches the two distinct values u_1 and u_2, respectively, then $u(x, y)$ does not have a limit as (x, y) approaches (x_0, y_0).

Example 2.15. Show that the function $u(x, y) = \frac{x y}{x^2 + y^2}$ does not have a limit as (x, y) approaches $(0, 0)$.

Solution. If we let (x, y) approach $(0, 0)$ along the x axis, then

$$\lim_{(x,0)\to(0,0)} u(x, y) = \lim_{(x,0)\to(0,0)} \frac{(x)(0)}{x^2 + 0^2} = 0.$$

But if we let (x, y) approach $(0, 0)$ along the line $y = x$, then

$$\lim_{(x,x)\to(0,0)} u(x, y) = \lim_{(x,x)\to(0,0)} \frac{(x)(x)}{x^2 + x^2} = \frac{1}{2}.$$

Because the value of the limit differs depending on how (x, y) approaches $(0, 0)$, we conclude that $u(x, y)$ does not have a limit as (x, y) approaches $(0, 0)$.

Explore Solution 2.15.

Let $f(z)$ be a complex function of the complex variable z that is defined for all values of z in some neighborhood of z_0, except perhaps at the point z_0. We say that f has the limit w_0 as z approaches z_0 provided that the value $f(z)$ can be made as close as we please to the value w_0 by taking z to be sufficiently close to z_0. When this happens we write

$$\lim_{z\to z_0} f(z) = w_0.$$

The distance between the points z and z_0 can be expressed by $|z - z_0|$, so we can give a precise definition similar to the one for a function of two variables.

Definition 2.4 (limit of f(z)). The expression $\lim_{z\to z_0} f(z) = w_0$ means that for each number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

Using Equations (1-49) and (1-51), we can also express the last relationship as

$$f(z) \in D_\epsilon(w_0) \text{ whenever } z \in D_\delta^*(z_0).$$

The formulation of limits in terms of open disks provides a good context for looking at this definition. It says that for each disk of radius $\epsilon > 0$ about the point w_0 (represented by $D_\epsilon(w_0)$)

there is a punctured disk of radius $\delta > 0$ about the point z_0 (represented by $D^*_\delta(z_0)$) such that the image of each point in the punctured δ-disk lies in the ϵ-disk. The image of the δ-disk does not have to fill up the entire ϵ-disk; but if z approaches z_0 along a curve that ends at z_0, then $w = f(z)$ approaches w_0. The situation is illustrated in Figure 2.17.

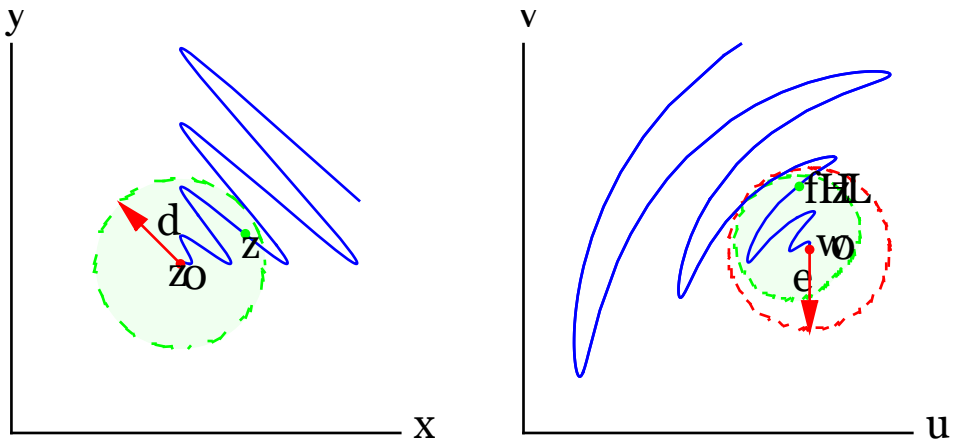

Figure 2.17 The limit $f(z) \to w_0$ as $z \to z_0$.

Example 2.16. Show that if $f(z) = \overline{z}$, then $\lim_{z \to z_0} f(z) = \overline{z_0}$, where z_0 is any complex number.

Solution. As f merely reflects points about the y axis, we suspect that any ϵ-disk about the point $\overline{z_0}$ would contain the image of the punctured δ-disk about z_0 if $\delta = \epsilon$. To confirm this conjecture, we let ϵ be any positive number and set $\delta = \epsilon$. Then we suppose that $z \in D^*_\delta(z_0) = D^*_\epsilon(z_0)$, which means that $0 < |z - z_0| < \epsilon$. The modulus of a conjugate is the same as the modulus of the number itself, so the last inequality implies that $0 < |\overline{z - z_0}| < \epsilon$. This is the same as $0 < |\overline{z} - \overline{z_0}| < \epsilon$. Since $f(z) = \overline{z}$ and $w_0 = \overline{z_0}$, this is the same as $0 < |f(z) - w_0| < \epsilon$, which in turn is the same as $f(z) \in D_\epsilon(w_0)$, which is what we needed to show.

Explore Solution 2.16.

If we consider $w = f(z)$ as a mapping from the z plane into the w plane and think about the previous geometric interpretation of a limit, then we are led to conclude that the limit of a function f should be determined by the limits of its real and imaginary parts, u and v. This conclusion also gives us a tool for computing limits.

Theorem 2.1. Let $f(z) = u(x, y) + i\, v(x, y)$ be a complex function that is defined in some neighborhood of z_0, except perhaps at $z_0 = x_0 + i\, y_0$. Then

$$\lim_{z \to z_0} f(z) = w_0 = u_0 + i\, v_0$$

iff both

$$\lim_{(x,y) \to (x_0, y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0, y_0)} v(x, y) = v_0.$$

Proof.

Proof of Theorem 2.1 is in the book.

Example 2.17. Show that $\lim_{z \to 1+i} (z^2 - 2z + 1) = -1$.

Solution. We have

$$f(z) = 1 - 2z + z^2$$

$$f(x + i y) = (x^2 - y^2 - 2x + 1) + i(2xy - 2y)$$

$$u(x, y) = x^2 - y^2 - 2x + 1$$

$$v(x, y) = 2xy - 2y$$

Computing the limits for u and v, we obtain

$$\lim_{(x,y)\to(1,1)} u(x, y) = \lim_{(x,y)\to(1,1)} (x^2 - y^2 - 2x + 1) = -1, \text{ and}$$

$$\lim_{(x,y)\to(1,1)} v(x, y) = \lim_{(x,y)\to(1,1)} (2xy - 2y) = 0,$$

so our previous theorem implies that $\lim_{z\to 1+i} f(z) = -1$.

Explore Solution 2.17.

Limits of complex functions are formally the same as those of real functions, and the sum, difference, product, and quotient of functions have limits given by the sum, difference, product, and quotient of the respective limits. We state this result as a theorem and leave the proof as an exercise.

Theorem 2.2. Suppose that $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$. Then

(2 – 18) $$\lim_{z\to z_0} f(z) + g(z) = A + B, \text{ and}$$
$$\lim_{z\to z_0} f(z) - g(z) = A - B,$$

(2 – 19) $$\lim_{z\to z_0} f(z)\, g(z) = AB,$$

(2 – 20) $$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \quad \text{where } B \neq 0.$$

Proof.

Proof of Theorem 2.2 is left as exercises in the book.

Definition 2.5 (continuity of u(x,y)). Let $u(x, y)$ be a real-valued function of the two real variables x and y. We say that u is continuous at the point (x_0, y_0) if the three conditions are satisfied:

(2 – 21) $$\lim_{(x,y)\to(x_0,y_0)} u(x, y) \text{ exists},$$

(2 – 22) $$u(x_0, y_0) \text{ exists},$$

(2 – 23) $$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u(x_0, y_0).$$

Condition (2-23) actually implies Conditions (2-21) and (2-22) because the existence of the quantity on each side of Equation (2-23) is implicitly understood to exist. For example, if $u(x, y) = \frac{x^3}{x^2+y^2}$ when $(x, y) \neq (0, 0)$ and if $u(0, 0) = 0$, then $\lim_{(x,y)\to(0,0)} u(x, y) = 0$ so that Conditions (2-21) , (2-22) , and (2-23) are satisfied. Hence $u(x, y)$ is continuous at $(0, 0)$.

There is a similar definition for complex valued functions.

Definition 2.6 (continuity of f(z)). Let $f(z)$ be a complex function of the complex variable z that is defined for all values of z in some neighborhood of z_0. We say that f is continuous at z_0 if three conditions are satisfied:

(2 – 24) $\lim_{z \to z_0} f(z)$ exists, exists,

(2 – 25) $f(z_0)$ exists,

(2 – 26) $\lim_{z \to z_0} f(z) = f(z_0)$.

Remark 2.3. Example 2.16 shows that the function $f(z) = \overline{z}$, is continuous.

A complex function f is continuous iff its real and imaginary parts, u and v, are continuous. The proof of this fact is an immediate consequence of Theorem 2.1. Continuity of complex functions is formally the same as that of real functions, and sums, differences, and products of continuous functions are continuous; their quotient is continuous at points where the denominator is not zero. These results are summarized by the following theorems. We leave the proofs as exercises.

Theorem 2.3. Let $f(z) = u(x, y) + i\, v(x, y)$ be a defined in some neighborhood of z_0. Then $f(z)$ is continuous at $z_0 = x_0 + i\, y_0$ iff $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0).

Proof.

Theorem 2.4. Suppose that $f(z)$ and $g(z)$ are continuous at the point z_0. Then the following functions are continuous at z_0.

The sum $f(z) + g(z)$,

The difference $f(z) - g(z)$,

The product $f(z)\, g(z)$,

The quotient $\frac{f(z)}{g(z)}$, provided that $g(z_0) \neq 0$.

The composition $f(g(z))$, provided that $f(z)$ is continuous in a neighborhood of the point $g(z_0)$.

Proof.

Example 2.18. Show that the polynomial function given by

$$w = P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

is continuous at each point z_0 in the complex plane.

Solution. If a_0 is the constant function, then $\lim_{z \to z_0} a_0 = a_0$; and if $a_1 \neq 0$, then we can use Definition 2.3 with $f(z) = a_1 z$ and the choice $\delta = \frac{\epsilon}{|a_1|}$ to prove that $\lim_{z \to z_0} (a_1 z) = a_1 z_0$. Using Property (2-19) and mathematical induction, we obtain

(2-27) $\lim_{z \to z_0} (a_k z^k) = a_k z_0^k$, for $k = 0, 1, 2, \ldots, n$.

We can extend Property (2-18) to a finite sum of terms and use the result of Equation (2-27) to get

$$\lim_{z \to z_0} P(z) = \lim_{z \to z_0} \left(\sum_{k=0}^{n} a_k z^k\right) = \sum_{k=0}^{n} a_k z_0^k = P(z_0).$$

Conditions (2-24), (2-25), and (2-26) are satisfied, so we conclude that P is continuous at z_0.

Explore Solution 2.18.

Extra Example 1. Show that the polynomial $P(z) = 1 - z - z^2 + z^3 - z^4 + z^5$ is continuous at the point $z_0 = 1$ in the complex plane.

Explore Solution for Extra Example 1.

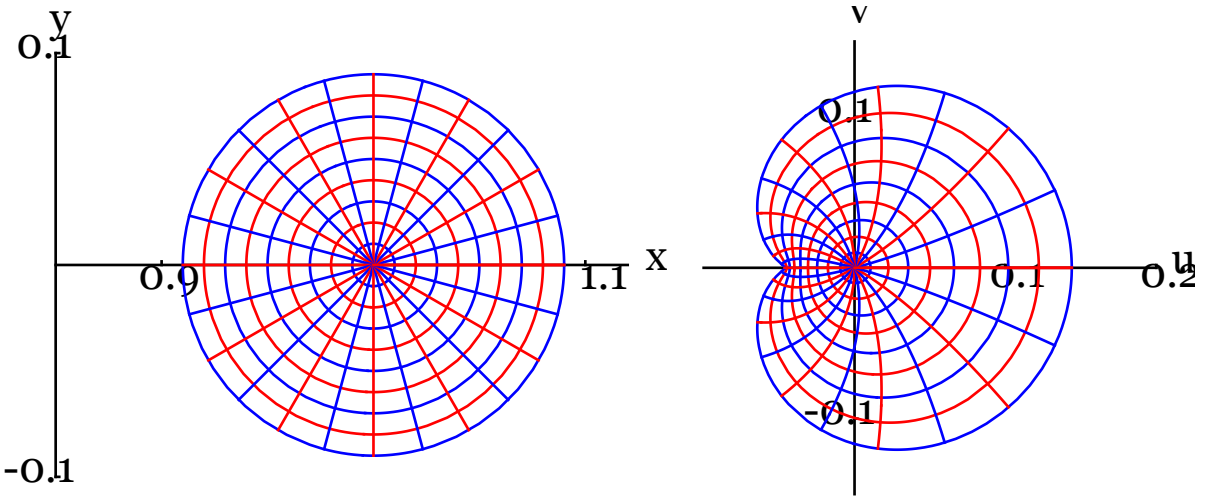

Figure 2.A The mapping $w = P(z) = 1 - z - z^2 + z^3 - z^4 + z^5$ where $P(1) = 0$.

Extra Example 2. Show that the polynomial $P(z) = 1 - z - z^2 + z^3 - z^4 + z^5$ is continuous at the point $z_0 = \frac{1}{2}$ in the complex plane.

Explore Solution for Extra Example 2.

Extra Example 3. Show that the polynomial $P(z) = -1 + 11z - 3z^2 + -4z^3 - 3z^4 + 2z^5$ is continuous at the point $z_0 = -1 + i$ in the complex plane.

Explore Extra Example 3.

One technique for computing limits is to apply Theorem 2.4 to quotients. If we let P and Q be polynomials and if $Q(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0)}.$$

Another technique, involves factoring polynomials. If both $P(z_0) = 0$ and $Q(z_0) = 0$, then P and Q can be factored as $P(z) = (z - z_0) P_1(z)$ and $Q(z) = (z - z_0) Q_1(z)$. If $Q(z_0) \neq 0$, then the limit is

$$\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{(z-z_0)\,P_1(z)}{(z-z_0)\,Q_1(z)} = \frac{P_1(z_0)}{Q_1(z_0)}.$$

Example 2.19. Show that $\lim_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} = 1 - i$.

Solution. Here P and Q can be factored in the form

$$P(z) = (z - 1 - i)(z + 1 + i),$$

and

$$Q(z) = (z - 1 - i)(z - 1 + i).$$

so that the limit is obtained by the calculation

$$\lim_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} = \lim_{z \to 1+i} \frac{(z-1-i)(z+1+i)}{(z-1-i)(z-1+i)}$$

$$= \lim_{z \to 1+i} \frac{z+1+i}{z-1+i}$$

$$= \frac{(1+i)+1+i}{(1+i)-1+i}$$

$$= \frac{2+2i}{2i}$$

$$= 1 - i$$

Explore Solution 2.19.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

by

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Jones and Bartlett Learning

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2.4 Branches of Functions

In Section 2.3 we defined the principal square root function and investigated some of its properties. We left unanswered some questions concerning the choices of square roots. We now look at these questions because they are similar to situations involving other elementary functions.

In our definition of a function in Section 2.1, we specified that each value of the independent variable in the domain is mapped onto one and only one value in the range. As a result, we often talk about a single-valued function, which emphasizes the "only one" part of the definition and allows us to distinguish such functions from multiple-valued functions, which we now introduce.

Let $w = f(z)$ denote a function whose domain is the set D and whose range is the set R. If w is a value in the range, then there is an associated inverse function $z = g(w)$ that assigns to each value w the value (or values) of z in D for which the equation $f(z) = w$ holds. But unless f takes on the value w at most once in D, then the inverse function g is necessarily many valued, and se say that g is a multivalued function. For example, the inverse of the function $w = f(z) = z^2$ is the square root function $z = f^{-1}(w) = w^{\frac{1}{2}}$. For each value z other than $z = 0$, then, the two points z and $-z$ are mapped onto the same point $w = f(z)$; hence g is in general a two-valued function.

The study of limits, continuity, and derivatives loses all meaning if an arbitrary or ambiguous assignment of function values is made. For this reason we did not allow multivalued functions to be considered when we defined these concepts. When working with inverse functions, you have to specify carefully one of the many possible inverse values when constructing an inverse function, as when you determine implicit functions in calculus. If the values of a function f are determined by an equation that they satisfy rather than by an explicit formula, then we say that the function is defined implicitly or that f is an implicit function. In the theory of complex variables we present a similar concept.

Let $w = f(z)$ be a multiple-valued function. A branch of f is any single-valued function f_0 that is continuous in some domain (except, perhaps, on the boundary). At each point z in the domain, assigns one of the values of $f(z)$. Associated with the branch of a function is the branch cut.

We now investigate the branches of the square root function.

Example 2.20. We consider some branches of the two-valued square root function $f(z) = z^{\frac{1}{2}}$, (where $z \neq 0$). Define the principal square root function as

(2-28) $$w = f_1(z) = |z|^{\frac{1}{2}} e^{i \frac{\text{Arg}(z)}{2}} = r^{\frac{1}{2}} e^{i \frac{\theta}{2}} = r^{\frac{1}{2}} \left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right) \right),$$

where $r = |z|^{\frac{1}{2}}$ and $\theta = \text{Arg}(z)$ so that $-\pi < \theta \leq \pi$. The function $f_1(z)$ is a branch of $f(z)$. Using the same notation, we can find other branches of the square root function. For example, if we let

(2-29)
$$w = f_2(z) = |z|^{\frac{1}{2}} e^{i \frac{\text{Arg}(z)+2\pi}{2}} = r^{\frac{1}{2}} e^{i \frac{\theta+2\pi}{2}} = r^{1/2} \left(\cos\frac{\theta+2\pi}{2} + i \sin\frac{\theta+2\pi}{2} \right),$$

then

$$\begin{aligned} f_2(z) &= r^{\frac{1}{2}} e^{i \frac{\theta+2\pi}{2}} \\ &= r^{1/2} e^{i \frac{\theta}{2}} e^{i\pi} \\ &= -r^{1/2} e^{i \frac{\theta}{2}} \\ &= -f_1(z) \end{aligned}$$

so $f_1(z)$ and $f_2(z)$ can be thought of as "plus" and "minus" square root functions. The negative real axis is called a branch cut for the functions $f_1(z)$ and $f_2(z)$. Each point on the branch cut is a point of discontinuity for both functions $f_1(z)$ and $f_2(z)$.

Explore Solution 2.20.

Example 2.21. Show that the function $f_1(z) = r^{\frac{1}{2}} \left(\cos\frac{\theta}{2} + i \sin\frac{\theta}{2} \right)$ is discontinuous along the negative real axis.

Solution. Let $z_0 = r_0 e^{i\pi}$ denote a negative real number. We compute the limit as z approaches z_0 through the upper half-plane $\{z : \text{Im}(z) > 0\}$ and the limit as z approaches z_0 through the lower half-plane $\{z : \text{Im}(z) < 0\}$. In polar coordinates these limits are given by

$$\lim_{(r,\theta)\to(r_0,\pi)} f_1(r e^{i\theta}) = \lim_{(r,\theta)\to(r_0,\pi)} r^{\frac{1}{2}} \left(\cos\frac{\theta}{2} + i \sin\frac{\theta}{2} \right) = i r_0^{1/2}, \text{ and}$$

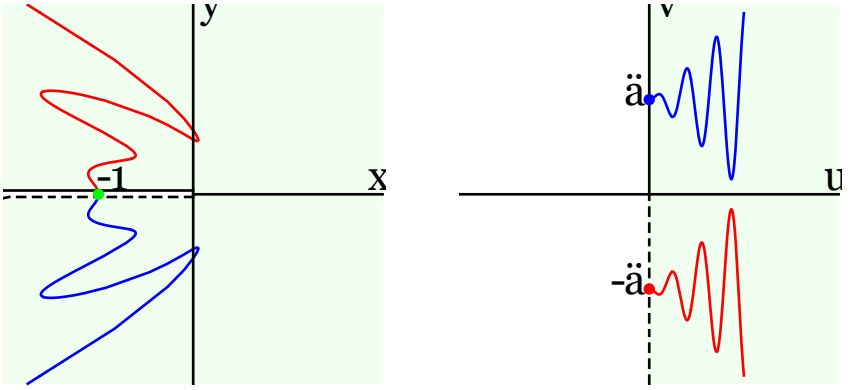
$$\lim_{(r,\theta)\to(r_0,-\pi)} f_1(r e^{i\theta}) = \lim_{(r,\theta)\to(r_0,-\pi)} r^{\frac{1}{2}} \left(\cos\frac{\theta}{2} + i \sin\frac{\theta}{2} \right) = -i r_0^{1/2}.$$

As the two limits are distinct, the function $f_1(z)$ is discontinuous at z_0.

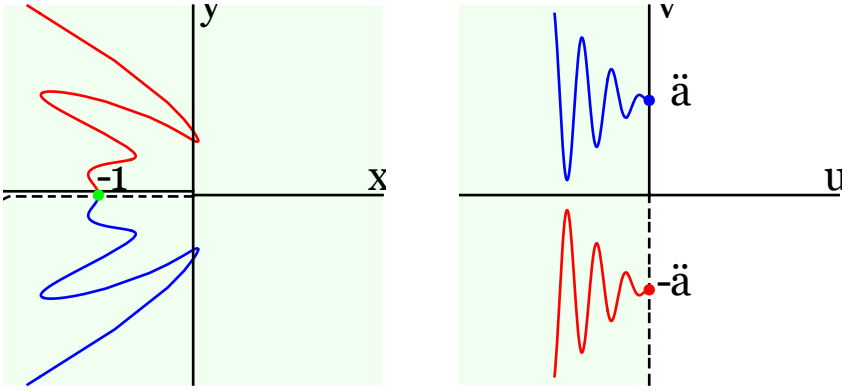
Remark 2.4 Likewise, $f_1(z)$ is discontinuous at z_0. The mappings $w = f_1(z)$, $w = f_2(z)$, and

the branch cut are illustrated in Figure 2.18.

Explore Solution 2.21.



(a) The branch $w = f_1(z) = \sqrt{z}$ (where $z = w^2$).



(b) The branch $w = f_2(z) = -\sqrt{z}$ (where $z = w^2$).

Figure 2.18 The branches $f_1(z)$ and $f_2(z)$ of $f(z) = z^{\frac{1}{2}}$.

We can construct other branches of the square root function by specifying that an argument of z given by $\theta = \arg z$ is to lie in the interval $\alpha < \theta \leq \alpha + 2\pi$. The corresponding branch, denoted $f_\alpha(z)$, is

(2-30) $$f_\alpha(z) = r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right),$$

where $z = r e^{i\theta} \neq 0$ and $\alpha < \theta \leq \alpha + 2\pi$.

The branch cut for $f_\alpha(z)$ is the ray $r \geq 0$, $\theta = \alpha$, which includes the origin. The point $z = 0$, common to all branch cuts for the multivalued square root function, is called a branch point. The mapping $w = f_\alpha(z)$ and its branch cut are illustrated in Figure 2.19.

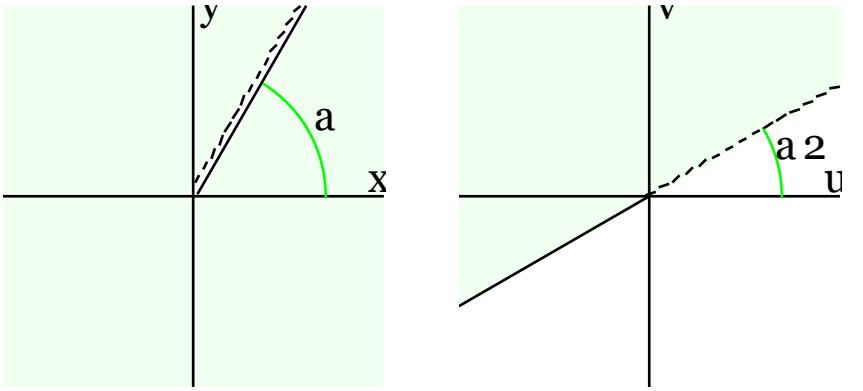


Figure 2.19 The branch $f_\alpha(z)$ of $f(z) = z^{\frac{1}{2}}$.

The Riemann Surface for $w = z^{\frac{1}{2}}$

A Riemann surface is a construct useful for visualizing a multivalued function. It was introduced by Georg Friedrich Bernhard Riemann (1826-1866) in 1851. The idea is ingenious - a geometric construction that permits surfaces to be the domain or range of a multivalued function. Riemann surfaces depend on the function being investigated. We now give a nontechnical formulation of the Riemann surface for the multivalued square root function.

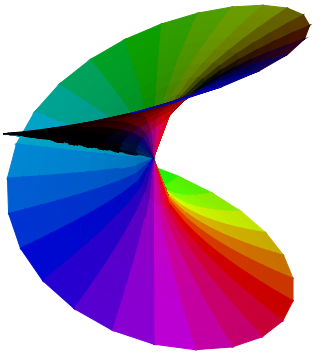
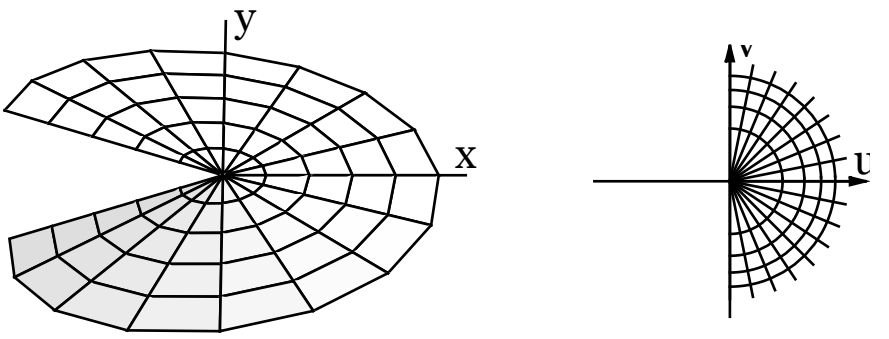


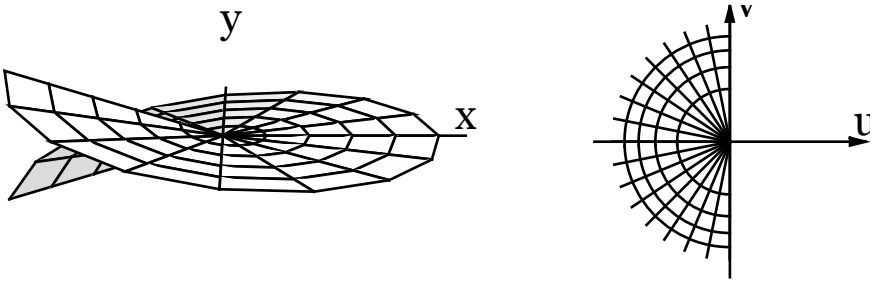
Figure 2.A A graphical view of the Riemann surface for $w = z^{\frac{1}{2}}$.

Consider $w = f(z) = z^{\frac{1}{2}}$, which has two values for any $z \neq 0$. Each function $f_1(z)$ and $f_2(z)$ in Figure 2.18 is single-valued on the domain formed by cutting the z plane along the negative x axis. Let D_1 and D_2 be the domains of $f_1(z)$ and $f_2(z)$, respectively. The range set for $f_1(z)$ is the set H_1 consisting of the right half-plane, and the positive v axis; the range set for $f_2(z)$ is the set H_2 consisting of the left half-plane and the negative v axis. The sets H_1 and H_2 are "glued together" along the positive v axis and the negative v axis to form the w plane with the origin deleted.

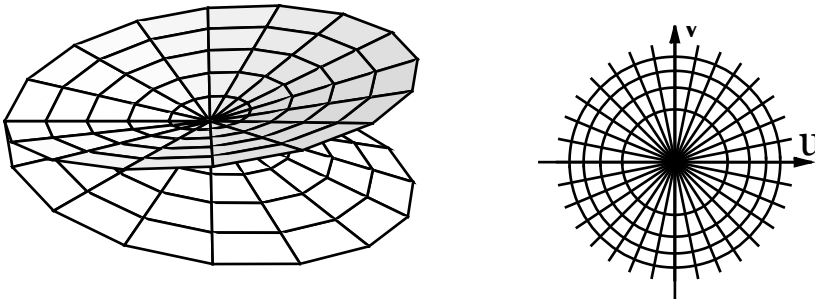
We stack D_1 directly above D_2. The edge of D_1 in the upper half-plane is joined to the edge of D_2 in the lower half-plane, and the edge of D_1 in the lower half-plane is joined to the edge of D_2 in the upper half-plane. When these domains are glued together in this manner, they form R, which is a Riemann surface domain for the mapping $w = f(z) = z^{\frac{1}{2}}$. The portions of D_1, D_2 and R that lie in $\{z : |z| < 1\}$ are shown in Figure 2.20.



(a) A portion of D_1 and its image under $w = z^{\frac{1}{2}}$.



(b) A portion of D_2 and its image under $w = z^{\frac{1}{2}}$.



(c) A portion of $\mathbb{R}$ and its image under $w = z^{\frac{1}{2}}$.

Figure 2.20 Formation of the Riemann surface for $w = z^{\frac{1}{2}}$.

Exploration

The beauty of this structure is that it makes this "full square root function" continuous for all $z \neq 0$. Normally, the principal square root function would be discontinuous along the negative real axis, as points near -1 but above that axis would get mapped to points close to i, and points near -1 but below the axis would get mapped to points close to $-i$.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

by

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Jones and Bartlett Learning

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2.5 The Reciprocal Transformation $w = \frac{1}{z}$

The mapping $w = \frac{1}{z}$ is called the reciprocal transformation and maps the z-plane one-to-one and onto the w-plane except for the point $z = 0$, which has no image, and the point $w = 0$, which has no preimage or inverse image. Use the exponential notation $w = \rho\, e^{i\phi}$ in the w-plane. If $z = r\, e^{i\theta} \neq 0$, we have

$$w = \rho\, e^{i\phi} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}.$$

The geometric description of the reciprocal transformation is now evident. It is an inversion (that is, the modulus of $(\frac{1}{z})$ is the reciprocal of the modulus of z) followed by a reflection through the x axis. The ray $r > 0, \theta = \alpha$, is mapped one-to-one and onto the ray $\rho > 0, \phi = -\alpha$. Points that lie inside the unit circle $C_1(0) = \{z : |z| < 1\}$ are mapped onto points that lie outside the unit circle and vice versa. The situation is illustrated in Figure 2.21.

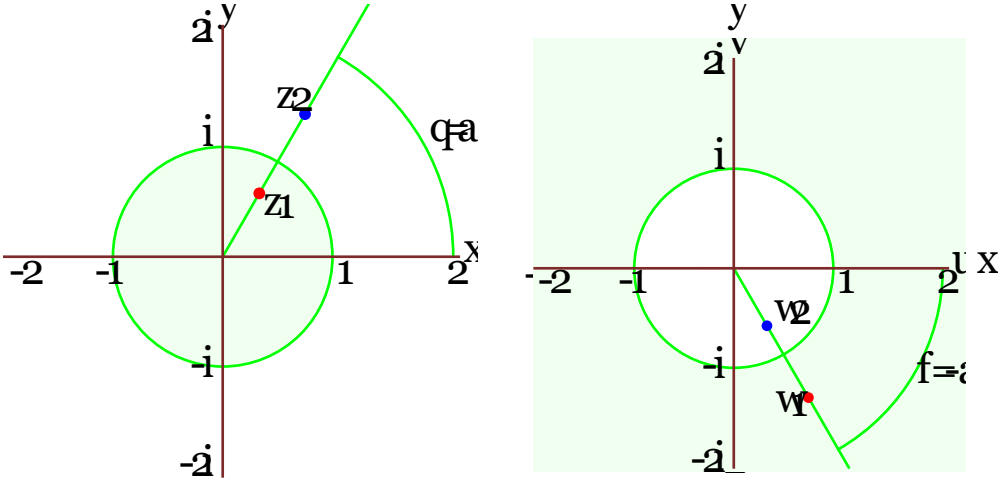

Figure 2.21 The reciprocal transformation

$$w = u + i\,v = \frac{1}{z} = \frac{1}{x+i\,y}.$$

We can extend the system of complex numbers by joining to it an "ideal" point denoted by ∞ and called the point at infinity. This new set is called the extended complex plane. You will see shortly that the point ∞ has the property, loosely speaking, that $\lim_{n\to\infty} z = \infty$ iff $\lim_{n\to\infty} |z| = \infty$.

An $\in$-neighborhood of the point at infinity is the set $\left\{z : |z| > \frac{1}{\in}\right\}$. The usual way to visualize the point at infinity is by using what we call the stereographic projection, which is attributed to Riemann. Let Ω be a sphere of diameter 1 that is centered at the point $\left(0, 0, \frac{1}{2}\right)$ in three-dimensional space where coordinates are specified by the triple of real numbers (x, y, ξ). Here the complex number $z = x + i\,y$ will be associated with the point $z = (x, y, 0)$.

The point $\mathbb{N} = (0, 0, 1)$ on Ω is called the north pole of Ω. If we let z be a complex number and consider the line segment L in three-dimensional space that joins z to the north pole $\mathbb{N} = (0, 0, 1)$, then L intersects in exactly one point $\mathbb{Z}$. The correspondence $z \leftrightarrow \mathbb{Z}$ is called the stereographic projection of the complex z plane onto the Riemann sphere Ω.

A point $z = x + i\,y = (x, y, 0)$ of unit modulus will correspond with $\mathbb{Z} = \left\{\frac{x}{2}, \frac{y}{2}, \frac{1}{2}\right\}$. If z has modulus greater than 1, then $\mathbb{Z}$ will lie in the upper hemisphere where for points $\mathbb{Z} = (x, y, \xi)$ we have $\xi > \frac{1}{2}$. If z has modulus less than 1, then $\mathbb{Z}$ will lie in the lower hemisphere where for points $\mathbb{Z} = (x, y, \xi)$ we have $\xi < \frac{1}{2}$. The complex number $z = 0 = 0 + 0\,i$ corresponds with the south pole, $\mathbb{S} = (0, 0, 0)$. Now you can see that indeed $z \to \infty$ iff $|z| \to \infty$ iff $\mathbb{Z} \to \mathbb{N}$. Hence $\mathbb{N}$ corresponds with the "ideal" point at infinity. The situation is shown in Figure 2.22.

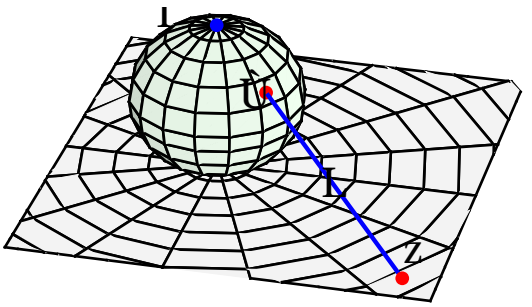

Figure 2.22 The complex plane and the Riemann sphere Ω.

Let us reconsider the mapping $w = \frac{1}{z}$. Let us assign the images $w = \infty$ and $w = 0$ to the points $z = 0$ and $z = \infty$, respectively. We now write the reciprocal transformation as

$$w = f(z) = \begin{cases} \frac{1}{z} & \text{when } z \neq 0, \infty \\ 0 & \text{when } z = \infty \\ \infty & \text{when } z = 0 \end{cases}$$

Note that the transformation $w = f(z)$ is a one-to-one mapping of the extended complex z plane onto the extended complex w plane. Further, f is a continuous mapping from the extended z plane onto the extended w plane. We leave the details to you.

Exploration.

Extra Example. Investigate the limits of $w = f[z] = \frac{1}{z}$ as $z \to 0$, and $z \to \infty$.

Explore Solution for Extra Example.

Example 2.22. Show that the image of the right half plane $A = \left\{z : \text{Re}(z) \geq \frac{1}{2}\right\}$ under the mapping $w = f(z) = \frac{1}{z}$ is the closed disk $\overline{D_1(1)} = \{w : |w - 1| \leq 1\}$ in the w-plane.

Solution. Proceeding as we did in Example 2.7, we get the inverse mapping of $u + i\,v = w = f(z) = \frac{1}{z}$ as $z = f^{-1}(w) = \frac{1}{w}$. Then

$$
\begin{aligned}
& u + i\,v = w = f(z) \in \overline{D_1(1)} \\
\Longleftrightarrow\ & f^{-1}(w) = x + i\,y \in A \\
\Longleftrightarrow\ & \frac{1}{u+i\,v} = x + i\,y \in A \\
\Longleftrightarrow\ & \frac{u}{u^2+v^2} + i\,\frac{-v}{u^2+v^2} = x + i\,y \in A \\
\Longleftrightarrow\ & \frac{u}{u^2+v^2} = x \geq \frac{1}{2}, \text{ and } \frac{-v}{u^2+v^2} = y \\
\Longleftrightarrow\ & \frac{u}{u^2+v^2} \geq \frac{1}{2} \\
\Longleftrightarrow\ & u^2 - 2\,u + 1 + v^2 \leq 1 \\
\Longleftrightarrow\ & (u-1)^2 + v^2 \leq 1
\end{aligned}
$$

which describes the disk $\overline{D_1(1)}$. As the reciprocal transformation is one-to-one, preimages of the points in the disk $\overline{D_1(1)}$ will lie in the right half-plane $\text{Re}(z) \geq \frac{1}{2}$. Figure 2.23 illustrates this result.

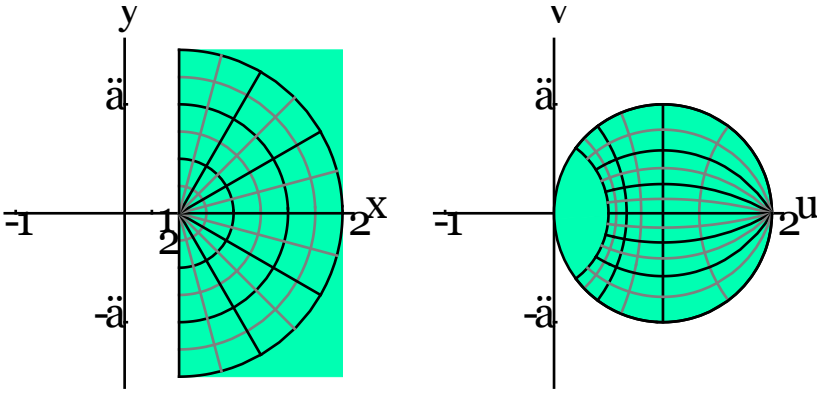

Figure 2.23 The image of $\text{Re}(z) \geq \frac{1}{2}$ under the mapping $w = \frac{1}{z}$.

Explore Solution 2.22.

Remark. Alas, there is a fly in the ointment here. As our notation indicates, $\frac{u}{u^2+v^2} \geq \frac{1}{2}$ and $u^2 - 2\,u + 1 + v^2 \leq 1$ are not equivalent. The former implies the latter, but not conversely. That is, $u^2 - 2\,u + 1 + v^2 \leq 1$ makes sense when $(u, v) = (0, 0)$, whereas $\frac{u}{u^2+v^2} \geq \frac{1}{2}$ does not. Yet Figure 2.23 seems to indicate that f maps $\text{Re}(z) \geq \frac{1}{2}$ onto the entire disk $D_1(1)$, including the point $(0, 0)$. Actually, it does not, because $(0, 0)$ has no preimage in the complex plane. The way out of this dilemma is to use the complex point at infinity. It is that quantity that gets mapped to the point $(u, v) = (0, 0)$, for as we have already indicated, the preimage of 0 under the "extended" mapping $w = \frac{1}{z}$ is indeed ∞.

Example 2.23. For the transformation $w = f(z) = \frac{1}{z}$, find the image of the portion of the right half plane $\mathrm{Re}(z) > \frac{1}{2}$ that lies inside the closed disk $\overline{D_1\left(\frac{1}{2}\right)} = \left\{z : \left|z - \frac{1}{2}\right| \le 1\right\}$.

Solution. Using the result of Example 2.22, we need only find the image of the closed disk $\overline{D_1\left(\frac{1}{2}\right)}$ and intersect it with the closed disk $\overline{D_1(1)}$. To begin, we note that

$$\overline{D_1\left(\frac{1}{2}\right)} = \left\{(x, y) : x^2 + y^2 - x \le \frac{3}{4}\right\}.$$

Because $z = f^{-1}(w) = \frac{1}{w}$, we have, as before,

$$\begin{aligned}
&u + i\,v = w \in f\left(\overline{D_1\left(\frac{1}{2}\right)}\right)\\
&\Leftrightarrow f^{-1}(w) = \frac{1}{w} \in \overline{D_1\left(\frac{1}{2}\right)}\\
&\Leftrightarrow \frac{1}{u+i\,v} \in \overline{D_1\left(\frac{1}{2}\right)}\\
&\Leftrightarrow \frac{u}{u^2+v^2} + i\,\frac{-v}{u^2+v^2} = x + i\,y \in \overline{D_1\left(\frac{1}{2}\right)}\\
&\Leftrightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 - \frac{u}{u^2+v^2} \le \frac{3}{4}\\
&\Leftrightarrow \frac{1}{u^2+v^2} - \frac{u}{u^2+v^2} \le \frac{3}{4}\\
&\Leftrightarrow \left(u + \frac{2}{3}\right)^2 + v^2 \ge \left(\frac{4}{3}\right)^2
\end{aligned}$$

which is an inequality that determines the set of points in the w plane that lie on and outside the circle $C_{\frac{4}{3}}\left(-\frac{2}{3}\right) = \left\{w : \left|w + \frac{2}{3}\right| = \frac{4}{3}\right\}$. Note that we do not have to deal with the point at infinity this time, as the last inequality is not satisfied when $(u, v) = (0, 0)$. When we intersect this set with $\overline{D_1(1)}$, we get the crescent-shaped region shown in Figure 2.24.

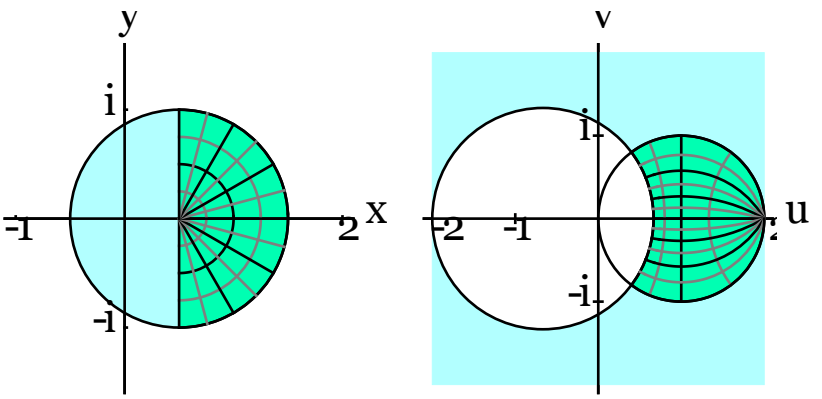

Figure 2.24 The image of the half disk under $w = \frac{1}{z}$ is a crescent-shaped region.

Explore Solution 2.23.

To study images of "generalized circles," we consider the equation

$$A(x^2 + y^2) + B\,x + C\,y + D = 0,$$

where A, B, C, and D are real numbers. This equation represents either a circle or a line, depending on whether $A \ne 0$ or $A = 0$, respectively. Transforming the equation to polar coordinates gives

$$A r^2 + r (B \cos\theta + C \sin\theta) + D = 0.$$

Using the polar coordinate form of the reciprocal transformation, we can express the image of the curve in the preceding equation as

$$A + \rho (B \cos\phi - C \sin\phi) + D \rho^2 = 0,$$

which represents either a circle or a line, depending on whether $D \neq 0$ or $D = 0$, respectively. Therefore, we have shown that the reciprocal transformation $w = \frac{1}{z}$ carries the class of lines and circles onto itself.

Example 2.24. Consider the mapping $w = f(z) = \frac{1}{z}$.

(a) Find the images of the vertical lines $x = a$. **(b)** Find the images the horizontal lines $y = b$.

Solution. Taking into account the point at infinity, we see that the image of the line $x = 0$ is the line $u = 0$; that is, the y axis is mapped onto the v axis.

Similarly, the x axis is mapped onto the u axis. Again, the inverse mapping is $z = \frac{1}{w} = \frac{u}{u^2+v^2} + i \frac{-v}{u^2+v^2}$, so if $a \neq 0$, the vertical line $x = a$ is mapped onto the set of (u, v) points satisfying $\frac{u}{u^2+v^2} = a$. For $(u, v) \neq (0, 0)$, this outcome is equivalent to

$$u^2 - \frac{1}{a} u + \frac{1}{4 a^2} + v^2 = \left(u - \frac{1}{2 a}\right)^2 + v^2 = \left(\frac{1}{2 a}\right)^2,$$

which is the equation of a circle in the w plane with center $w_0 = \frac{1}{2 a}$ and radius $\left|\frac{1}{2 a}\right|$. The point at infinity is mapped to (u,v)=(0,0).

Similarly, the horizontal line $y = b$ is mapped onto the circle

$$u^2 + v^2 + \frac{1}{b} v + \frac{1}{4 b^2} = u^2 + \left(v + \frac{1}{2 b}\right)^2 = \left(\frac{1}{2 b}\right)^2$$

which has center $w_0 = -\frac{i}{2 b}$ and radius $\left|\frac{1}{2 b}\right|$. Figure 2.25 illustrates the images of several lines.

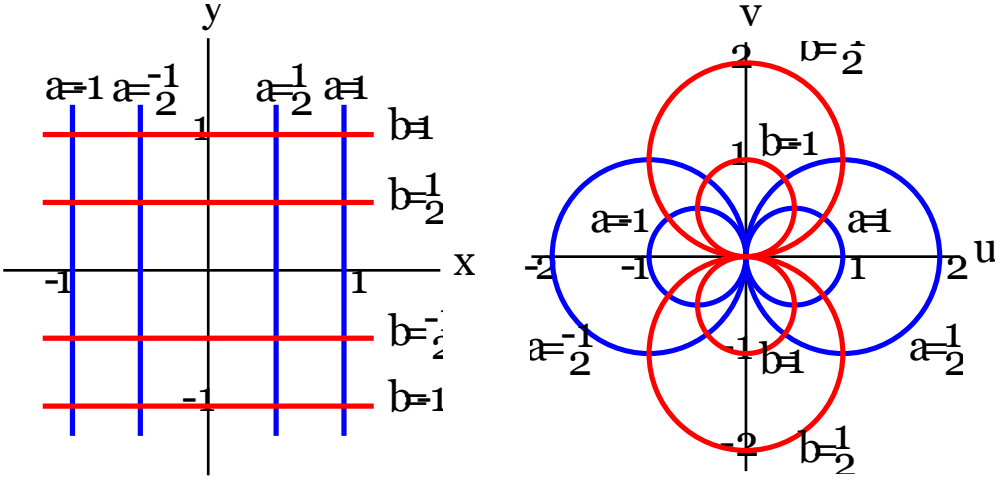

Figure 2.25 The images of horizontal and vertical lines under the reciprocal transformation.

Explore Solution 2.24.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

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Chapter 3 Analytic and Harmonic Functions

Overview

Does the notion of a derivative of a complex function make sense? If so, how should it be defined and what does it represent? These and similar questions are the focus of this chapter. As you might guess, complex derivatives have a meaningful definition, and many of the standard derivative theorems from from calculus (such as the product rule and chain rule) carry over for complex functions. There are also some interesting applications. But not everything is symmetric. You will learn in this chapter that the mean value theorem for derivatives does not extend to complex functions. In later chapters you will see that differentiable complex functions are, in some sense, much more "differentiable" than differentiable real functions.

Section 3.1 Differentiable and Analytic Functions

Using our imagination, we take our lead from elementary calculus and define the derivative of $f(z)$ at z_0, written $f'(z_0)$, by

(3-1) $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided that the limit exists. If it does, we say that the function $f(z)$ is differentiable at z_0. If we write $\triangle z = z - z_0$, then we can express Equation (3-1) in the form

(3-2) $$f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0+\triangle z) - f(z_0)}{\triangle z}.$$

If we let $w = f(z)$ and $\triangle w = f(z) - f(z_0)$, then we can use the Leibniz's notation $\frac{dw}{dz}$ for the derivative:

(3-3) $$f'(z_0) = \frac{dw}{dz} = \lim_{\triangle z \to 0} \frac{\triangle w}{\triangle z}.$$

Extra Example 1. Use the limit definition to find the derivative $f'(z_0)$.

Explore Extra Example 1.

Example 3.1. Use the limit definition to find the derivative of $f(z) = z^3$.

Solution. Using Equation (3-1), we have

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{(z - z_0)^3}{z - z_0} \\ &= \lim_{z \to z_0} \frac{\left(z^2 + z\, z_0 + z_0^2\right)(z - z_0)}{z - z_0} \\ &= \lim_{z \to z_0} \left(z^2 + z\, z_0 + z_0^2\right) \\ &= \left(z_0^2 + z_0\, z_0 + z_0^2\right) \\ &= 3\, z_0^2 \end{aligned}$$

We can drop the subscript on z_0 to obtain $f'(z) = 3\, z^2$ as a general formula.

Explore Solution 3.1.

Pay careful attention to the complex value $\triangle z$ in Equation (3-3); the value of the limit must be independent of the manner in which $\triangle z \to 0$. If we can find two curves that end at z_0 along which $\frac{\triangle w}{\triangle z}$ approaches distinct values, then $\frac{\triangle w}{\triangle z}$ does not have a limit as $\triangle z \to 0$ and $f(z)$ does not have a derivative at z_0. The same observation applies to the limits in Equations (3-2) and (3-1).

Example 3.2. Show that the function $f(z) = \overline{z}$ is nowhere differentiable.

Solution. We choose two approaches to the point $z_0 = x_0 + i\, y_0$ and compute limits of the difference quotients. First, we approach $z_0 = x_0 + i\, y_0$ along a line parallel to the x axis by forcing z to be of the form $z = x + i\, y_0$.

$$\begin{aligned} \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{(x + i\, y_0) \to (x_0 + i\, y_0)} \frac{f(x + i\, y_0) - f(x_0 + i\, y_0)}{(x + i\, y_0) - (x_0 + i\, y_0)} \\ &= \lim_{(x + i\, y_0) \to (x_0 + i\, y_0)} \frac{(x - i\, y_0) - (x_0 - i\, y_0)}{(x + i\, y_0) - (x_0 + i\, y_0)} \\ &= \lim_{(x + i\, y_0) \to (x_0 + i\, y_0)} \frac{(x - i\, y_0) - (x_0 - i\, y_0)}{(x - x_0) + i\,(y_0 - y_0)} \\ &= \lim_{(x + i\, y_0) \to (x_0 + i\, y_0)} \frac{x - x_0}{x - x_0} \\ &= 1. \end{aligned}$$

Next, we approach z_0 along a line parallel to the y axis by forcing z to be of the form $z = x_0 + i\, y$.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x_0+i\,y) \to (x_0+i\,y_0)} \frac{f(x_0+i\,y) - f(x_0+i\,y_0)}{(x_0+i\,y) - (x_0+i\,y_0)}$$

$$= \lim_{(x_0+i\,y) \to (x_0+i\,y_0)} \frac{(x_0-i\,y) - (x_0-i\,y_0)}{(x_0+i\,y) - (x_0+i\,y_0)}$$

$$= \lim_{(x_0+i\,y) \to (x_0+i\,y_0)} \frac{(x_0-i\,y) - (x_0-i\,y_0)}{(x_0-x_0) + i\,(y-y_0)}$$

$$= \lim_{(x_0+i\,y) \to (x_0+i\,y_0)} \frac{-i\,(y-y_0)}{i\,(y-y_0)}$$

$$= -1.$$

The limits along the two paths are different, so there is no possible value for the right side of Equation (3-1). Therefore $f(z) = \overline{z}$ is not differentiable at the point z_0, and since z_0 was arbitrary, $f(z) = \overline{z}$ is nowhere differentiable.

Explore Solution 3.2.

Remark 3.1. In Section 2.3 we showed that $f(z) = \overline{z}$ is continuous for all z. Thus, we have a simple example of a function that is continuous everywhere but differentiable nowhere. Such functions are hard to construct in real variables. In some sense, the complex case has made pathological constructions simpler!

We seldom are interested in studying functions that aren't differentiable, or are differentiable at only a single point. Complex functions that have a derivative at all points in a neighborhood of z_0 deserve further study. In Chapter 7 we demonstrate that, if the complex function $f(z)$ can be represented by a Taylor series at z_0, then it must be differentiable in some neighborhood of z_0. Functions that are differentiable in neighborhoods of points are pillars of the complex analysis edifice; we give them a special name, as indicated in the following definition.

Definition 3.1 (Analytic Function). The complex function $f(z)$ is analytic at the point z_0 provided there is some $\epsilon > 0$ such that $f'(z)$ exists for all $z \in D_\epsilon(z_0)$. In other words, $f(z)$ must be differentiable not only at z_0, but also at all points in some ϵ-neighborhood of z_0.

If $f(z)$ is analytic at each point in the region R, then we say that $f(z)$ is analytic on R. Again, we have a special term if $f(z)$ is analytic on the whole complex plane.

Definition 3.2 (Entire Function). If $f(z)$ is analytic on the whole complex plane then $f(z)$ is said to be entire.

Points of nonanalyticity for a function are called singular points. They are important for certain applications in physics and engineering.

Our definition of the derivative for complex functions is formally the same as for real functions and is the natural extension from real variables to complex variables. The basic differentiation formulas are identical to those for real functions, and we obtain the same rules for differentiating powers, sums, products, quotients, and compositions of functions. We can easily establish the proof of the differentiation formulas by using the limit theorems.

The Rules for Differentiation.

Suppose that $f(z)$ and $g(z)$ are differentiable. From Equation (3-2) and the technique exhibited in the solution to Example 3.1 we can establish the following rules, which are virtually

identical to those for real-valued functions.

(3 – 4) $\frac{d}{dz} C = 0$, where C is a constant,

(3 – 5) $\frac{d}{dz} z^n = n z^{n-1}$, where n is a positive integer,

(3 – 6) $\frac{d}{dz}[C f(z)] = C f'(z)$,

(3 – 7) $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$,

(3 – 8) $\frac{d}{dz}[f(z)\, g(z)] = f(z)\, g'(z) + g(z)\, f'(z)$,

(3 – 9) $\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{g(z) f'(z) - f(z) g'(z)}{[g(z)]^2}$, provided that $g(z) \neq 0$,

(3 – 10) $\frac{d}{dz} f(g(z)) = f'(f(g(z)))\, g'(z)$.

Important particular cases of Equations (3-9) and (3-10), respectively, are

(3 – 11) $\frac{d}{dz}\frac{1}{z^n} = \frac{-n}{z^{n+1}}$, for $z \neq 0$, and n is a positive integer,

(3 – 12) $\frac{d}{dz}[f(z)]^n = n\,[f(z)]^{n-1} f'(z)$, where n is a positive integer.

Exploration for the Rules for Differentiation.

Example 3.3. If we use Equation (3-12) with $f(z) = z^2 + i\,2\,z + 3$, and $f'(z) = 2\,z + 2\,i$, then we get

$$\frac{d}{dz}(z^2 + i\,2\,z + 3)^4 = 4\,(z^2 + i\,2\,z + 3)^3\,(2\,z + 2\,i)$$

$$= 8\,(z^2 + i\,2\,z + 3)^3\,(z + i).$$

Explore Solution 3.3.

The proofs of the rules given in Equations through (3-10) depend on the validity of extending theorems for real functions to their complex companions. Equation (3-8), for example, relies on Theorem 3.1.

Theorem 3.1. If $f(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0.

Proof. From Equation (3-1), we obtain

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

Using the multiplicative property of limits given by Formula (2-19), we get

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} (z - z_0)$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) * 0 = 0.$$

This result implies that $\lim_{z \to z_0} f(z) = f(z_0)$, which is equivalent to showing that $f(z)$ is continuous at z_0.

Proof.

Proof of Theorem 3.1 is also in the book.

We can establish Equation (3-8) from Theorem 3.1. Letting $h(z) = f(z)\, g(z)$ and using Definition 3.1, we write

$$h'(z_0) = \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(z)\, g(z) - f(z_0)\, g(z_0)}{z - z_0}.$$

If we subtract and add the term $f(z_0)\, g(z)$ in the numerator, we get

$$h'(z_0) = \lim_{z \to z_0} \frac{1}{z - z_0} (f(z)\, g(z) - f(z_0)\, g(z) + f(z_0)\, g(z) - f(z_0)\, g(z_0))$$

$$= \lim_{z \to z_0} \frac{f(z)\, g(z) - f(z_0)\, g(z)}{z - z_0} + \lim_{z \to z_0} \frac{f(z_0)\, g(z) - f(z_0)\, g(z_0)}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{(f(z) - f(z_0))\, g(z)}{z - z_0} + \lim_{z \to z_0} \frac{f(z_0)\, (g(z) - f(z_0)\, g(z_0))}{z - z_0}$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} g(z) + \lim_{z \to z_0} f(z_0) \lim_{z \to z_0} \frac{g(z) - f(z_0)\, g(z_0)}{z - z_0}$$

Using the definition of the derivative given by Equation (3-1) and the continuity of $g(z)$, we obtain $h'(z_0) = f'(z_0)\, g(z_0) + f(z_0)\, g'(z_0)$, which is what we wanted to establish. We leave the proofs of the other rules as exercises.

The rule for differentiating polynomials carries over to the complex case as well. If we let P(z) be a polynomial of degree n, so

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n,$$

then mathematical induction, along with Equations (3-5) and (3-7), gives

$$P'(z) = a_1 + 2\, a_2 z + 3\, a_3 z^2 + \ldots + a_{n-1} z^{n-1}.$$

Again, we leave the proof of this result as an exercise.

We can use the differentiation rules as aids in determining when functions are analytic. For example, Equation (3-9) tells us that if $P(z)$ and $Q(z)$ are polynomials, then their quotient $\frac{P(z)}{Q(z)}$ is analytic at all points where $Q(z) \neq 0$. This condition implies that the function $f(z) = \frac{1}{z}$ is analytic for all $z \neq 0$.

The square root function is more complicated. If $f(z) = |z|^{\frac{1}{2}} e^{i \frac{\text{Arg}(z)}{2}}$, then $f(z)$ is analytic at all points except $z = 0$ (because $\text{Arg}(0)$ is undefined) and at points that lie along the negative x axis. The argument function, and therefore the function $f(z)$ itself, are not continuous at points that lie along the negative x axis.

We close this section with a complex extension of a famous theorem, which is attributed to Guillaume François Antoine Marquis de L'Hôpital (1661-1704). The proof of will be given in Chapter 7.

Theorem 3.2, (L'Hôpital's Rule). Assume that $f(z)$ and $g(z)$ are both analytic at z_0. If $f(z_0) = 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$ then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Proof.

Proof of Theorem 3.2. We defer the proof until Chapter 7, where we will learn some amazing things about analytic functions.

Exploration for L'Hospital's Rule.

Extra Example 2. Use L'Hopital's rule to find $\lim_{z \to 1+i} \frac{z^2 + z - 1 - 3i}{z^2 - 2z + 2}$.

Explore Extra Example 2.

Complex Analysis for Mathematics and Engineering

by

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Section 3.2 The Cauchy-Riemann Equations

In Section 3.1 we showed that computing the derivative of complex functions written in a form such as $f(z) = z^2$ is a rather simple task. But life isn't always so easy.
Many times we encounter complex functions written as $f(z) = u(x, y) + i\, v(x, y)$. For example, suppose we had

(3-13)

$$f(z) = f(x + i\, y) = u(x, y) + i\, v(x, y) = (x^3 - 3\, x\, y^2) + i\, (3\, x^2\, y - y^3).$$

Is there some criterion - perhaps involving the partial derivatives for $u(x,y)$, and $v(x,y)$ - that we can use to determine whether f is differentiable, and if so, to find the value of $f(z)$?

The answer to this question is yes, thanks to the independent discovery of two important equations by the French mathematician Augustin Louis Cauchy (1789-1857) and the German mathematician Georg Friedrich Bernhard Riemann (1826-1866).

First, let's reconsider the derivative of $f(z) = z^2$. As we have stated, the limit given in Equation (3-1) must not depend on how z approaches z_0. We investigate two such approaches: a horizontal approach and a vertical approach to z_0. Recall from our graphical analysis of $w = z^2$ that the image of a square is a "curvilinear quadrilateral." For convenience, we let the square have vertices $z_0 = 2 + i$, $z_1 = 2.01 + i$, $z_2 = 2 + 1.01\, i$, and $z_3 = 2.01 + 1.01\, i$. Then the image points are $w_0 = 3 + 4\, i$, $w_1 = 3.0401 + 4.02\, i$, $w_2 = 2.9799 + 4.04\, i$, and $w_3 = 3.02 + 4.0602\, i$, as shown in Figure 3.1.

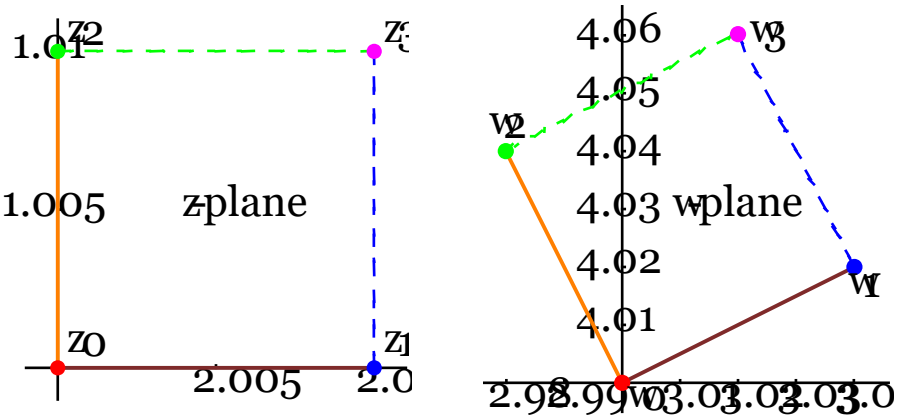

Figure 3.1 The image of a small square with vertex $z_0 = 2 + i$, under the mapping $w = f(z) = z^2$.

Exploration

We know that f is differentiable, so the limit of the difference quotient $\frac{f(z)-f(z_0)}{z-z_0}$ exists no matter how we approach $z_0 = 2 + i$.

Thus we can approximate $f'(z_0) = f'(2 + i)$ by using a horizontal increment in z:

$$f'(2+i) \approx \frac{f(z_1)-f(z_0)}{z_1-z_0} = \frac{f(2.01+i)-f(2+i)}{(2.01+i)-(2+i)} = \frac{(3.0401+4.02\,i)-(3+4\,i)}{(2.01+i)-(2+i)}$$

$$= \frac{(3.0401-3)+i\,(4.02-4)}{(2.01-2)-i\,(1-1)} = \frac{0.0401+0.02\,i}{0.01} = 4.01+2\,i$$

And, we can approximate $f'(z_0) = f'(2 + i)$ by using a vertical increment in z:

$$f'(2+i) \approx \frac{f(z_2)-f(z_0)}{z_2-z_0} = \frac{f(2+1.01\,i)-f(2+i)}{(2+1.01\,i)-(2+i)} = \frac{(2.9799+4.04\,i)-(3+4\,i)}{(2+1.01\,i)-(2+i)}$$

$$= \frac{(2.9799-3)+i\,(4.04-4)}{(2-2)+i\,(1.01-1)} = \frac{-0.0201+0.04\,i}{0.01\,i} = 4+2.01\,i$$

Comparing these two results we see that

$$f'(2+i) \approx \frac{f(z_1)-f(z_0)}{z_1-z_0} = 4.01+2\,i$$

and

$$f'(2+i) \approx \frac{f(z_2)-f(z_0)}{z_2-z_0} = 4+2.01\,i$$

which leads us to speculate that $f'(2 + i) = 4 + 2\,i$.

These computations lead to the idea of taking limits along the horizontal and vertical directions. When we do so, we get

$$f'(2+i) = \lim_{h\to 0}\frac{f(2+h+i)-f(2+i)}{h} = \lim_{h\to 0}\frac{(2+h+i)^2-(3+4\,i)}{h}$$

$$= \lim_{h\to 0}\frac{3+4\,h+h^2+i\,(4+2\,h)-(3+4\,i)}{h} = \lim_{h\to 0}\frac{4\,h+h^2+2\,h\,i}{h}$$

$$= \lim_{h\to 0}(4+h+2\,i) = 4+2\,i$$

and

$$f'(2+i) = \lim_{h\to 0} \frac{f(2+i+i\,h) - f(2+i)}{i\,h} = \lim_{h\to 0} \frac{(2+i+i\,h)^2 - (3+4\,i)}{i\,h}$$

$$= \lim_{h\to 0} \frac{3-2\,h-h^2+(4+4\,h)\,i - (3+4\,i)}{i\,h} = \lim_{h\to 0} \frac{-2\,h-h^2+4\,h\,i}{i\,h}$$

$$= \lim_{h\to 0} (4 + (2+h)\,i) = 4 + 2\,i$$

We now generalize this idea by taking limits of an arbitrary differentiable complex function and obtain an important result.

Explore Numerical Computations

Theorem 3.3 (Cauchy-Riemann Equations). Suppose that

$$f(z) = f(x+iy) = u(x, y) + i\,v(x, y)$$

is differentiable at the point $z_0 = x_0 + i\,y_0$. Then the partial derivatives of u and v exist at the point (x_0, y_0), and

(3-14) $f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0)$, and also

(3-15) $f'(z_0) = v_y(x_0, y_0) - i\,u_y(x_0, y_0)$.

Equating the real and imaginary parts of Equations (3-14) and (3-15) gives

(3-16) $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

Proof.

Derivation of the Cauchy-Riemann Equations with Mathematica.

Exploration for the Cauchy – Riemann Equations.

Note some of the important implications of this theorem.

(i). If f is differentiable at z_0, then the Cauchy-Riemann Equations (3-16) will be satisfied at z_0, and we can use either either Equation (3-14) or (3-15) to evaluate $f'(z_0)$.

(ii). Taking the contrapositive, if Equations (3-16) are not satisfied at z_0, then we know automatically that $f(z)$ is not differentiable at z_0.

(iii). Even if Equations (3-16) are satisfied at z_0, we cannot necessarily conclude that f is differentiable at z_0.

We now illustrate each of these points.

Example 3.4. We know that $f(z) = z^2 = x^2 - y^2 + i\,2\,x\,y$ is differentiable and that $f'(z) = 2\,z$. We also have

$$f(z) = z^2 = (x + i\,y)^2 = (x^2 - y^2) + i\,(2\,x\,y).$$

It is easy to verify that Equations (3-16) are indeed satisfied:

$$u_x(x, y) = 2\,x = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -2\,y = -v_x(x, y).$$

Using Equations (3-14) and (3-15), respectively, to compute $f'(z)$ gives

$$f'(z) = u_x(x, y) + i\,v_x(x, y) = 2\,x + i\,2\,y = 2\,z, \quad \text{and}$$

$$f'(z) = v_y(x, y) - i\,u_y(x, y) = 2\,x - i\,(-2\,y) = 2\,z,$$

as expected.

Explore Solution 3.4.

Example 3.5. Show that $f(z) = \overline{z}$ is nowhere differentiable.

Solution. We have $f(z) = f(x + i\,y) = x - i\,y = u(x, y) + i\,v(x, y)$, where $u(x, y) = x$ and $v(x, y) = -y$. Thus, for any point (x, y), $u_x(x, y) = 1$ and $v_y(x, y) = -1$. The Cauchy-Riemann equations are not satisfied at any point $z = (x, y)$, so we conclude that $f(z) = \overline{z}$ is nowhere differentiable.

Explore Solution 3.5.

Example 3.6. Show that the function defined by

$$f(z) = \begin{cases} \frac{(\overline{z})^2}{z} = \frac{x^3 - 3\,x\,y^2}{x^2 + y^2} + i\,\frac{y^3 - 3\,x^2\,y}{x^2 + y^2} & \text{when } z \neq 0, \text{ and} \\ 0 & \text{when } z \neq 0. \end{cases}$$

is not differentiable at the point $z_0 = 0$ even though the Cauchy-Riemann equations are satisfied at $(0, 0)$.

Solution. We must use limits to calculate the partial derivatives at $(0, 0)$.

$$u_x(0, 0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{\frac{x^3 - 0}{x^2 + 0} - 0}{x} = \lim_{x \to 0} \frac{x}{x} = 1,$$

$$u_y(0, 0) = \lim_{y \to 0} \frac{u(0,y) - u(0,y)}{y} = \lim_{y \to 0} \frac{\frac{0 - 0}{0 + y^2} - 0}{y} = \lim_{y \to 0} \frac{0}{y} = 0,$$

$$v_x(0, 0) = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{\frac{0 - 0}{x^2 + 0} - 0}{x} = \lim_{x \to 0} \frac{0}{x} = 1,$$

$$v_y(0, 0) = \lim_{y \to 0} \frac{v(0,y) - v(0,y)}{y} = \lim_{y \to 0} \frac{\frac{y^3 - 0}{0 + y^2} - 0}{y} = \lim_{y \to 0} \frac{y}{y} = 1.$$

Thus we have shown that $u_x(0, 0) = 1$, $u_y(0, 0) = 0$, $v_x(0, 0) = 0$, $v_y(0, 0) = 1$

Hence the Cauchy-Riemann equations hold at the point $(0, 0)$.

We now show that f is not differentiable at $z_0 = 0$. Letting z approach 0 along the x axis gives

$$\lim_{(x,0)\to(0,0)} \frac{f(x+0\,i)-f(0)}{x+0\,i-0} = \lim_{(x,0)\to(0,0)} \frac{\frac{x^3-0}{x^2+0}+i\,\frac{0-0}{x^2+0}-0}{x} = \lim_{(x,0)\to(0,0)} \frac{\frac{x^3}{x^2}+0\,i-0}{x}$$

$$= \lim_{(x,0)\to(0,0)} \frac{x}{x} = 1$$

But if we let z approach 0 along the line $y = x$ given by the parametric equations $x = t$ and $y = t$, then

$$\lim_{(t,t)\to(0,0)} \frac{f(t+i\,t)-f(0)}{t+i\,t-0} = \lim_{(t,t)\to(0,0)} \frac{\frac{t^3-3\,t^3}{t^2+t^2}+i\,\frac{t^3-3\,t^3}{t^2+t^2}-0}{t+i\,t} = \lim_{(t,t)\to(0,0)} \frac{\frac{-2\,t^3}{2\,t^2}+i\,\frac{-2\,t^3}{2\,t^2}}{t+i\,t}$$

$$= \lim_{(t,t)\to(0,0)} \frac{-t-i\,t}{t+i\,t} = -1$$

The two limits are distinct, so f is not differentiable at the origin.

Explore Solution 3.6.

Example 3.6 reiterates that the mere satisfaction of the Cauchy-Riemann equations is not sufficient to guarantee the differentiability of a function. The following theorem, however, gives conditions that guarantee the differentiability of f at z_0, so that which we can use Equation (3-14) or (3-15) to compute $f'(z_0)$. They are referred to as the Cauchy-Riemann conditions for differentiability.

Theorem 3.4 (Cauchy-Riemann conditions for differentiability). Let $f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$ be a continuous function that is defined in some neighborhood of the point $z_0 = x_0 + i\,y_0$. If all the partial derivatives u_x, u_y, v_x and v_y are continuous at the point (x_0, y_0) and if the Cauchy-Riemann equations $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$ hold at (x_0, y_0), then f(z) is differentiable at z_0 and the derivative $f'(z_0)$ can be computed with either formula (3-14) or (3-15), i.e.

$$f'(z_0) = u_x(x_0, y_0) + i\,v_x(x_0, y_0), \text{ or}$$

$$f'(z_0) = v_y(x_0, y_0) - i\,u_y(x_0, y_0).$$

Proof.

Proof of Theorem 3.4 is in the book.

Example 3.7. At the beginning of this section (Equation (3-13)) we defined the function

$$f(z) = u(x, y) + i\,v(x, y) = x^3 - 3\,x\,y^2 + i\,(3\,x^2\,y - y^3).$$

Show that this function is differentiable for all z, and find its derivative.

Solution. We compute $u_x(x, y) = 3\,x^3 - 3\,y^2 = v_y(x, y)$ and $u_y(x, y) = -6\,x\,y = -v_x(x, y)$, so the Cauchy-Riemann Equations (3-16), are satisfied.

Moreover, u, v, u_x, u_y, v_x and v_y are continuous everywhere. By Theorem 3.4, f is differentiable everywhere, and, from Equation (3-14),

$$f'(z) = u_x(x, y) + i\, v_x(x, y) = 3x^3 - 3y^2 + i\, 6xy$$

$$= 3(x^2 - y^2 + 2i\, xy) = 3(x + i\, y)^2 = 3z^2$$

Alternatively, from Equation (3-15),

$$f'(z) = v_y(x, y) - i\, u_y(x, y) = 3x^3 - 3y^2 - i(-6xy)$$

$$= 3(x^2 - y^2 + 2i\, xy) = 3(x + i\, y)^2 = 3z^2$$

This result isn't surprising because $(x + i\, y)^3 = x^3 - 3xy^2 + i(3x^2 y - y^3)$ and so the function f is really our old friend $f(z) = z^3$.

Explore Solution 3.7.

Extra Example 1. Given $f(z) = x^6 - 15x^4 y^2 + 15x^2 y^4 - y^6 + i(6x^5 y - 20x^3 y^3 + 6xy^5)$.

Show that this function is differentiable for all z, and find its derivative.

Explore Extra Solution 1.

Example 3.8. Show that the function $f(z) = f(x + i\, y) = e^{-y}\cos x + i\, e^{-y}\sin x$ is differentiable for all $z = x + i\, y$ and find its derivative.

Solution. We first write $u(x, y) = e^{-y}\cos x$ and $v(x, y) = e^{-y}\sin x$ and then compute the partial derivatives.

$$u_x(x, y) = v_y(x, y) = -e^{-y}\sin x, \text{ and}$$

$$v_x(x, y) = -u_y(x, y) = e^{-y}\cos x.$$

We note that u, v, u_x, u_y, v_x and v_y are continuous functions and that the Cauchy-Riemann equations hold for all values of (x, y). Hence, using Equation (3-14), we write

$$f'(z) = u_x(x, y) + i\, v_x(x, y) = -e^{-y}\sin x + i\, e^{-y}\cos x.$$

Aside. Can you guess the "complex" form of $f(z)$?

Explore Solution 3.8.

Extra Example 2. Show that the function $f(z) = e^{x^2 - y^2}\cos(2xy) + i\, e^{x^2 - y^2}\sin(2xy)$ is differentiable for all $z = x + i\, y$ and find its derivative.

Explore Extra Solution 2.

The Cauchy-Riemann conditions are particularly useful in determining the set of points for which a function f is differentiable.

Example 3.9. Show that the function $f(z) = x^3 + 3xy^2 + i(3x^2y + y^3)$ is differentiable at points that lie on the x and y axes but analytic nowhere.

Solution. Recall (Definition 3.1) that when we say a function is analytic at a point z_0 we mean that the function is differentiable not only at z_0, but also at every point in some neighborhood of z_0. With this in mind, we proceed to determine where the Cauchy-Riemann equations are satisfied. We write $u(x, y) = x^3 + 3xy^2$ and $v(x, y) = 3x^2y + y^3$ and compute the partial derivatives:

$$u_x(x, y) = 3x^3 + 3y^2, \quad v_y(x, y) = 3x^3 + 3y^2, \quad \text{and}$$

$$u_y(x, y) = 6xy, \quad v_x(x, y) = 6xy.$$

Here $u, v, u_x,$ and v_y are continuous, and $u_x(x, y) = v_y(x, y)$ holds for all (x, y). But $u_y(x, y) = -v_x(x, y)$ iff $6xy = -6xy$, which is equivalent to $12xy = 0$. The Cauchy-Riemann equations hold only when $x = 0$ or $y = 0$, and according to Theorem 3.4, f is differentiable only at points that lie on the coordinate axes. But this means that f is nowhere analytic because any ϵ-neighborhood about a point on either axis contains points that are not on those axes.

Explore Solution 3.9.

When polar coordinates (r, θ) are used to locate points in the plane, we use Expression (2-2) for a complex function for convenience; that is,

$$f(z) = u(x, y) + i\,v(x, y)$$

$$f(re^{i\theta}) = u(r\cos\theta, r\sin\theta) + i\,v(r\cos\theta, r\sin\theta)$$

$$= U(r, \theta) + i\,V(r, \theta)$$

where $U(r, \theta)$ and $V(r, \theta)$ are real functions of the real variables r and θ. The polar form of the Cauchy-Riemann equations and a formula for finding f'(z) in terms of the partial derivatives of U(r,) and V(r,) are given in Theorem 3.5, which we ask you to prove in Exercise 10. This theorem makes use of the validity of the Cauchy-Riemann equations for the functions u and v, so the relation between them and the functions $U(r, \theta)$ and $V(r, \theta)$ - namely, $u(x, y) = u(r\cos\theta, r\sin\theta) = U(r, \theta)$ and $v(x, y) = v(r\cos\theta, r\sin\theta) = V(r, \theta)$ - is important.

Theorem 3.5 (Polar Form of the Cauchy-Riemann equations). Let $f(z) = f(re^{i\theta}) = U(r, \theta) + i\,V(r, \theta)$ be a continuous function that is defined in some neighborhood of the point $z_0 = r_0 e^{i\theta_0}$. If all the partial derivatives U, V, U_r, U_θ, V_r and V_θ are continuous at the point (r_0, θ_0) and if the polar form of the Cauchy-Riemann equations,

(3-22) $$U_r(r_0, \theta_0) = \frac{1}{r_0} V_\theta(r_0, \theta_0) \quad \text{and} \quad V_r(r_0, \theta_0) = \frac{-1}{r_0} U_\theta(r_0, \theta_0),$$

holds, then $f(z_0)$ is differentiable at z_0, and we can compute the derivative $f(z_0)$ by using either

(3-23) $$f'(z_0) = f'(r_0 e^{i\theta_0}) = e^{-i\theta_0}(U_r(r_0,\theta_0) + i V_r(r_0,\theta_0)), \quad \text{or}$$

(3-24) $$f'(z_0) = f'(r_0 e^{i\theta_0}) = \frac{1}{r_0} e^{-i\theta_0}(V_\theta(r_0,\theta_0) - i U_\theta(r_0,\theta_0)).$$

Proof.

Example 3.10. Show that, if f is is the principal square root function given by

$$f(r e^{i\theta}) = f(z) = z^{\frac{1}{2}} = r^{\frac{1}{2}}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

where the domain is restricted to be $\{r e^{i\theta} : r > 0 \text{ and } -\pi < \theta < \pi\}$, then the derivative is given by

$$f'(z) = \frac{1}{2z^{1/2}} = \frac{1}{2} r^{-\frac{1}{2}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$

for every point in the domain $\{r e^{i\theta} : r > 0 \text{ and } -\pi < \theta < \pi\}$.

Solution. We write

$$U(r,\theta) = r^{\frac{1}{2}}\cos\frac{\theta}{2}, \text{ and}$$

$$V(r,\theta) = r^{\frac{1}{2}}\sin\frac{\theta}{2}.$$

Thus,

$$U_r(r,\theta) = \frac{1}{r}V_\theta(r,\theta) = \frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2}, \text{ and}$$

$$V_r(r,\theta) = \frac{-1}{r}U_\theta(r,\theta) = \frac{1}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}.$$

Since U, V, U_r, U_θ, V_x and V_θ are continuous at every point in the domain (note the strict inequality in $-\pi < \theta < \pi$), we use Theorem 3.5 and Equation (3-23) to get

$$f'(z) = f'(r e^{i\theta}) = e^{-i\theta}(U_r(r,\theta) + i V_r(r,\theta))$$

$$= e^{-i\theta}\left(\frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} + i\frac{1}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}\right)$$

$$= e^{-i\theta}\left(\frac{1}{2}r^{-\frac{1}{2}}e^{i\frac{\theta}{2}}\right) = \frac{1}{2}r^{-\frac{1}{2}}e^{-i\frac{\theta}{2}}$$

$$= \frac{1}{2z^{\frac{1}{2}}} = \frac{1}{2}z^{-\frac{1}{2}}$$

Note that $f(z)$ is discontinuous on the negative real axis and is undefined at the origin. Using the terminology of Section 2.4, the negative real axis is a branch cut, and the origin is a branch point for this function.

Explore Solution 3.10.

Two important consequences of the Cauchy-Riemann equations close this section.

Theorem 3.6. Let $f = u + i\,v$ be an analytic function on the domain D. Suppose for all $z \in D$ that $|f(z)| \equiv K$, where K is a constant. Then f is constant on D.

Proof.

Theorem 3.7. Let $f = u + i\,v$ be an analytic function on the domain D. If $f'(z) \equiv 0$ for all $z \in D$, then f is constant on D.

Proof.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

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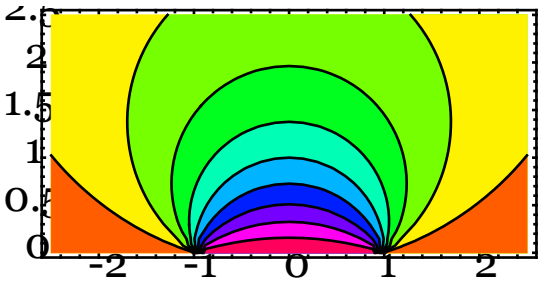
Section 3.3 Harmonic Functions

Let $\phi(x, y)$ be a real-valued function of the two real variables x and y defined on a domain D. (Recall that a domain is a connected open set.) The partial differential equation

(3-26) $$\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0$$

is known as Laplace's equation and is sometimes referred to as the potential equation. If $\phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yx}$ and ϕ_{yy} are all continuous and if $\phi(x, y)$ satisfies Laplace's equation, then $\phi(x, y)$ is called a harmonic function.

Harmonic functions are important in the areas of applied mathematics, engineering, and mathematical physics. In calculus we might have been asked to show that the functions $x^3 - 3xy^2$ and $3x^2y - y^3$ are harmonic (and also that the functions $e^x \cos y$ and $e^x \sin y$ are harmonic). Harmonic functions are used to solve problems involving steady state temperatures, two-dimensional electrostatics, and ideal fluid flow. In Chapter 11 we will see how complex analysis techniques can be used to solve some problems involving harmonic functions. For example, the function $\phi(x, y) = \frac{1}{\pi}\left(\arctan\frac{y}{x-1} - \arctan\frac{y}{x+1}\right)$ is harmonic in the upper half plane and takes on the boundary values $\phi(x, 0) = 1$ when $|x| < 1$ and $\phi(x, 0) = 0$ when $|x| > 1$, as shown in Figure 3.A.



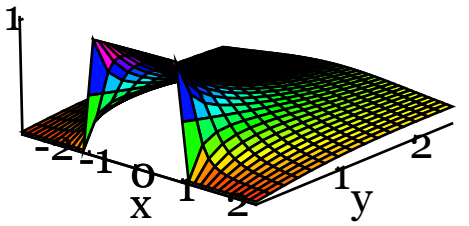


Figure 3.A The harmonic function $\phi(x, y) = \frac{1}{\pi}\left(\arctan\frac{y}{x-1} - \arctan\frac{y}{x+1}\right)$.

Exploration

These pairs of functions are not chosen at random, and there is an intimate relationship between them. We begin with an important theorem relating analytic and harmonic functions.

Theorem 3.8. Let $f(z) = f(x+iy) = u(x, y) + i\, v(x, y)$ be an analytic function on a domain D. Then both $u(x, y)$ and $v(x, y)$ are harmonic functions on D. In other words, the real and imaginary parts of an analytic function are harmonic.

Proof. In Corollary 6.3 we will show that, if f(z) is analytic, then all partial derivatives of u and v are continuous. Using that result here, we see that, as f is analytic, u and v satisfy the Cauchy-Riemann equations

$$u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -v_x(x, y).$$

Taking the partial derivative with respect to x of each side of these equations gives

$$u_{xx}(x, y) = v_{yx}(x, y) \quad \text{and} \quad u_{yx}(x, y) = -v_{xx}(x, y).$$

Similarly, taking the partial derivative of each side with respect to y yields

$$u_{xy}(x, y) = v_{yy}(x, y) \quad \text{and} \quad u_{yy}(x, y) = -v_{xy}(x, y).$$

The partial derivatives u_{xy}, u_{yx}, v_{xy} and v_{yx} are all continuous, so we use a theorem from the calculus of real functions that states that the mixed partial derivatives are equal; that is,

$$u_{xy}(x, y) = u_{yx}(x, y) \quad \text{and} \quad v_{xy}(x, y) = v_{yx}(x, y).$$

Combining all these results finally gives

$$u_{xx}(x, y) + u_{yy}(x, y) = v_{yx}(x, y) - v_{xy}(x, y) = 0, \text{ and}$$

$$v_{xx}(x, y) + v_{yy}(x, y) = -u_{yx}(x, y) + u_{xy}(x, y) = 0.$$

Therefore both u and v are harmonic functions on D.

Proof.

If we have a function $u(x, y)$ that is harmonic on the domain D and if we can find another harmonic function $v(x, y)$ such that the partial derivatives for u and v satisfy the Cauchy-Riemann equations throughout D, then we say that $v(x, y)$ is a harmonic conjugate of $u(x, y)$. It then follows that the function $f(z) = u(x, y) + i\, v(x, y)$ is analytic on D.

Example 3.11. If $u(x, y) = x^2 - y^2$, then $u_{xx}(x, y) + u_{yy}(x, y) = 2 - 2 = 0$; hence u is a harmonic function for all z. We find that $v(x, y) = 2\,x\,y$ is also a harmonic function and that

$$u_x(x, y) = v_y(x, y) = 2\,x, \quad \text{and}$$

$$u_y(x, y) = -v_x(x, y) = -2\,y.$$

Therefore v is a harmonic conjugate of u, and the function f given by

$$f(z) = x^2 - y^2 + i\,2\,x\,y$$

$$= (x + i\,y)^2 = z^2$$

is an analytic function.

Explore Solution 3.11.

Theorem 3.8 makes the construction of harmonic functions from known analytic functions an easy task.

Example 3.12. The function $f(z) = z^3 = x^3 - 3\,x\,y^2 + i\,(3\,x^2\,y - y^3)$ is analytic for all values of z, hence it follows that

$$u(x, y) = \text{Re}[f(z)] = x^3 - 3\,x\,y^2$$

is harmonic, and

$$v(x, y) = \text{Im}[f(z)] = 3\,x^2\,y - y^3$$

is a harmonic conjugate of u(x,y).

Explore Solution 3.12.

Figures 3.2 and 3.3 show the graphs of these two functions. The partial derivatives are $u_x(x, y) = 3\,x^2 - 3\,y^2$, $u_y(x, y) = -6\,x\,y$, $v_x(x, y) = 6\,x\,y$, and $v_y(x, y) = 3\,x^2 - 3\,y^2$. They satisfy the Cauchy-Riemann equations because they are the real and imaginary parts of an analytic function. At the point $(x, y) = (2, -1)$, we have $u_x(2, -1) = v_y(2, -1) = 9$, and these partial derivatives appear along the edges of the surfaces for u and v where $x = 2$ and $y = -1$. Similarly, $u_y(2, -1) = 12$ and $v_x(2, -1) = -12$ also appear along the edges of the surfaces for u and v where $x = 2$ and $y = -1$.

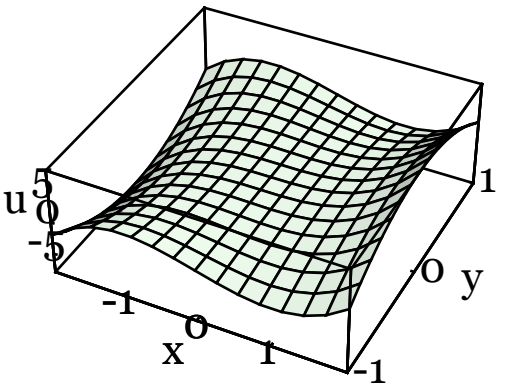


Figure 3.2 $u(x, y) = x^3 - 3 x y^2$.

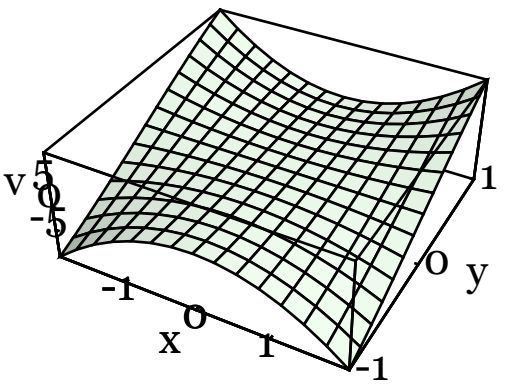


Figure 3.3 $v(x, y) = 3 x^2 y - y^3$.

We can use complex analysis to show easily that certain combinations of harmonic functions are harmonic. For example, if v is a harmonic conjugate of u, then their product $\phi(x, y) = u(x, y)\, v(x, y)$ is a harmonic function. This can be verified directly by computing the partial derivatives and showing that Laplace's equation (3-26) holds, but the details are tedious. If we use complex variable techniques instead, we can start with the fact that $f(z) = u(x, y) + i\, v(x, y)$ is an analytic function. Then we observe that the square of f is also an analytic function, which is

$$[f(z)]^2 = [u(x, y)]^2 - [v(x, y)]^2 + i\, 2\, u(x, y)\, v(x, y).$$

We then know immediately that the imaginary part, $2\, u(x, y)\, v(x, y)$, is a harmonic function by Theorem 3.8. Since a constant multiple of a harmonic function is harmonic, it follows that $\phi(x, y) = u(x, y)\, v(x, y)$ is harmonic. It is left as an exercise to show that if $u_1(x, y)$ and $u_2(x, y)$ are two harmonic functions that are not related in the preceding fashion, then their product need not be harmonic.

Theorem 3.9 (Construction of a Conjugate). Let $u(x, y)$ be harmonic in an ϵ-neighborhood of the point (x_0, y_0). Then there exists a conjugate harmonic function $v(x, y)$ defined in this neighborhood such that $f(z) = u(x, y) + i\, v(x, y)$ is an analytic function.

Proof. A conjugate harmonic function v will satisfy the Cauchy-Riemann equations $u_x(x, y) = v_y(x, y)$ and $u_y(x, y) = -v_x(x, y)$. Assuming that such a function exists, we determine what it would have to look like by using a two-step process. First, we integrate $v_y(x, y)$ (which should equal $u_x(x, y)$) with respect to y and get

(3-27)
$$v(x, y) = \int v_y(x, y)\, dy + C(x)$$

$$v(x, y) = \int u_x(x, y)\, dy + C(x)$$

where $C(x)$ is a function of x alone that is yet to be determined. Second, we compute $C'(x)$ by differentiating both sides of this equation with respect to x and replacing $v_x(x, y)$ with $-u_y(x, y)$ on the left side, which gives

$$v_x(x,y) = \frac{d}{dx}\int u_x(x,y)\,dy + C'(x)$$

$$-u_y(x,y) = \frac{d}{dx}\int u_x(x,y)\,dy + C'(x)$$

It can be shown (we omit the details) that because u is harmonic, all terms except those involving x in the last equation will cancel, revealing a formula for $C'(x)$ involving x alone. Elementary integration of the single-variable function $C'(x)$ can then be used to discover $C(x)$. We finally observe that the function v so created indeed has the properties we seek.

The functions $C(x)$ and $v(x,y)$ are computed with the formulas:

$$C(x) = \int\left(-u_y(x,y) - \frac{d}{dx}\left(\int u_x(x,y)\,dy\right)\right)dx, \quad \text{and}$$

$$v(x,y) = \int u_x(x,y)\,dy + C(x).$$

Proof.

Technically we should always specify the domain of function when defining it. When no such specification is given, it is assumed that the domain is the entire complex plane, or the largest set for which the expression defining the function makes sense.

Method. Construction of the Harmonic Conjugate of u(x,y).

Execute this cell to activate the construction of the harmonic conjugate subroutine.

Example 3.13. Show that $u(x, y) = x\,y^3 - x^3\,y$ is a harmonic function and find the harmonic conjugate $v(x, y)$.

Solution. We follow the construction process of Theorem 3.9. The first partial derivatives are

(3-28) $$u_x(x,y) = y^3 - 3\,x^2\,y \quad \text{and} \quad u_y(x,y) = 3\,x\,y^2 - x^3.$$

To verify that u is harmonic, we compute the second partial derivatives and note that $u_{xx}(x,y) + u_{yy}(x,y) = -6\,x\,y + 6\,x\,y = 0$, so u satisfies Laplace's Equation (3-26). To construct v, we start with Equation (3-27) and the first of Equations (3-28) to get

$$v(x,y) = \int (y^3 - 3\,x^2\,y)\,dy + C(x)$$

$$= \frac{1}{4}y^4 - \frac{3}{2}x^2\,y^2 + C(x)$$

Differentiating the left and right sides of this equation with respect to x and using $-u_y(x,y) = v_x(x,y)$ and Equations (3-28) on the left side yields

$$-3\,x\,y^2 + x^3 = 0 - 3\,x\,y^2 + C'(x)$$

which implies that

$$C'(x) = x^3,$$

then an easy integration yields $C(x) = \frac{1}{4}x^4 + C$, where C is a constant. Now we substitute

C (x) and obtain the desired solution

$$v(x, y) = \frac{1}{4} x^4 + \frac{1}{4} y^4 - \frac{3}{2} x^2 y^2 + C.$$

Explore Solution 3.13.

Contour plots for the families of curves {u (x, y) = C : C is a real constant} and {v (x, y) = C : C is a real constant} are shown in Figures 3.B.

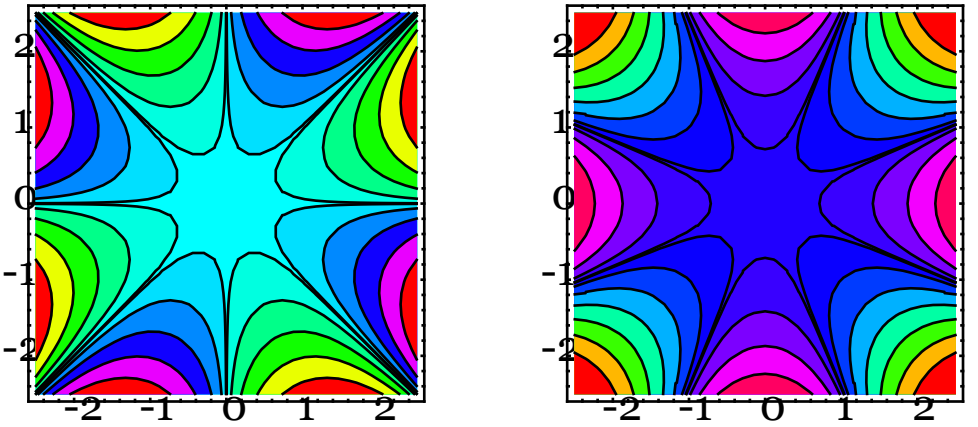


Figure 3.B Contour plot for $u(x, y) = x^3 - 3 x y^2 = C$ and $v(x, y) = 3 x^2 y - y^3 = C$.

In Chapter 11 we will show that if f (z) = u (x, y) + i v (x, y) is an analytic function. Then the two families of level curves {u (x, y) = C : C is a real constant} and {v (x, y) = C : C is a real constant}, are orthogonal in the sense that if (x_0, y_0) is a point in common to the two curves $u(x, y) = C_1$ and $v(x, y) = C_2$, and if $f'(x_0 + i\, y_0) \neq 0$, then these two curves intersect orthogonally. The orthogonal grid shown in Figure 3.C.

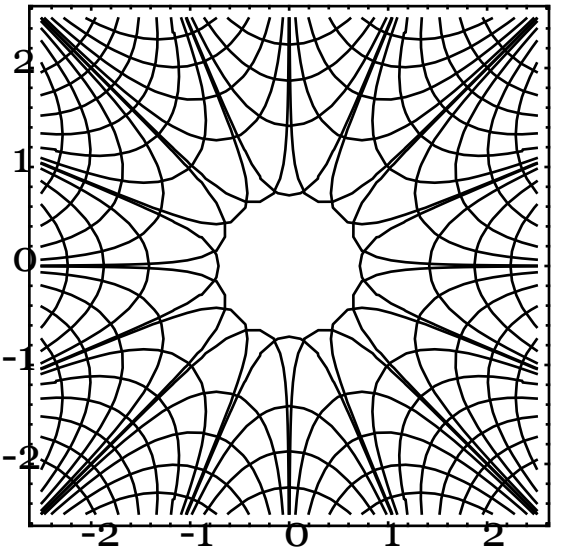


Figure 3.C The orthogonal grid formed by $x^3 - 3 x y^2 = C$ and $3 x^2 y - y^3 = C$.

Extra Example 1. Show that $u(x, y) = x - 2 x^3 - 6 x y - 4 x^3 y + 6 x y^2 + 4 x y^3$ is a harmonic function and find the harmonic conjugate v (x, y).

Explore Extra Solution 1.

Fluid Flow

Harmonic functions are solutions to many physical problems. Applications include two-dimensional models of heat flow, electrostatics, and fluid flow. We now give an example of the latter.

We assume that an incompressible and frictionless fluid flows over the complex plane and that all cross sections in planes parallel to the complex plane are the same. Situations such as this occur when fluid is flowing in a deep channel. The velocity vector at the point (x, y) is

(3-29) $$\mathbf{V}(x, y) = p(x, y) + i\, q(x, y).$$

The assumption that the flow is irrotational and has no sources or sinks implies that both the curl and divergence vanish, that is, $q_x - p_y = 0$ and $p_x + q_y = 0$. Hence p and q obey the partial differential equations

(3-30) $$p_x(x, y) = -q_y(x, y), \quad \text{and}$$

$$p_y(x, y) = q_x(x, y).$$

Equations (3-30) are similar to the Cauchy-Riemann equations and permit us to define a special complex function:

(3-31) $$f(z) = u(x, y) + i\, v(x, y) = p(x, y) - i\, q(x, y).$$

Here we have $u_x = p_x$, $u_y = p_y$, $v_x = -q_x$ and $v_y = -q_y$. We can use Equations (3-30) to verify that the Cauchy-Riemann equations are satisfied for f:

$$u_x(x, y) = p_x(x, y) = -q_y(x, y) = v_y(x, y), \quad \text{and}$$

$$u_y(x, y) = p_y(x, y) = q_x(x, y) = -v_x(x, y).$$

Assuming the functions p and q have continuous partials, Theorem 3.4 guarantees that function f defined in Equation (3-31) is analytic, and that the fluid flow of Equation (3-29) is the conjugate of an analytic function, that is,

$$\mathbf{V}(x, y) = \overline{f(z)}.$$

In Chapter 6 we will prove that every analytic function f has an analytic antiderivative F; assuming this to be the case, we can write

(3-32) $$F(z) = \phi(x, y) + i\, \psi(x, y),$$

where $F'(z) = f(z)$.

Theorem 3.8 tells us that $\phi(x, y)$ is a harmonic function. If we use the vector interpretation of a complex number we see that the gradient of ϕ can be written as

$$\text{grad}\, \phi(x, y) = \phi_x(x, y) + i\, \phi_y(x, y).$$

The Cauchy-Riemann equations applied to $F(z)$ give $\phi_y(x, y) = -\psi_x(x, y)$; making

this substitution in the last equation yields

$$\text{grad}\,\phi\,(x, y) = \phi_x\,(x, y) - i\,\psi_x\,(x, y) = \overline{\phi_x\,(x, y) + i\,\psi_x\,(x, y)}.$$

Equation (3-14) says that $\phi_x\,(x, y) + i\,\psi_x\,(x, y) = F'\,(z)$, which by the preceeding equation and Equation (3-32) imply that

$$\text{grad}\,\phi\,(x, y) = \overline{F'\,(z)} = \overline{f\,(z)}.$$

Finally, from Equation (3-29), $\phi\,(x, y)$ is the scalar potential function for the a fluid flow, so

$$\mathbf{V}\,(x, y) = \text{grad}\,\phi\,(x, y).$$

The curves given by $\{(x, y) : \phi\,(x, y) = \text{constant}\}$ are called equipotentials. The curves $\{(x, y) : \psi\,(x, y) = \text{constant}\}$ are called streamlines and describe the path of fluid flow. In Chapter 11 we will see that the family of equipotentials is orthogonal to the family of streamlines.

Example 3.14. Show that the harmonic function $\phi\,(x, y) = x^2 - y^2$ is the scalar potential function for the fluid flow $\mathbf{V}\,(x, y) = 2\,x - 2\,i\,y$.

Solution. We can write the fluid flow expression as

$$\mathbf{V}\,(x, y) = 2\,x - 2\,i\,y = 2\,z.$$

An antiderivative of $f\,(z) = 2\,z$ is $F\,(z) = z^2$, and the real part of $F\,(z)$ is the desired harmonic function:

$$\phi\,(x, y) = \text{Re}[F\,(z)] = \text{Re}[x^2 - y^2 + 2\,i\,x\,y] = x^2 - y^2.$$

Note that the hyperbolas $\phi\,(x, y) = x^2 - y^2 = C$ are the equipotential curves and that the hyperbolas $\psi\,(x, y) = 2\,x\,y = C$ are the streamline curves; these curves are orthogonal, as shown in Figure 3.6.

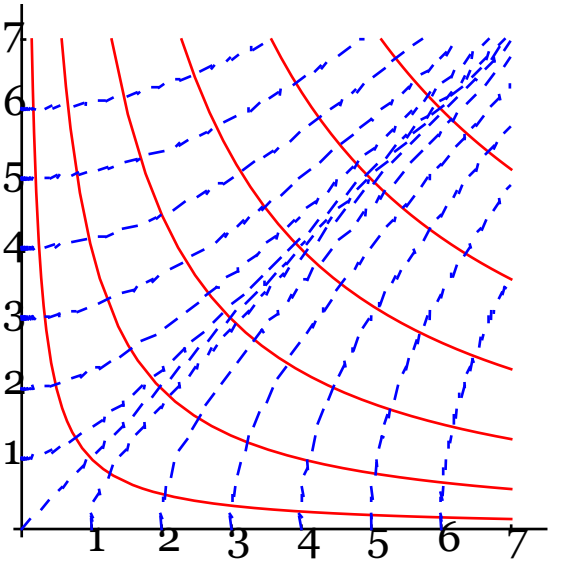


Figure 3.6 Blue equipotential curves $x^2 - y^2 = C$, and red streamline curves $2\,x\,y = C$, for

$F(z) = z^2$.

Explore Solution 3.14.

Complex Analysis for Mathematics and Engineering

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Chapter 4 Sequences, Julia and Mandelbrot Sets, and Power Series

Overview

In 1980 Benoit Mandelbrot led a team of mathematicians in producing some stunning computer graphics from very simple rules for manipulating complex numbers. This event marked the beginning of a new branch of mathematics, known as fractal geometry, that has some amazing applications. Many of the tools needed to appreciate Mandelbrot's work are contained in this chapter. We look at extensions to the complex domain of sequences and series, ideas that are familiar to students who have completed a standard calculus course.

4.1 Sequences and Series

In formal terms, a complex sequence is a function whose domain is the positive integers and whose range is a subset of the complex numbers. The following are examples of sequences:

$$f(n) = \left(2 - \frac{1}{n}\right) + \left(5 + \frac{1}{n}\right) i \quad (n = 1, 2, 3, \ldots); \tag{4-1}$$

$$g(n) = e^{i\frac{\pi n}{4}} \quad (n = 1, 2, 3, \ldots); \tag{4-2}$$

$$h(k) = 5 + 3i + \left(\frac{1}{1+i}\right)^k \quad (k = 1, 2, 3, \ldots); \tag{4-3}$$

$$r(n) = \left(\frac{1}{4} + \frac{i}{2}\right)^n \quad (n = 1, 2, 3, \ldots). \tag{4-4}$$

Exploration

For convenience, at times we use the term sequence rather than complex sequence. If we want a function s to represent an arbitrary sequence, we can specify it by writing $s(1) = z_1$, $s(2) = z_2$, $s(3) = z_3$, and so on. The values $z_1, z_2, z_3, \ldots$, are called the terms of a sequence, and mathematicians, being generally lazy when it comes to such things, often refer to z_1, z_2, z_3, etc. as the sequence itself, even though they are really speaking of the range of the sequence when they do so. You will usually see a sequence written as $\{z_n\}_{n=1}^{\infty}$, $\{z_n\}_1^{\infty}$, or when the indices are understood, as $\{z_n\}$. Mathematicians are also not so fussy about starting a

sequence at z_1, so that $\{z_n\}_{n=-1}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$, etc., would also be acceptable notation, provided all terms were defined. For example, the sequence r given by Equation (4-4) could be written in a variety of ways:

$$\left\{\left(\frac{1}{4}+\frac{i}{2}\right)^n\right\}_{n=1}^{\infty},\ \left\{\left(\frac{1}{4}+\frac{i}{2}\right)^n\right\}_{1}^{\infty},\ \left\{\left(\frac{1}{4}+\frac{i}{2}\right)^n\right\}_{n=1}^{\infty},\ \left\{\left(\frac{1}{4}+\frac{i}{2}\right)^{n+2}\right\}_{n=-2}^{\infty},\ \left\{\left(\frac{1}{4}+\frac{i}{2}\right)^k\right\}_{k=1}^{\infty},\ \ldots$$

The sequences f and g given by Equations (4-1) and (4-2) behave differently as n gets larger. The terms in Equation (4-1) approach $2+5i=(2,5)$, but those in Equation (4-2) do not approach any particular number, as they oscillate around the eight eighth roots of unity on the unit circle. Informally, the sequence $\{z_n\}_1^{\infty}$ has ζ as its limit as n approaches infinity, provided the terms z_n can be made as close as we want to ζ by making n large enough. When this happens, we write

(4-5) $$\lim_{n\to\infty} z_n=\zeta \quad \text{or} \quad z_n\to\zeta \ \text{ as } \ n\to\infty.$$

If $\lim_{n\to\infty} z_n=\zeta$, we say that the sequence $\{z_n\}_1^{\infty}$ converges to ζ.

We need a rigorous definition for Statement (4-5), however, if we are to do honest mathematics.

Definition 4.1 (Limit of a Sequence). $\lim_{n\to\infty} z_n=\zeta$ means that for any real number $\epsilon>0$ there corresponds a positive integer N_ϵ (which depends on ϵ) such that $z_n\in D_\epsilon(\zeta)$ whenever $n>N_\epsilon$. That is $|\zeta-z_n|<\epsilon$ whenever $n>N_\epsilon$. Figure 4.1 illustrates a convergent sequence.

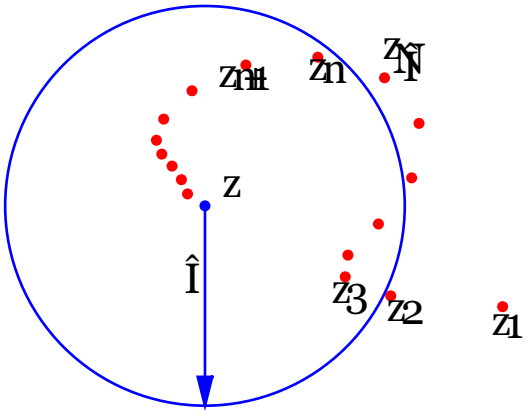

Figure 4.1 A sequence $\{z_n\}_1^{\infty}$ that converges to ζ. (If $n>N_\epsilon$ then $z_n\in D_\epsilon(\zeta)$.)

Remark 4.1. The reason we use the notation N_ϵ is to emphasize the fact that this number depends on our choice of ϵ. Sometimes it will be convenient to drop the subscript.

In form, Definition 4.1 is exactly the same as the corresponding definition for limits of real sequences. In fact, a simple criterion casts the convergence of complex sequences in terms of the convergence of real sequences.

Theorem 4.1. Let $z_n=x_n+i\,y_n$ and $\zeta=u+i\,v$. Then

(4-6) $$\lim_{n\to\infty} z_n=\zeta, \ \text{iff}$$

(4-7) $$\lim_{n\to\infty} x_n=u \quad \text{and} \quad \lim_{n\to\infty} y_n=v.$$

Proof.

Example 4.1. Find the limit of the sequence $\{z_n\} = \left\{\frac{\sqrt{n} + i\,(n+1)}{n}\right\}$.

Solution. We write $z_n = x_n + i\,y_n = \frac{1}{\sqrt{n}} + i\,\frac{n+1}{n}$. Using results concerning sequences of real numbers, we find that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

Therefore $\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{\sqrt{n} + i\,(n+1)}{n} = 0 + i = i$.

Explore Solution 4.1.

Example 4.2. Show that the sequence $\{z_n\} = \{(1+i)^n\}$ diverges.

Solution. We have

$$z_n = (1+i)^n = x_n + i\,y_n$$

$$z_n = \left(\sqrt{2}\right)^n \cos\frac{n\pi}{4} + i\,\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$$

The real sequences $x_n = \left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$ and $y_n = \left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$ both exhibit divergent oscillations, so we conclude that $z_n = (1+i)^n$ diverges.

Explore Solution 4.2.

Definition 4.2 (Bounded Sequence). A complex sequence $\{z_n\}$ is bounded provided that there exists a positive real number R and an integer N such that $|z_n| < R$ for all $n > N$. In other words, for $n > N$, the sequence $\{z_n\}$ is contained in the disk $D_R(0)$.

Bounded sequences play an important role in some newer developments in complex analysis that are discussed in Section 4.2. A theorem from real analysis stipulates that convergent sequences are bounded. The same result holds for complex sequences.

Theorem 4.2. If $\{z_n\}$ is a convergent sequence, then $\{z_n\}$ is bounded.

Proof.

As with real numbers, we also have the following definition.

Definition 4.3 (Cauchy Sequence). The sequence $\{z_n\}$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists a positive integer N_ϵ, such that if $n, m > N_\epsilon$, then $|z_n - z_m| < \epsilon$, or, equivalently, $z_n - z_m \in D_\epsilon(0)$.

The following should now come as no surprise.

Theorem 4.3, (Cauchy Sequences Converge). If $\{z_n\}$ is a Cauchy sequence, then $\{z_n\}$ converges.

Proof.

Proof of Theorem 4.3 is in the book.

One of the most important notions in analysis (real or complex) is a theory that allows us to add up infinitely many terms. To make sense of such an idea we begin with a sequence $\{z_n\}$, and

form a new sequence $\{S_n\}$, called the sequence of partial sums, as follows.

$$S_1 = z_1,$$

$$S_2 = z_1 + z_2,$$

$$S_3 = z_1 + z_2 + z_3,$$

$$\vdots$$

$$S_n = z_1 + z_2 + \ldots + z_n = \sum_{k=1}^{\infty} z_k,$$

$$\vdots$$

Definition 4.4 (Infinite Series). The formal expression $\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \ldots + z_n + \ldots$ is called an infinite series, and $z_1, z_2, \ldots, z_n, \ldots,$ are called the terms of the series.

If there is a complex number S for which

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} z_k,$$

we will say that the infinite series $\sum_{k=1}^{\infty} z_k$ converges to S, and that S is the sum of the infinite series. When this occurs, we write

$$S = \sum_{k=1}^{\infty} z_k.$$

The series $\sum_{k=1}^{\infty} z_k$ is said to be absolutely convergent provided that the (real) series of magnitudes $\sum_{k=1}^{\infty} |z_k|$ converges.

If a series does not converge, we say that it diverges.

Remark 4.2. The first finitely many terms of a series do not affect its convergence or divergence and, in this respect, the beginning index of a series is irrelevant. Thus, we will without comment conclude that if a series $\sum_{k=N+1}^{\infty} z_k$ converges, then so does $\sum_{k=1}^{\infty} z_k$, where $z_1, z_2, \ldots, z_N$ is any finite collection of terms. A similar remark holds for determining divergence of a series.

As you might expect, many of the results concerning real series carry over to complex series. We now give several of the more standard theorems for complex series, along with examples of how they are used.

Theorem 4.4. Let $z_n = x_n + i\, y_n$ and $S = U + i\, V$. Then

$$S = \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (x_n + i\, y_n) \quad \text{(converges)}$$

if and only if both

$$U = \sum_{n=1}^{\infty} x_n \quad \text{and} \quad V = \sum_{n=1}^{\infty} y_n \quad \text{(converge)}.$$

Proof.

Proof of Theorem 4.4 is in the book.

Theorem 4.5. If $\sum_{n=1}^{\infty} z_n$ is a convergent complex series, then $\lim_{n\to\infty} z_n = 0$.

Proof.

Example 4.3. Show that the series $\sum_{n=1}^{\infty} \frac{1+i\,n\,(-1)^n}{n^2} = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} + i\,\frac{(-1)^n}{n}\right]$ is convergent.

Solution. Recall that the real series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ are convergent. Hence, Theorem 4.4 implies that the given complex series is convergent.

Explore Solution 4.3.

Example 4.4. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n+i}{n} = \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n} + i\,\frac{1}{n}\right]$ is divergent.

Solution. We know that the real series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Hence, Theorem 4.4 implies that the given complex series is divergent.

Explore Solution 4.4.

Example 4.5. Show that the series $\sum_{n=1}^{\infty} (1+i)^n$ is divergent.

Solution. Here we set $z_n = (1+i)^n$ and observe that

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} |(1+i)^n| = \lim_{n\to\infty} \left(\sqrt{2}\right)^n = \infty.$$

Thus $\lim_{n\to\infty} z_n \neq 0$, and Theorem 4.5 implies that the series is not convergent; hence it is divergent.

Explore Solution 4.5.

Theorem 4.6. Let $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ be convergent series, and let c be a complex number. Then

$$\sum_{n=1}^{\infty} c\,z_n = c \sum_{n=1}^{\infty} z_n$$

and

$$\sum_{n=1}^{\infty} (z_n + w_n) = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n.$$

Proof.

Definition 4.5 (Cauchy Product of Series). Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series, where a_n and b_n are complex numbers. The Cauchy product of the two series is defined to be the series $\sum_{n=1}^{\infty} c_n$, where $c_n = \sum_{k=1}^{n} a_k\, b_{n-k}$.

Theorem 4.7. If the Cauchy product converges, then

$$\sum_{n=1}^{\infty} c_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right),$$

where

$$c_n = \sum_{k=1}^{n} a_k\, b_{n-k}.$$

Proof.

Theorem 4.8 (Comparison Test). Let $\sum_{n=1}^{\infty} M_n$ be a convergent series of real nonnegative terms. If $\{z_n\}$ is a sequence of complex numbers and $|z_n| \leq M_n$ holds for all n, then $\sum_{n=1}^{\infty} z_n$ converges.

Proof.

Corollary 4.1. If $\sum_{n=1}^{\infty} |z_n|$, then $\sum_{n=1}^{\infty} z_n$ converges. In other words, absolute convergence implies convergence for complex series as well as for real series.

Proof.

Example 4.6. Show that the series $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$ is convergent.

Solution. We calculate $|z_n| = \left|\frac{(3+4i)^n}{5^n n^2}\right| = \frac{1}{n^2} = M_n$. Using the comparison test and the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, we determine that $\sum_{n=1}^{\infty} \left|\frac{(3+4i)^n}{5^n n^2}\right|$ converges and hence, by Corollary 4.1, so does $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$.

Explore Solution 4.6.

Complex Analysis for Mathematics and Engineering
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4.2 Julia and Mandelbrot Sets

An impetus for studying complex analysis is the comparison of properties of real numbers and functions with their complex counterparts. In this section we take a look at Newton's method for finding solutions to the equation $f(x) = 0$. Then, by examining the more general topic of iteration, we will plunge into a breathtaking world of color and imagination. The mathematics surrounding this topic has generated a great deal of popular attention in the past few years.

Recall from calculus that Newton's method method proceeds by starting with a function $f(x)$ and an initial "guess" x_0 as a solution to $f(x) = 0$. We then generate a new guess x_1 by the computation $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. Using x_1 in place of x_0, this process is repeated, giving us $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. Thus we obtain a sequence of points $\{x_k\}$, where $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$. The points $\{x_k\}_{k=0}^{\infty}$ are called the iterates of x_0. For functions defined on the real numbers, this method gives remarkably good results so that the sequence $\{x_k\}$ often converges to a solution of $f(x) = 0$ rather quickly. In the late 1800's the british mathematician Arthur Cayley investigated the question as to whether Newton's method can be applied to complex functions. He wrote a paper giving an analysis for how this method works for quadratic polynomials and indicated his intention to publish a subsequent paper for cubic polynomials. Unfortunately, Cayley died before producing this paper. As you will see, the extension of Newton's method to the complex domain and the more general question of iteration are quite complicated.

Example 4.7. Trace out the next five iterates of Newton's method given an initial guess of $z_0 = \frac{1}{4} + \frac{i}{4}$ as a solution to the equation $f(z) = 0$, where $f(z) = z^2 + 1$.

Solution. For any guess z for a solution, Newton's method gives as the next guess the number $z - \frac{f(z)}{f'(z)} = z - \frac{z^2+1}{2z} = \frac{z^2-1}{2z}$. Table 4.1 gives the required iterates.

Figure 4.2 shows the relative positions of these points on the z plane. Note that the points z_4 and z_5 are so close together that they appear to coincide, and that the value for z_5 agrees to six decimal places with the actual solution $z = i$.

k	z_k	$f(z_k)$
0	0.2500000000 + 0.2500000000 i	1.0000 000 000 + 0.1250000000 i

Table 4.1 The iterates of $z_0 = \frac{1}{4} + \frac{i}{4}$ for Newton's method applied to $f(z) = z^2 + 1$.

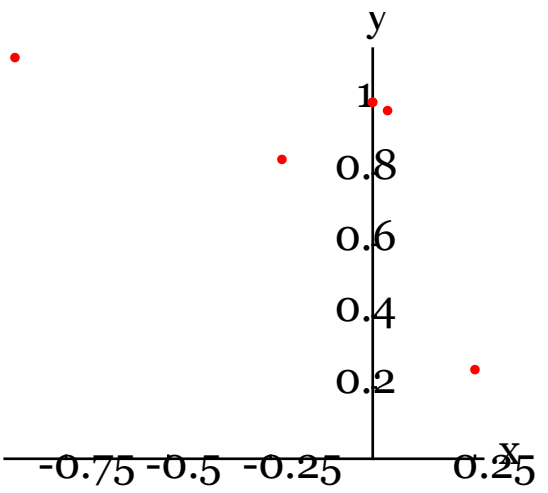


Figure 4.2 The iterates of $z_0 = \frac{1}{4} + \frac{i}{4}$ for Newton's method applied to $f(z) = z^2 + 1$.

Explore Solution 4.7.

The complex version of Newton's method also appears to work quite well. Recall, however, that with functions defined on the reals, not every initial guess produces a sequence that converges to a solution. Example 4.8 shows that the same is true in the complex case.

Example 4.8. Show that Newton's method fails for the function $f(z) = z^2 + 1$ if the initial guess is a real number.

Solution. From Example 4.7 we know that, for any guess z as a solution of $z^2 + 1 = 0$, the next guess at a solution is $N(z) = z - \frac{f(z)}{f'(z)} = \frac{z^2-1}{2z}$. We let z_0 be any real number and $\{z_k\}$ be the sequence of iterations produced by the initial seed z_0. If for any k, $z_k = 0$, the procedure terminates, as z_{k+1} will be undefined. If all the terms of the sequence $\{z_k\}$ are defined, an easy induction argument shows that all the terms of the sequence are real. Because the solutions of $z^2 + 1 = 0$ are $z = \pm i$, the sequence $\{z_k\}$ cannot possibly converge to either solution. In the exercises we ask you to explore in detail what happens when z_0 is in the upper or lower half-plane.

Extra Example 1. Investigate Newton's method for finding the roots of $f(z) = z^3 + 1$. Given the initial seed z_0 determine if the sequence $\{z_k\}$ converges to one of the roots $z = -1, \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}$.

Explore Extra Solution 1.

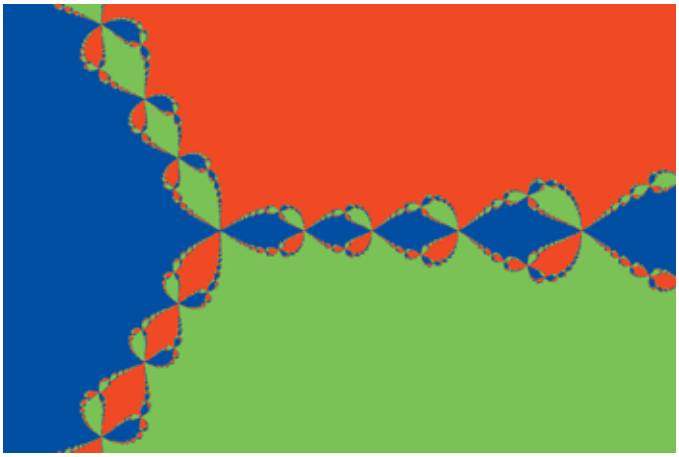
The case for cubic polynomials is more complicated than that for quadratics. Fortunately, we can get an idea of what's going on by doing some experimentation with computer graphics. We begin with the cubic polynomial $f(z) = z^3 + 1$. (Recall that the roots of this polynomial are at $z = -1, \frac{1}{2} + \frac{\sqrt{3}}{2} i, \frac{1}{2} - \frac{\sqrt{3}}{2} i$.) We associate a color with each root (blue, red, and green, respectively). We form a rectangular region R, which contains the three roots of $f(z)$, and partition this region into equal rectangles R_{ij}. We then choose a point z_{ij} at the center of each rectangle and

for each of these points we apply the following algorithm.

1. With $N(z) = z - \frac{f(z)}{f'(z)} = z - \frac{z^3+1}{3z^2} = \frac{2z^3-1}{3z^2}$, compute $N(z_{ij})$. Continue computing successive iterates of this initial point until we either are within a certain preassigned tolerance (say,) of one of the roots of $f(z) = 0$, or until the number of iterations has exceeded a preassigned maximum.

2. If Step 1 leaves us within of one of the roots of $f(z)$, we color the entire rectangle R_{ij} with the color associated with that root.
Otherwise, we assume that the initial point z_{ij} does not converge to any root, and we color the entire rectangle yellow.

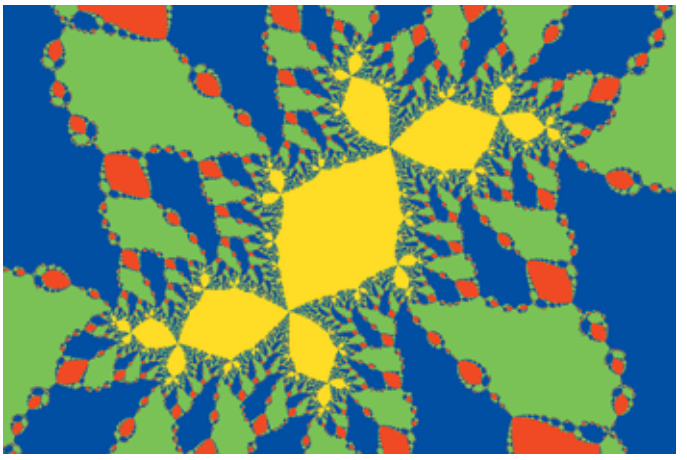
Note that this algorithm doesn't prove anything. In Step 2, there is no a priori reason to justify the assumption mentioned, nor is there any necessity for an initial point z_{ij} to have its sequence of iterates converging to one of the roots of $f(z) = 0$, just because a particular iteration is within of that root. Finally, the fact that one point in a rectangle behaves in a certain way does not imply that all the points in that rectangle behave in a like manner. Nevertheless, we can use this algorithm as a basis for mathematical explorations. Indeed, computer experiments such as the one described have contributed to a lot of exciting mathematics during the past 20 years. The color plates located on the inside front and back covers of this book illustrate the results of applying our algorithm to various functions. Color plate 1shows the results for the cubic polynomial $f(z) = z^3 + 1$. The points in the blue, red, and green regions are those "initial guesses" that will converge to the roots $z = -1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i$ -1, respectively. (The roots themselves are located in the middle of the three largest colored regions.) The complexity of this picture becomes apparent when you observe that, wherever two colors appear to meet, the third color emerges between them. But then, a closer inspection of the area where this third color meets one of the other colors reveals again a different color between them. This process continues with an infinite complexity.



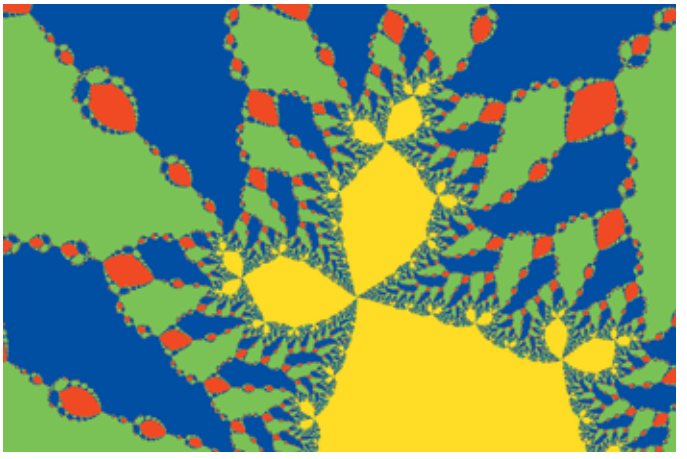
Color Plate 1. Newton's method applied to $f(z) = z^3 + 1$.

There appear to be no yellow regions with any area in Color plate 1, indicating that at least most initial guesses z_0 at a solution to $z^3 + 1 = 0$ will produce a sequence $\{z_k\}$ that converges to one of the three roots. Color plate 2 demonstrates that this outcome does not always occur. It shows the results of applying the preceding algorithm to the polynomial $f(z) = z^3 + (-0.26 + 0.02i)z + (-0.74 + 0.02i)$. The yellow area shown is often referred to as the rabbit. It consists of a main body and two ears. Upon closer inspection (color plate 3) you

can see that each of the ears consists of a main body and two ears. Color plate 2 is an example of a fractal image. Mathematicians use the term fractal to indicate an object that have this kind of recursive structure.



Color Plate 2. The rabbit fractal of
$f(z) = z^3 + (-0.26 + 0.02\,i)\,z - 0.74 + 0.02\,i$.



Color Plate 3. A zoom of the rabbit fractal of
$f(z) = z^3 + (-0.26 + 0.02\,i)\,z - 0.74 + 0.02\,i$.

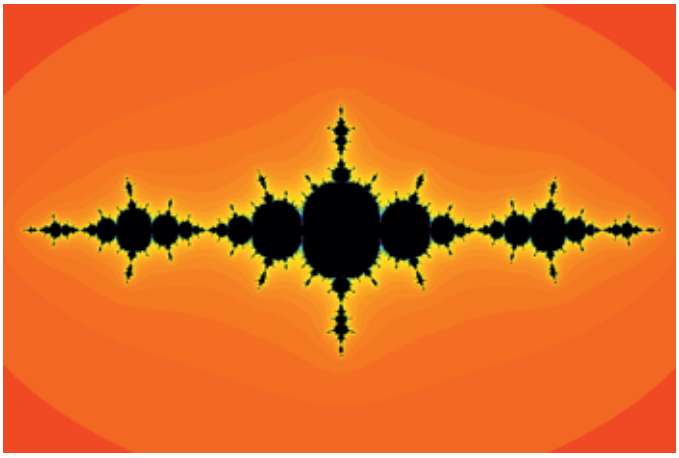
In 1918 the French mathematicians Gaston Julia and Pierre Fatou noticed the fractal phenomenon when exploring iterations of functions not necessarily connected with Newton's method. Beginning with a function f(z) and a point z_0, they computed the iterates $z_1 = f(z_0)$, $z_2 = f(z_1)$, ... , $z_{k+1} = f(z_k)$, and investigated properties of the sequences $\{z_k\}$. Their findings did not receive a great deal of attention, in part because computer graphics were not available at this time. With the recent proliferation of computers, it is not surprising that these investigations were revived in the 1980's. Detailed studies of Newton's method and the more general topic of iteration were undertaken by a host of mathematicians including Curry, Douady, Garnett, Hubbard, Mandelbrot, Milnor and Sullivan. We now turn our attention to some of their results by focusing on the iterates produced by quadratics of the form $f_c(z) = z^2 + c$. You will be surprised at the startling pictures that graphical iterates of such a simple functions produce.

Example 4.9. For $f_c(z) = z^2 + c$, analyze all possible iterations when $c = 0$, that is, for the function f_0 defined by $f_0(z) = z^2 + 0$.

Solution. We leave as an exercise the claim that, if $|z_0| < 1$, the sequence will converge to 0;

if $|z_0| > 1$, the sequence will be unbounded; and if $|z_0| = 1$, the sequence will either oscillate around the unit circle or converge to 1.

For the function $f_c(z)$ defined by $f_c(z) = z^2 + c$, and an initial seed z_0, the set of iterates given by $z_1 = f_c(z_0)$, $z_2 = f_c(z_1)$, etc., are also called the orbits of z_0 generated by $f_c(z)$. We let K_c denote the set of points with bounded orbits for $f_c(z)$. Example 4.9 shows that K_0 is the closed unit disk $\overline{D_1}(0)$. The boundary of K_c is known as the Julia set for the function $f_c(z)$. Thus the Julia set for $f_0(z)$ is the unit circle $C_1(0)$. It turns out that K_c is a nice simple set only when $c = 0$ or $c = -2$. Otherwise, K_c is fractal. Color plate 4 shows $K_{-1.25}$. The variation in colors indicate the length of time it takes for points to become "sufficiently unbounded" according to the following algorithm, which uses the same notation as our algorithm for iterations via Newton's method.



Color Plate 4. The Julia set for $f_c(z) = f_{-1.25}(z) = z^2 - 1.25$.

Extra Example 2. Plot the Julia set for $f_c(z) = f_{-1.25}(z) = z^2 - 1.25$.

Explore Extra Solution 2.

1. Compute $f_c(z_{ij})$. Continue computing successive iterates of this initial point until the absolute value of one of the iterations exceeds a certain bound (say, L), or until the number of iterations has exceeded a preassigned maximum.
2. If Step 1 leaves us with an iteration whose absolute value exceeds L, we color the entire rectangle R_{ij} with a color indicating the number of iterations needed before this value was attained (the more iterations required, the darker the color). Otherwise, we assume that the orbits of the initial point z_{ij} do not diverge to infinity, and we color the entire rectangle black.

Note, again, that this algorithm doesn't prove anything. It merely guides the direction of our efforts to do rigorous mathematics.

Color plate 5 shows the Julia set for the function $f_c(z)$, where $c = -0.11 - 0.67\,i$. The boundary of this set is different from the boundaries of the other sets we have seen, in that it is disconnected. Julia and Fatou independently discovered a simple criterion that can be used to tell when the Julia set for $f_c(z)$ is connected or disconnected. We state their result, but omit the proof, as it is beyond the scope of this text.

Color Plate 5. A disconnected Julia set for $f_c(z) = z^2 - 0.11 - 0.67\,i$.

Theorem 4.9. The boundary of K_c is connected if and only if $0 \in K_c$. In other words, the Julia set for $f_c(z)$ is connected if and only if the orbits of 0 are bounded.

Proof.

Example 4.10. Show that the Julia set for $f_i(z) = z^2 + i$ is connected.

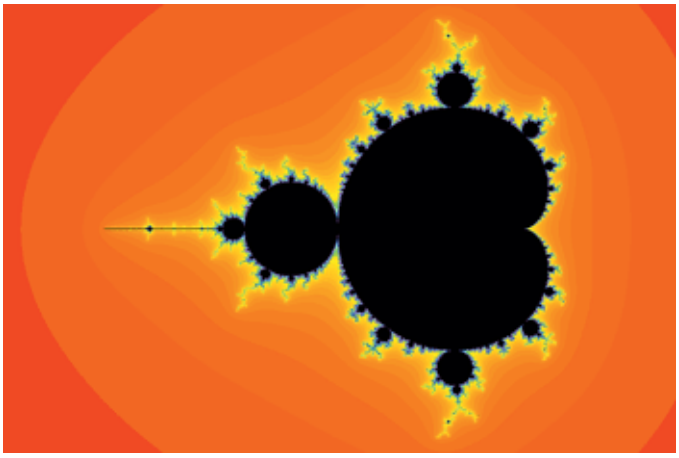
Solution. We apply Theorem 4.9 and compute the orbits of 0 for $f_i(z) = z^2 + i$. We have $f_i(0) = i$, $f_i(0) = -1 + i$, $f_i(-1 + i) = -i$, and $f_i(-i) = -1 + i$. Thus the orbits of 0 are the sequence $\{0, i, -1 + i, -i, -1 + i, -i, -1 + i, -i, \ldots\}$, which is clearly a bounded sequence. Thus, by Theorem 4.9, the Julia set for $f_i(z)$ is connected.

Explore Solution 4.10.

In 1980, the Polish born mathematician Benoit Mandelbrot used computer graphics to study the following set:

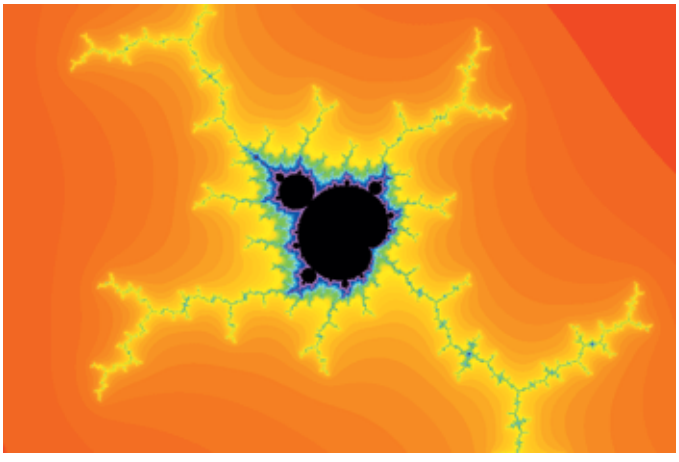
$$M = \{c : \text{the Julia set for } f_c(z) \text{ is connected}\}$$
$$M = \{c : \text{the orbits of } 0 \text{ determined by } f_c(z) \text{ are bounded}\}$$

The set M has come to be known as the Mandelbrot Set. Color plate 6 shows its intricate nature. Technically, the Mandelbrot set is not fractal because it is not self-similar (although it may look that way). However, it is infinitely complex. Color plate 7 shows a zoom over the upper portion of the set shown in color plate 6. Likewise, color plate 8 zooms in on the upper portion of color plate 7. In color plate 8 you can see the emergence of another structure very similar to the Mandelbrot set that we began with. Although it isn't an exact replica, if you zoomed in on this set at almost any spot, you would eventually see yet another "Mandelbrot clone" and so on ad infinitum! In the remainder of this section we look at some of the properties of this amazing set.



Color Plate 6. The Mandelbrot set M.

Color Plate 7. A zoom of the upper portion of the Mandelbrot set M.



Color Plate 8. A zoom of the upper portion of the Color Plate 7.

Example 4.11. Show that $\left\{c : \left| z \right| \leq \frac{1}{4}\right\}$ is a subset of the Mandelbrot set M.

Solution. Let $\{a_n\}_{n=0}^{\infty}$ be the orbits of 0 generated by $f_c(z) = z^2 + c$, where $\left| z \right| \leq \frac{1}{4}$. Then

$$a_0 = f(0) = 0 + c = c$$
$$a_1 = f(a_0) = a_0^2 + c$$
$$a_2 = f(a_1) = a_1^2 + c$$
in general
$$a_{n+1} = f(a_n) = a_n^2 + c$$

We show that $\{a_n\}_{n=0}^{\infty}$ is bounded, and, in particular, we show that $|a_n| \le \frac{1}{2}$ for all n by mathematical induction. Clearly $|a_n| \le \frac{1}{2}$ if $n = 0$ or 1. We assume that $|a_n| \le \frac{1}{2}$ for some value of $n \ge 1$ (our goal is to show $|a_{n+1}| \le \frac{1}{2}$). Now,

$$\begin{aligned} |a_{n+1}| &= |a_n^2 + c| \\ &\le |a_n^2| + |c| \\ &\le \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \end{aligned}$$

In the exercises, we ask you to show that, if $|c| > 2$, then c is not in the set M. Thus the Mandelbrot set depicted in color plate 6 contains the disk $\overline{D_{\frac{1}{4}}}(0)$ and is contained in the disk $\overline{D_2}(0)$.

We can use other methods to determine which points belong to M. To do so, we need some additional vocabulary.

Definition 4.6 (Fixed Point). The point z_0 is a fixed point for the function f if $f(z_0) = z_0$.

Definition 4.7 (Attracting Point). The point z_0 is an attracting point for the function f if $|f'(z_0)| < 1$.

Theorems 4.10 and 4.11 explain the significance of these terms.

Theorem 4.10. Suppose that z_0 is an attracting fixed point for the function f. Then there is a disk $D_r(z_0)$ about z_0 such that the iterates of all the points in $D_r(z_0)$ are dawn toward the point z_0 in the sense that, if $z \in D_r^*(z_0)$, then $|f(z) - z_0| < |z - z_0|$. In fact, if z_k is the k^{th} iterate of $z \in D_r^*(z_0)$, then $\lim_{k\to\infty} z_k = z_0$.

Proof.

In 1905, Pierre Fatou showed that, if the function $f_c(z)$ defined by $f_c(z) = z^2 + c$ has attracting fixed points, then the orbits of 0 determined by $f_c(z)$ must converge to one of them. Because a convergent sequence is bounded, this condition implies that c must belong to M. In the exercises we ask you to show that the main cardioid-shaped body of M in color plate 6 is composed of those points c for which $f_c(z)$ has attracting fixed points. You will find Theorem 4.10 to be a useful characterization of those points.

Theorem 4.11. The function $f_c(z)$ defined by $f_c(z) = z^2 + c$ has attracting fixed points iff $|1 + (1 - 4c)^{1/2}| < 1$ or $|1 - (1 - 4c)^{1/2}| < 1$, where the square root designates the principal square root function.

Proof.

Definition 4.8 (n-Cycle). An n-cycle for a function f is a set $\{z_0, z_1, \ldots, z_{n-1}\}$ of n complex numbers such that $z_k = f(z_{k-1})$ for $k = 1, 2, \ldots, n-1$, and $f(z_{n-1}) = z_0$.

Definition 4.9 (Attracting n-Cycle). An n-cycle $\{z_0, z_1, \ldots, z_{n-1}\}$ for a function f is said to be attracting if $|g_n'(z_0)| < 1$, where g_n is the composition of f with itself n times. For example, if n = 2, then $g_2(z) = f(f(z))$.

Example 4.12. Example 4.10 shows that $\{-1+i, -i\}$ is a 2-cycle for the function $f_i(z) = z^2 + i$. It is not an attracting 2-cycle because $g_2(z) = z^4 + 2iz^2 + (-1+i)$ and $g_2'(z) = 4z^3 + 4iz$. Hence $|g_2'(-1+i)| = |4(-1+i)^3 + 4i(-1+i)| = |4+4i|$, so that $|g_2'(-1+i)| > 1$.

Explore Solution 4.12.

In the exercises, we ask you to show that, if $\{z_0, z_1, \ldots, z_{n-1}\}$ is an attracting n-cycle for a function f, then not only does z_0 satisfy $|g_n'(z_0)| < 1$, but also that $|g_n'(z_k)| < 1$, for $k = 1, 2, \ldots, n-1$.

It turns out that the large disk to the left of the cardioid in color plate 6 consists of those points c for which $f_c(z)$ has a 2-cycle. The large disks above and below the main cardioid disk are the points c for which $f_c(z)$ has a 3-cycle.

Continuing with this scheme, we see that the idea of n-cycles explains the appearance of the "buds" that you see on color plate 6. It does not, however, begin to do justice to the enormous complexity of the entire set. Even color plates 7 and 8 are mere glimpses into its awesome beauty. In the exercises, we suggest several references for projects that you could pursue for a more detailed study of topics relating to those covered in this section.

Extra Example 3. Plot the Mandelbrot set.

Explore Extra Solution 3.

Complex Analysis for Mathematics and Engineering

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4.3 Geometric Series and Convergence Theorems

We begin this section by presenting a series of the form $\sum_{n=0}^{\infty} z^n$, which is called a geometric series and is one of the most important series in mathematics.

Real Exploration.

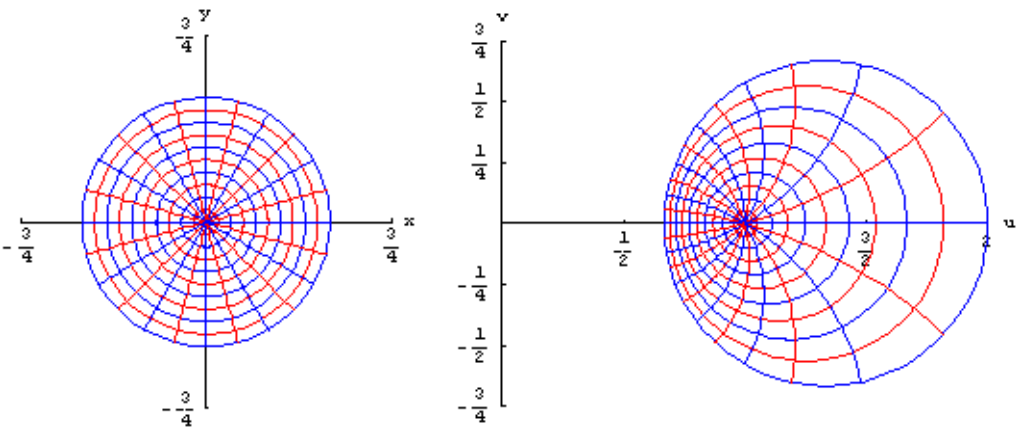

Figure 4.A The image of the disk $|z| \le \frac{1}{2}$ under the mapping $w = f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$.

Theorem 4.12 (Geometric Series). If $|z| < 1$, the series $\sum_{n=0}^{\infty} z^n$ converges to $f(z) = \frac{1}{1-z}$. That is, if $|z| < 1$ then

(4-11) $$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots + z^k + \ldots = \frac{1}{1-z}.$$

If $|z| \ge 1$, the series diverges.

Proof.

Exploration.

Corollary 4.2. If $|z| > 1$, the series $\sum_{n=1}^{\infty} z^{-n}$ converges to $f(z) = \frac{1}{z-1}$. That is, if $|z| > 1$ then

$$\sum_{n=1}^{\infty} z^{-n} = z^{-1} + z^{-2} + \ldots + z^{-4} + \ldots = \frac{1}{z-1},$$

or equivalently,

$$-\sum_{n=1}^{\infty} z^{-n} = -z^{-1} - z^{-2} - \ldots - z^{-4} - \ldots = \frac{1}{1-z}.$$

If $|z| \leq 1$, the series diverges.

Proof.

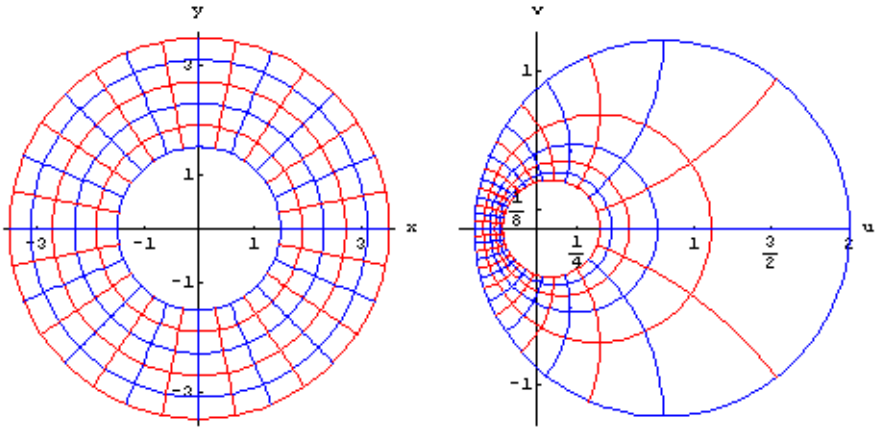

Figure 4.B The annulus of the disk $\frac{3}{2} \leq |z| \leq \frac{7}{2}$ under the mapping $w = f(z) = \frac{1}{z-1} = \sum_{n=1}^{\infty} z^{-n}$.

Exploration.

Corollary 4.3. If $z \neq 1$, then for all n,

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots + z^{n-1} + \frac{z^n}{1-z}.$$

Proof.

Exploration.

Example 4.13. Show that $\sum_{n=0}^{\infty} \frac{(1-i)^n}{2^n} = 1 - i$.

Solution. If we set $z = \frac{1-i}{2}$, then $|z| = \left|\frac{1-i}{2}\right| = \frac{\sqrt{2}}{2} < 1$. By Theorem 4.12, the sum is

$$1 \Big/ \left(1 - \frac{1-i}{2}\right) = \frac{2}{2-1+i} = \frac{2}{1+i} = 1 - i.$$

Explore Solution 4.13.

Example 4.14. Evaluate $\sum_{n=3}^{\infty} \frac{i^n}{2^n}$.

Solution. We can put this expression in the form of a geometric series:

$$\sum_{n=3}^{\infty} \frac{i^n}{2^n} = \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^n = \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^3 \left(\frac{i}{2}\right)^{n-3}$$

$$= \left(\frac{i}{2}\right)^3 \sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^{n-3} = \left(\frac{i}{2}\right)^3 \frac{1}{1-\frac{i}{2}}$$

$$= -\frac{i}{8} \frac{1}{1-\frac{i}{2}} = \frac{-i}{8-4i}$$

$$= \frac{-i}{8-4i} \frac{8+4i}{8+4i} = \frac{4-8i}{80}$$

$$= \frac{1}{20} - \frac{i}{10}$$

Explore Solution 4.14.

Remark 4.3. The equality given in Example 4.14 illustrates an important point when evaluating a geometric series whose beginning index is other than zero. The value of $\sum_{n=r}^{\infty} z^n$ equals $\frac{z^r}{1-z}$. If we think of z as the "ratio" by which a given term of the series is multiplied to generate successive terms, we see that the sum of a geometric series equals $\frac{\text{first term}}{1-\text{ratio}}$, provided $|\,\text{ratio}\,| < 1$.

The geometric series is used in the proof of Theorem 4.12, which is known as the ratio test. It is one of the most commonly used tests for determining the convergence or divergence of series. The proof is similar to the one used for real series, and we leave it for you to do.

Theorem 4.13 (d'Alembert's Ratio Test). If $\sum_{n=0}^{\infty} \zeta_n$ is a complex series with the property that

$$\lim_{n\to\infty} \frac{|\zeta_{n+1}|}{|\zeta_n|} = L,$$

then the series converges absolutely if $L < 1$ and diverges if $L > 1$.

Proof.

Example 4.15. Show that $\sum_{n=0}^{\infty} \frac{(1-i)^n}{n!}$ converges.

Solution. Using the ratio test, we find that

$$\lim_{n\to\infty} \left| \frac{(1-i)^{n+1}}{(n+1)!} \Big/ \frac{(1-i)^n}{n!} \right| = \lim_{n\to\infty} \left| \frac{n!\,(1-i)}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n!\,(1-i)}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \frac{n!\,|1-i|}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{n!\,|1-i|}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{|1-i|}{n+1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{2}}{n+1}$$

$$= 0 = L$$

Because $L < 1$, the series converges.

Explore Solution 4.15.

Example 4.16. Show that $\sum_{n=0}^{\infty} \frac{(z-i)^n}{2^n}$ converges for all z in the disk $|z-i| < 2$.

Solution. Using the ratio test, we find that

$$\lim_{n\to\infty} \left| \frac{(z-i)^{n+1}}{2^{n+1}} \Big/ \frac{(z-i)^n}{2^n} \right| == \lim_{n\to\infty} \left| \frac{z-i}{2} \right| == \frac{|z-i|}{2} == L.$$

If $|z-i| < 2$, then $\frac{|z-i|}{2} = L < 1$, and the series converges.

If $|z-i| > 2$, then $L > 1$, and the series diverges.

Explore Solution 4.16.

Our next result, known as the root test, is slightly more powerful than the ratio test. Before we present this test, we need to discuss a rather sophisticated idea used with it-the limit supremum.

Definition 4.10 (Limit Supremum). Let $\{t_n\}$ be a sequence of positive real numbers. The limit supremum of the sequence (denoted by $\lim_{n\to\infty} \sup t_n$) is the smallest real number L with the property that for any $\in > 0$ there are at most finitely many terms in the sequence that are larger than $L + \in$. If there is no such number L, then we set $\lim_{n\to\infty} \sup t_n = \infty$.

Example 4.17. The limit supremum of the sequence $\{t_n\} = \{4.1, 5.1, 4.01, 5.01, 4.001, 5.001, \ldots\}$ is $\lim_{n\to\infty} \sup t_n = 5$, because if we set $L = 5$, then for any $\in > 0$, there are only finitely many terms in the sequence larger than $L + \in = 5 + \in$. Additionally, if L is smaller than 5, then by setting $\in = 5 - L$, we can find infinitely many terms in the sequence larger than $L + \in$ (because $L + \in = 5$).

Explore Solution 4.17.

Example 4.18. The limit supremum of the sequence $\{t_n\} = \{1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots\}$ is $\lim_{n\to\infty} \sup t_n = 3$, because if we set $L = 3$, then for any $\in > 0$, there are only finitely many terms (actually, there are none) in the sequence larger than $L + \in = 3 + \in$. Additionally, if L is smaller than 3, then by setting $\in = \frac{3-L}{2}$ we can find infinitely many terms in the sequence larger than $L + \in$, because $L + \in < 3$, as the following calculation shows:

$$L + \in = L + \frac{3-L}{2} = \frac{3+L}{2} = \frac{3}{2} + \frac{L}{2} < \frac{3}{2} + \frac{3}{2} = 3.$$

Explore Solution 4.18.

Example 4.19. The limit supremum of the Fibonacci sequence $\{t_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ is $\lim_{n\to\infty} \sup t_n = \infty$.

(The Fibonacci sequence satisfies the relation $t_n = t_{n-1} + t_{n-2}$ for $n > 2$).

Explore Solution 4.19.

The limit supremum is a powerful idea because the limit supremum of a sequence always exists, which is not true for the ordinary limit. However, Example 4.20 illustrates the fact that, if the limit of a sequence does exist, then it will be the same as the limit supremum.

Example 4.20. The sequence $\{t_n\} = \left\{1 + \frac{1}{n}\right\} = \left\{2, 1.5, 1.3\overline{3}, 1.25, 1.2, \ldots\right\}$ has $\limsup_{n\to\infty} t_n = 1$.

We leave verification of this as an exercise.

Explore Solution 4.20.

Theorem 4.14 (Root Test). Suppose that the series $\sum_{n=0}^{\infty} \zeta_n$, has $\limsup_{n\to\infty} |\zeta_n|^{\frac{1}{n}} = L$ (i.e. $\limsup_{n\to\infty} \sqrt[n]{|\zeta_n|} = L$).

Then the series is absolutely convergent if $L < 1$ and divergent if $L > 1$.

Proof.

Note that in applying either Theorems 4.13 or 4.14, if $L = 1$ the convergence or divergence of the series is unknown, and further analysis is required to determine the true state of affairs.

Extra Example 1. Find the radius of convergence of the infinite series

$$S(z) = 1 + \frac{z}{2} + \frac{z^2}{3^2} + \frac{z^3}{2^3} + \frac{z^4}{3^4} + \frac{z^5}{2^5} + \frac{z^6}{3^6} + \frac{z^7}{2^7} + \frac{z^8}{3^8} + \frac{z^9}{2^9} + \frac{z^{10}}{3^{10}} + \frac{z^{11}}{2^{11}} + \ldots.$$

Explore Extra Solution 1.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

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4.4 Power Series Functions

Suppose that we have a series $\sum_{n=0}^{\infty} \xi_n$ where $\xi_n = c_n (z-\alpha)^n$. If α and the collection of c_n are fixed complex numbers, we will get different series by selecting different values for z. For example, if $\alpha = 2$ and $c_n = \frac{1}{n!}$ for all n, we get the series $\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2} - 2\right)^n$ if $z = \frac{i}{2}$, and $\sum_{n=0}^{\infty} \frac{1}{n!} (2+i)^n$ if $z = 4 + i$. Note that when $\alpha = 0$ and $c_n = 1$ for all n, we get a geometric series. The collection of points for which the series $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ converges is the domain of a function $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$, which we call a power series function. Technically, this series is undefined if $z = \alpha$ and $n = 0$, since 0^0 is undefined. We get around this difficulty by stipulating that the series $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ is really compact notation for $c_0 + \sum_{n=1}^{\infty} c_n (z-\alpha)^n$. In this section we present some results that are useful in helping establish properties of functions defined by power series.

Definition (Power Series). The function $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ is called a power series, with center $z = \alpha$.

Theorem 4.15. Suppose $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$. Then the set of points z for which the series converges is one of the following:

(i) The single point $z = \alpha$.

(ii) The disk $D_\rho(\alpha) = \{z : |z-\alpha| < \rho\}$, along with part (either none, or some or all) of the circle $C_\rho(\alpha) = \{z : |z-\alpha| = \rho\}$.

(iii) The entire complex plane.

Proof.

Another way to phrase case (ii) of Theorem 4.15 is to say that the power series $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ converges if $|z-\alpha| < \rho$ and diverges if $|z-\alpha| > \rho$. We call the number ρ the radius of convergence of the power series (see Figure 4.3). For case (i) of Theorem 4.15, we say that the radius of convergence is zero and that the radius of convergence is infinity for case (iii).

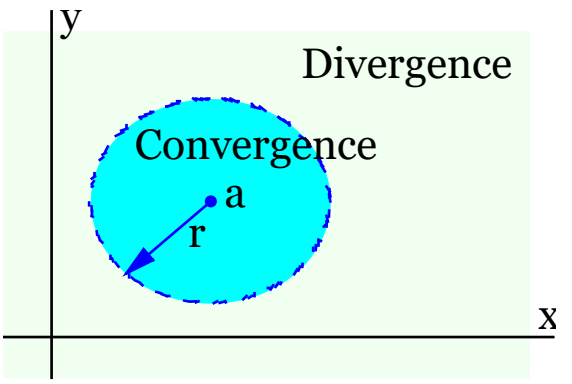

Figure 4.3 The radius of convergence of a power series.
What happens on the boundary circle may be unknown.

Theorem 4.16 (Radius of Convergence). For the power series function $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$, we can find ρ, its radius of convergence, by any of the following methods:

(i) Cauchy's Root Test: $\rho = \frac{1}{\lim_{n\to\infty} |c_n|^{\frac{1}{n}}}$ (provided that the limit exists.)

(ii) Cauchy-Hadamard Formula: $\rho = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$ (the limit superior always exists.)

(iii) d'Alembert's Ratio Test: $\rho = \frac{1}{\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|}$ (provided that the limit exists.)

We set $\rho = \infty$ if the limit equals 0, and $\rho = 0$ if the limit equals ∞.

Proof.

We now give an example illustrating each of these cases.

Example 4.21. The infinite series $f(z) = \sum_{n=0}^{\infty} \left(\frac{n+2}{3n+1}\right)^n (z-4)^n$ has radius of convergence $\rho = 3$ Cauchy's root test because

$$\lim_{n\to\infty} |c_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{n+2}{3n+1}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n+2}{3n+1} = \frac{1}{3},$$

hence $\rho = \frac{1}{\lim_{n\to\infty} |c_n|^{\frac{1}{n}}} = \frac{1}{\frac{1}{3}} = 3.$

Explore Solution 4.21.

Example 4.22. The infinite series
$f(z) = \sum_{n=1}^{\infty} c_n z^n = 4z + 5^2 z^2 + 4^3 z^3 + 5^4 z^4 + 4^5 z^5 + 5^6 z^6 + \ldots$ has radius of convergence $\rho = \frac{1}{5}$ by the Cauchy-Hadamard formula. We see this by calculating

$$\left\{ |c_n|^{\frac{1}{n}} \right\} = \{4, 5, 4, 5, 4, 5, \ldots\}, \text{ so}$$

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \limsup_{n\to\infty} \{4, 5, 4, 5, 4, 5, \ldots\} = 5,$$

hence $\rho = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}} = \frac{1}{5}$.

Explore Solution 4.22.

Example 4.23. The infinite series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has radius of convergence $\rho = 8$ by the ratio test because

$$\lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since the limit equals 0, we set $\rho = \infty$.

Explore Solution 4.23.

Extra Example 1. Find the radius of convergence of the infinite series
$S(z) = 1 + \frac{z}{2} + \frac{z^2}{3^2} + \frac{z^3}{2^3} + \frac{z^4}{3^4} + \frac{z^5}{2^5} + \frac{z^6}{3^6} + \frac{z^7}{2^7} + \frac{z^8}{3^8} + \frac{z^9}{2^9} + \frac{z^{10}}{3^{10}} + \frac{z^{11}}{2^{11}} + \ldots$.

Explore Extra Solution 1.

We come now to the main result of this section.

Theorem 4.17. Suppose the function $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has radius of convergence $\rho > 0$. Then

(i) $f(z)$ is infinitely differentiable for all $z \in D_\rho(\alpha)$. In fact

(ii) for all k, $f^{(k)}(z) = \sum_{n=0}^{\infty} n(n-1)\ldots(n-k+1) c_n (z-\alpha)^{n-k}$; and

(iii) $c_k = \frac{f^{(k)}(\alpha)}{k!}$ where $f^{(k)}(\alpha)$ denotes the k^{th} derivative of f. (When $k = 0$, $f^{(k)}$ denotes the function f itself so that $f^{(0)}(z) = f(z)$ for all z.)

Proof.

Example 4.24. Show that $g(z) = \sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}$ for all $z \in D_1(0)$

Solution. We know from Theorem 4.12 that $f(z) = \frac{1}{(1-z)} = \sum_{n=0}^{\infty} z^n$ for all $z \in D_1(0)$. If we set k=1 in Theorem 4.16, part (ii), then

$$f'(z) = \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n,$$

for all $z \in D_1(0)$.

Explore Solution 4.24.

Extra Example 2. Show that $h(z) = \sum_{n=0}^{\infty} (n+2)(n+1) z^n = \frac{2}{(1-z)^3}$ for all $z \in D_1(0)$

Explore Extra Solution 2.

Example 4.25. The Bessel function $J_0(z)$ of order zero is defined by

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} = 1 - \frac{z^2}{(2)^2} + \frac{z^4}{(2)^2 (4)^2} - \frac{z^6}{(2)^2 (4)^2 (6)^2} + \frac{z^8}{(2)^2 (4)^2 (6)^2 (8)^2} - \ldots,$$

and termwise differentiation shows that its derivative is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{d}{dz} \left(\frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}\right) &= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(n!)^2} 2n \left(\frac{z}{2}\right)^{2n-1} \left(\frac{1}{2}\right)\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(n-1)!\, n!} \left(\frac{z}{2}\right)^{2n-1}\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{(n-1)!\, n!} \left(\frac{z}{2}\right)^{2n-1}\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^{n+1}}{n!\, (n+1)!} \left(\frac{z}{2}\right)^{2n+1}\right) \\
&= -\frac{z}{2} + \frac{1}{1! \times 2!} \left(\frac{z}{2}\right)^3 - \frac{1}{2! \times 3!} \left(\frac{z}{2}\right)^5 + \frac{1}{3! \times 4!} \left(\frac{z}{2}\right)^7 - \ldots
\end{aligned}$$

We leave as an exercise to show that the radius of convergence of these series is infinity. The Bessel function $J_1(z)$ of order 1 is known to satisfy the differential equation $J_1(z) = -J_0'(z)$.

Explore Solution 4.25.

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Chapter 5 Elementary Functions

Overview

How should complex-valued functions such as e^z, $\log(z)$, $\sin(z)$, etc., be defined ? Clearly, any responsible definition should satisfy the following criteria:

(i) The functions so defined must give the same values as the corresponding functions for real variables when the number z is a real number.

(ii) As far as possible, the properties of these new functions must correspond with their real counterparts. For example, we would want $e^{z_1} e^{z_2} = e^{z_1+z_2}$ to be valid regardless of whether z were real or complex.

These requirements may seem like a tall order to fill. There is a procedure, however that offers promising results. It is to put in complex form the expansion of the real functions e^x, $\ln(x)$, $\sin(x)$, etc., as power series into complex form. We use this strategy in this chapter.

5.1 The Complex Exponential Function

Recall that the real exponential function can be represented by the power series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Thus it is only natural to define the complex exponential e^z, also written as $\exp(z)$, in the following way:

Definition 5.1 (Exponential Function). The definition of exp(z) is

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

Demonstration for Definition 5.1.

Clearly, this definition agrees with that of the real exponential function when z is a real

number. We now show that this complex exponential has two of the key properties associated with its real counterpart and verify the identity $e^{i\theta} = \cos\theta + i \sin\theta$, which, back in Chapter 1 (see identity (1-30) of Section 1.4), we promised to establish.

Theorem 5.1 (The exponential function). The function $\exp(z) = e^z$ is an entire function satisfying the following conditions:

(i). $\frac{d}{dz} \exp(z) = \exp(z) = e^z$, using Leibniz notation $\frac{d}{dz} e^z = e^z$.

(ii). $\exp(z_1 + z_2) = \exp(z_1) \exp(z_2)$, i.e. $e^{z_1+z_2} = e^{z_1} e^{z_2}$.

(iii). If θ is a real number, then $e^{i\theta} = \cos\theta + i \sin\theta$.

The exponential function is a solution to the differential equation $f'(z) = f(z)$ with the initial condition $f(0) = 1$.

Proof of Theorem 5.1.

Demonstration for Theorem 5.1 (i).

Demonstration for Theorem 5.1 (ii).

Demonstration for Theorem 5.1 (iii).

Note that parts (ii) and (iii) of the Theorem 5.1 combine to verify De Moivre's formula, which we introduced in Section 1.5.

If $z = x + i y$, we also see from parts (ii) and (iii) that

$$\exp(z) = e^z = e^{x+iy} = e^x e^{iy}$$

(5-1) $$\exp(z) = e^x (\cos y + i \sin y)$$

Some texts start with this identity as their definition for e^z. In the exercises, we show that this is a natural approach from the standpoint of differential equations.

The notation $\exp(z)$ is preferred over e^z in some situations. For example, $\exp\left(\frac{1}{5}\right)$ = "1.22140275816016983392..." is the value of $\exp(z)$ when $z = \frac{1}{5}$ and equals the positive fifth root of $e = 2.71828182845904523536\ldots$. Thus the notation $e^{1/5}$ is ambiguous and might be interpreted as any of the complex fifth roots of the number e that we discussed in Section 1.5:

$$e^{1/5} = 1.22140275816016983392 \left(\cos\frac{2\pi k}{5} + i \sin\frac{2\pi k}{5}\right) \text{ for } k = 0, 1, 2, 3, 4.$$

To prevent this confusion, we often use $\exp(z)$ to denote the single-valued exponential function.

We now explore some additional properties of $\exp(z) = e^z$. Using identity (5-1), we can easily establish that

(5-2) $e^{z+i\,2\,n\,\pi} = e^{z}$, for all z, provided n is an integer,

(5-3) $e^{z} = 1$, if and only if $z = i\,2\,n\,\pi$, where n is an integer, and

(5-4) $e^{z_1} = e^{z_2}$, if and only if $z_2 = z_1 + i\,2\,n\,\pi$, for some integer n.

For example, because Identity (5-1) involves the periodic functions cos y and sin y, any two points in the z plane that lie on the same vertical line with their imaginary parts differing by an integral multiple of $2\,\pi$ are mapped onto the same point in the w plane. Thus the complex exponential function is periodic with period $2\,\pi$, which establishes Equation (5-2). We leave the verification of Equations (5-3) and (5-4) as exercises for the reader.

Example 5.1. For any integer n, the points $z_n = \frac{5}{4} + i\left(\frac{11\,\pi}{6} + 2\,n\,\pi\right)$ are mapped onto a single point

$$w_0 = e^{z_n} = \exp(z_n)$$

$$= \exp\left[\frac{5}{4} + i\left(\frac{11\,\pi}{6} + 2\,n\,\pi\right)\right]$$

$$= e^{5/4}\left[\text{Cos}\left(\frac{11\,\pi}{6} + 2\,n\,\pi\right) + i\,\text{Sin}\left(\frac{11\,\pi}{6} + 2\,n\,\pi\right)\right]$$

$$= e^{5/4}\left(\cos\frac{11\,\pi}{6} + i\,\sin\frac{11\,\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2}\,e^{5/4} - i\,\frac{1}{2}\,e^{5/4}$$

$$= 3.02272566908 - 1.74517147873\,i$$

in the w plane, as indicated in Figure 5.1.

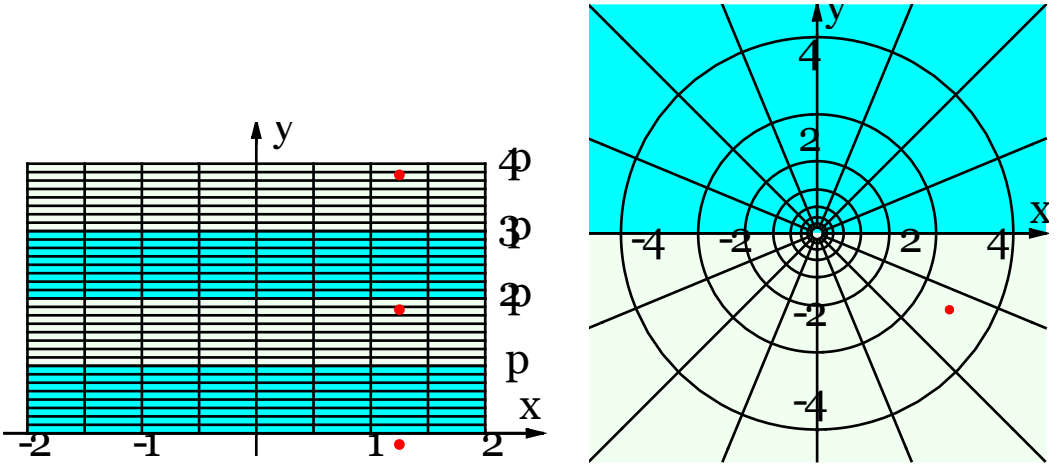


Figure 5.1 The points $\{z_n\}$ in the z plane (i.e., the xy plane) and their image $w_0 = \exp(z_n)$ in the w plane (i.e., the uv plane).

Explore Solution 5.1.

Let's look at the range of the exponential function. If $z = x + i\,y$, we see from identity (5-1) that e^{z} can never equal zero, as e^{x} is never zero, and the cosine and sine functions are never

zero at the same point. Suppose, then, that $w = e^z \neq 0$. If we write w in its exponential form as $w = \rho\, e^{i\phi}$, identity (5-1) gives

$$\rho\, e^{i\phi} = e^x\, e^{iy}.$$

Using identity (5-1), and property (1-39) of Section 1.5 we get

$$\rho = e^x \text{ and } \phi = y + 2n\pi, \text{ where n is an integer.}$$

Therefore,

$$\rho = |e^z| = e^x,$$

and

$$\phi \in \arg(e^z) = \{\operatorname{Arg}(e^z) + 2n\pi : n \text{ is an integer}\}.$$

Solving these equations for x and y, yields

$$x = \ln \rho \text{ and } y = \phi + 2n\pi,$$

where n is an integer. Thus, for any complex number $w \neq 0$, there are infinitely many complex numbers $z = x + iy$ such that $w = e^z$. From the previous equations, we see that the numbers z are

$$z = x + iy = \ln \rho + i(\phi + 2n\pi)$$

or

$$z = \ln|w| + i(\operatorname{Arg} w + 2n\pi),$$

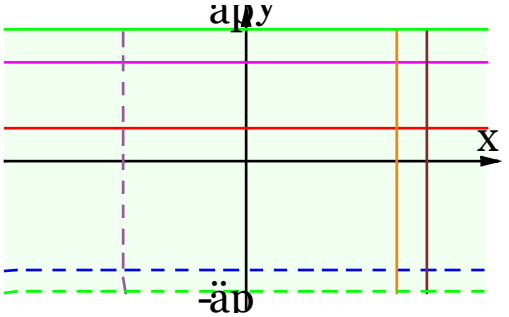
where n is an integer. Hence

$$\exp[\ln|w| + i(\operatorname{Arg} w + 2n\pi)] = w$$

$$e^{\ln|w| + i(\operatorname{Arg} w + 2n\pi)} = w$$

In summary, the transformation $w = e^z$ maps the complex plane (infinitely often) onto the set of nonzero complex numbers.

If we restrict the solutions in equation (5-9) so that only the principal value of the argument, $-\pi < \operatorname{Arg} w \leq \pi$, is used, the transformation $w = e^z = e^{x+iy}$ maps the horizontal strip $\{(x, y) : -\pi < y \leq \pi\}$, one-to-one and onto the range set $S = \{w : w \neq 0\}$. This strip is called the fundamental period strip and is shown in Figure 5.2.



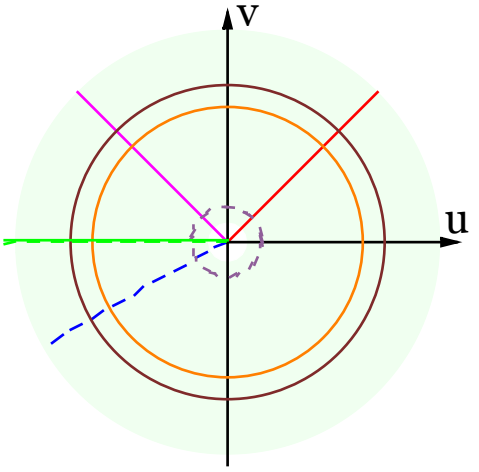


Figure 5.2 The fundamental period strip for the mapping $w = e^z = \exp(z)$.

The horizontal line $z = t + i\,b$, for $-\infty < t < \infty$ in the z plane, is mapped onto the ray $w = e^t\,e^{i\,b} = e^t\,(\cos b + i \sin b)$ that is inclined at an angle $\phi = b$ in the w plane. The vertical segment $z = a + i\,\theta$, for $-\pi < \theta \le \pi$ in the z plane, is mapped onto the circle centered at the origin with radius e^a in the w plane. That is, $w = e^a\,e^{i\,\theta} = e^a\,(\cos\theta + i \sin\theta)$.

Example 5.2. Consider a rectangle $R = \{(x, y) : a \le x \le b \text{ and } c \le y \le d\}$, where $-\pi < c < d \le \pi$. Show that the transformation $w = e^z = e^{x+i\,y}$ maps the rectangle R onto a portion of an annular region bounded by two rays.

Solution. The image points in the w plane satisfy the following relationships involving the modulus and argument of w:

$$e^a = |e^{a+i\,y}| \le |e^{x+i\,y}| \le |e^{b+i\,y}| = e^b, \text{ and}$$

$$c = \text{Arg}\,(e^{x+i\,c}) \le \text{Arg}\,(e^{x+i\,y}) \le \text{Arg}\,(e^{x+i\,d}) \le d,$$

which is a portion of the annulus $\{\rho\,e^{i\,\phi} : e^a \le \rho \le e^b\}$ in the w plane subtended by the rays $\phi = c$ and $\phi = d$. In Figure 5.3, we show the image of the rectangle

$$R^* = \left\{(x, y) : -1 \le x \le 1 \text{ and } \frac{-\pi}{4} \le y \le \frac{\pi}{3}\right\}.$$

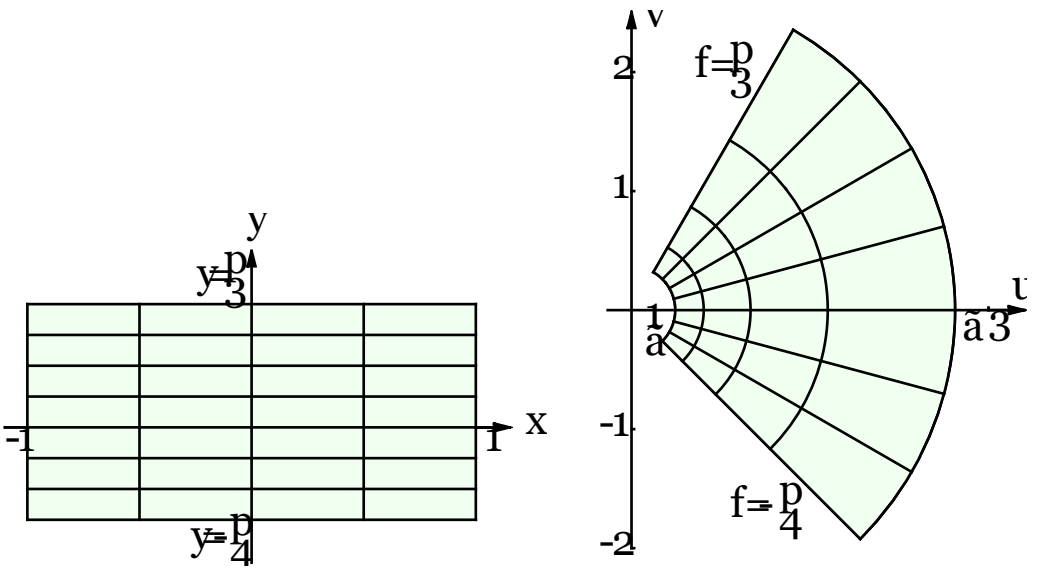


Figure 5.3 The image of R* under the transformation w = e^z = exp(z).

Explore Solution 5.2.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

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5.2 The Complex Logarithm

In Section 5.1, we showed that, if w is a nonzero complex number, then the equation $w = e^z$ has infinitely many solutions. Because the function $f(z) = e^z$ is a many-to-one function, its inverse (the logarithm) is multivalued.

Definition 5.2. (Multivalued Logarithm) For $z \neq 0$, we define the function $\log(z)$ as the inverse of the exponential function; that is,

(5-10) $\log(z) = w$ if and only if $z = e^w$.

If we go through the same steps as we did in equations (5-8) and (5-9), we find that, for any complex number $z \neq 0$, the solutions w to equation (5-10) take the form

(5-11) $w = \log(z) = \ln|z| + i\theta$, for $z \neq 0$,

where $\theta \in \arg(z)$ and $\ln|z|$ denotes the natural logarithm of the positive number $|z|$. Because $\arg(z)$ is the set $\arg(z) = \{\text{Arg}(z) + 2n\pi : \text{where n is an integer}\}$, we can express the set of values comprising $\log(z)$ as

(5-12) $\log(z) = \{\ln|z| + i(\text{Arg} + 2n\pi) : \text{where } n \text{ is an integer}\}$,
or
(5-13) $\log(z) = \ln|z| + i\arg(z)$ for $z \neq 0$,

where it is understood that identity (5-13) refers to the same set of numbers given in identity (5-12).

Recall that Arg is defined so that for $z \neq 0$, we have $-\pi < \text{Arg}(z) \leq \pi$. We call any one of the values given in Identities (5-12) or (5-13) a logarithm of z. Notice that the different values of $\log(z)$ all have the same real part and that their imaginary parts differ by the amount $2\pi n$, where n is an integer. When $n = 0$, we have a special situation.

Definition 5.3. (Principal Value of the Logarithm) For $z \neq 0$, we define the principal value of the logarithm as follows:

(5-14) $\quad \text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$ where $|z| > 0$ and $-\pi < \text{Arg}(z) \leq \pi$.

The domain for the function $w = f(z) = \text{Log}(z)$ is the set of all nonzero complex numbers in the z-plane, and its range is the horizontal strip $\{w : -\pi < \text{Im}(w) \leq \pi\}$ in the w-plane, and is shown in Figure 5.A. We stress again that $\text{Log}(z)$ is a single-valued function and corresponds to setting $n = 0$ in equation (5-12). As we demonstrated in Chapter 2, the function $\text{Arg}(z)$ is discontinuous at each point along the negative x-axis, hence so is the function $\text{Log}(z)$. In fact, because any branch of the multi-valued function $\text{Arg}(z)$ is discontinuous along some ray, a corresponding branch of the logarithm will have a discontinuity along that same ray.

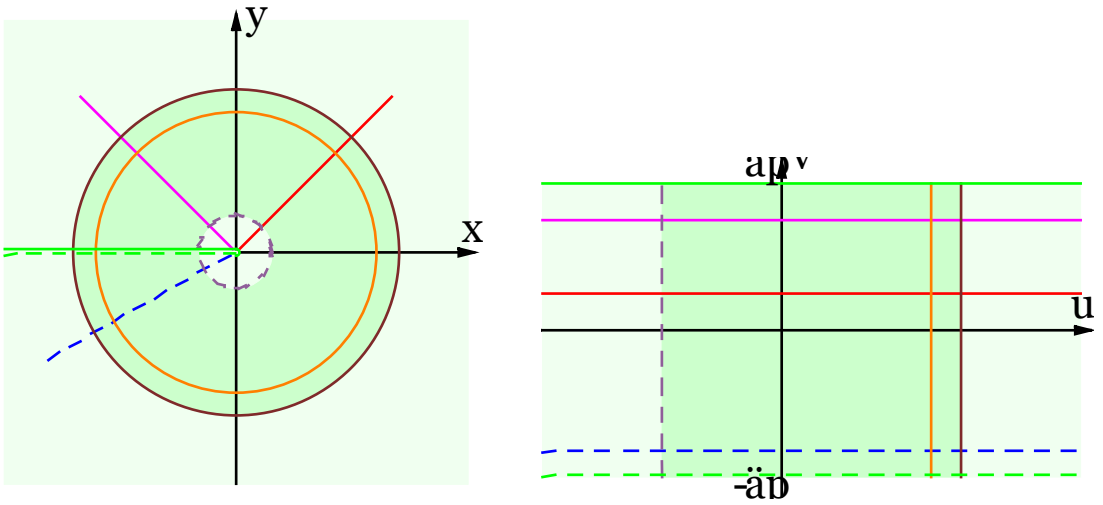

Figure 5.A The principal branch of the logarithm $w = \text{Log}(z)$.

Caution. A phenomenon inherent in constructing an logarithm function: It must have a discontinuity! This is the case because as we saw in Chapter 2, any branch we choose for $\arg(z)$ is necessarily a discontinuous function. The principal branch, $\text{Log}(z)$, is discontinuous at each point along the negative x-axis.

Extra Example 1. Investigate the complex logarithm function.

Explore Solution for Extra Example 1.

Example 5.3. Find the values of $\text{Log}(1+i)$ and $\log(1+i)$.

Solution. By standard computations, we have

$$\log(1+i) = \{\ln|1+i| + i\,(\text{Arg}(1+i) + 2n\pi) : \text{where } n \text{ is an integer}\}$$

$$= \left\{\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right) : \text{where } n \text{ is an integer}\right\}$$

and

$$\log(i) = \{\ln|i| + i\,(\text{Arg}(i) + 2n\pi) : \text{where } n \text{ is an integer}\}$$

$$= \left\{i\left(\frac{\pi}{2} + 2n\pi\right) : \text{where } n \text{ is an integer}\right\}$$

The principal values are

$$\text{Log}(1+i) = \ln\sqrt{2} + i\frac{\pi}{4} = \frac{\ln 2}{2} + i\frac{\pi}{4}$$

and

$$\text{Log}(i) = i\frac{\pi}{2}$$

Explore Solution 5.3.

Extra Example 2. The transformation $w = \text{Log}(z)$ maps the z-plane punctured at the origin onto the horizontal strip in the w-plane.

Explore Solution for Extra Example 2.

We now investigate some of the properties of $\log(z)$ and $\text{Log}(z)$. From Equations (5-10), (5-12), and (5-14), it follows that

(5-15) $\exp(\text{Log } z) = z$ for all $z \neq 0$

and

(5-16) $\text{Log}(\exp z) = z$, provided $-\pi < \text{Arg}(z) \leq \pi$,

and that the mapping $w = \text{Log}(z)$ is one-to-one from domain $D = \{z : |z| > 0\}$ in the z plane onto the horizontal strip $\{w : -\pi < \text{Im}(w) \leq \pi\}$ in the w plane.

The following example illustrates that, even though $\text{Log}(z)$ is not continuous along the negative real axis, it is still defined there.

Example 5.4. Identity reveals that

(a) $\text{Log}(-e) = \ln|-e| + i\,\text{Arg}(-e) = 1 + i$, and

(b) $\text{Log}(-1) = \ln|-1| + i\,\text{Arg}(-1) = i$.

Explore Solution 5.4.

When $z = x + i\,0$, where x is a positive real number, the principal value of the complex logarithm of z is

$$\text{Log}(x + i\,0) = \ln x + i\,\text{Arg}(x) = \ln x + i\,0 = \ln x,$$

where $x > 0$. Hence Log is an extension of the real function $\ln(x)$ to the complex case. Are there other similarities? Let's use complex function theory to find the derivative of $\text{Log}(z)$.
When we use polar coordinates for $z = r\,e^{i\theta} \neq 0$, equation (5-14) becomes

$$\begin{aligned}\text{Log}(z) &= \text{Log}(r\,e^{i\theta}) \\ &= \ln|r\,e^{i\theta}| + i\,\text{Arg}(r\,e^{i\theta}) \\ &= \ln r + i\,\theta, \quad \text{for } r > 0 \text{ and } -\pi < \theta \leq \pi \\ &= U(r,\theta) + i\,V(r,\theta)\end{aligned}$$

where $U(r, \theta) = \ln r$ and $V(r, \theta) = \theta$. Because Arg(z) is discontinuous only at points in its domain that lie on the negative real axis, U and V have continuous partials for any point (r, θ) in their domain, provided $r\, e^{i\theta}$ is not on the negative real axis, that is, provided $-\pi < \theta < \pi$ (Note the strict inequality for here.). In addition, the polar form of the Cauchy-Riemann equations holds in this region (see Equation (3-22) of Section 3.2), since
$U_r(r, \theta) = \frac{1}{r}$, $U_\theta(r, \theta) = 0$, $V_r(r, \theta) = 0$, $V_\theta(r, \theta) = 1$ it follows that

$$U_r(r, \theta) = \frac{1}{r} = \frac{1}{r} V_\theta(r, \theta)$$

and

$$U_\theta(r, \theta) = 0 = \frac{-1}{r} 0 = -\frac{-1}{r} V_r(r, \theta)$$

Using Theorem 3.5 of Section 3,2, we see that

$$\frac{d}{dz} \text{Log}(z) = e^{-i\theta} (U_r(r, \theta) + i\, U_\theta(r, \theta))$$

$$= e^{-i\theta} \left(\frac{1}{r} + 0\, i\right) = \frac{1}{r\, e^{i\theta}}$$

$$= \frac{1}{z}$$

provided $r > 0$ and $-\pi < \theta < \pi$. Thus the principal branch of the complex logarithm has the derivative we would expect. Other properties of the logarithm carry over, but only in specified regions of the complex plane.

Example 5.5. Show that the identity $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$ is not always valid.

Solution. Let $z_1 = -\sqrt{3} + i$ and $z_2 = -1 + i\sqrt{3}$. Then

$$\text{Log}(z_1 z_2) = \text{Log}\left[\left(-\sqrt{3} + i\right)\left(-1 + i\sqrt{3}\right)\right]$$
$$= \text{Log}(-4\, i) = \ln 4 + i \frac{-\pi}{2}$$
$$= \ln 4 - i \frac{\pi}{2}$$

but

$$\text{Log}(z_1) + \text{Log}(z_2) = \left(\text{Log}[2] + i \frac{5\pi}{6}\right) + \left(\text{Log}[2] + i \frac{2\pi}{3}\right)$$
$$= 2\,\text{Log}[2] + i \left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)$$
$$= \ln 4 + i \frac{3\pi}{2}$$

Since $\ln 4 - i \frac{\pi}{2} \neq \ln 4 + i \frac{3\pi}{2}$, this is a counterexample for which
$\text{Log}(z_1 z_2) \neq \text{Log}(z_1) + \text{Log}(z_2)$.

Explore Solution 5.5.

Our next result explains why the identity $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$ did not hold for the particular numbers we chose.

Theorem 5.2. The identity $\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$ holds true if and only if

$-\pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le \pi$.

Proof of Theorem 5.2.

Suppose first that $-\pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le \pi$. By definition, $\text{Log}(z_1 z_2) = \ln|z_1 z_2| + i\,\text{Arg}(z_1 z_2) = \ln|z_1| + \ln|z_2| + i\,\text{Arg}(z_1 z_2)$. As $-\pi < \text{Arg}(z_1) + \text{Arg}(z_2) \le \pi$, it follows that $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ (explain!), and so $\text{Log}(z_1 z_2) = \ln|z_1| + \ln|z_2| + i\,\text{Arg}(z_1) + i\,\text{Arg}(z_2) = \text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$.

The "only if" part is left as an exercise.

As Example 5.5 and Theorem 5.2 illustrate, properties of the complex logarithm don't carry over when arguments of products combine in such a way that they drop down to $-\pi$ or rise above $-\pi$. This is because of the restrictions placed on the domain of the function $\text{Arg}(z)$. From the set of numbers associated with the multivalued logarithm, however, we can formulate properties that look exactly the same as those corresponding with the real logarithm.

Theorem 5.3. Let z_1 and z_2 be nonzero complex numbers. The multivalued function $\log(z)$ obeys the familiar properties of logarithms:

(5-17) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$,

(5-18) $\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$, and

(5-19) $\log\left(\frac{1}{z}\right) = -\log(z)$.

Proof of Theorem 5.3.

Property of Log(z). We know that $\log(z_1 z_2) = \log(z_1) + \log(z_2)$, what about $\text{Log}(z)$?

Demonstration of Property of Log(z).

We can construct many different branches of the multivalued logarithm function that are continuous and differentiable except at points along any preassigned ray $\{r e^{i\alpha} : r > 0\}$. If we let α denote a real fixed number and choose the value of $\theta \in \arg(z)$, that lies in the range $\alpha < \theta \le \alpha + 2\pi$, then the function $\log_\alpha(z)$ defined by

$$\log_\alpha(z) = \ln r + i\theta$$

where $z = r e^{i\theta} \neq 0$, and $\alpha < \theta \le \alpha + 2\pi$, is a single-valued branch of the logarithm function. The branch cut for $\log_\alpha(z)$ is the ray $\{r e^{i\alpha} : r > 0\}$, and each point along this ray is a point of discontinuity of $\log_\alpha(z)$. Because $\exp(\log_\alpha(z)) = z$, we conclude that the mapping $w = \log_\alpha(z)$ is a one-to-one mapping of the domain $|z| > 0$ onto the horizontal strip $\{w : \alpha < \text{Im}(w) \le \alpha + 2\pi\}$. If $\alpha < c < d < \alpha + 2\pi$, then the function $w = \log_\alpha(z)$ maps the set $D = \{r e^{i\alpha} : 0 < a < r < b, \text{ and } c < \theta < d\}$ one-to-one and onto the rectangle $R = \{u + iv : \ln a < u < \ln b, \text{ and } c < v < d\}$. Figure 5.4 shows the mapping $w = \log_\alpha(z)$, its branch cut $\{r e^{i\alpha} : r > 0\}$, the set D, and its image R.

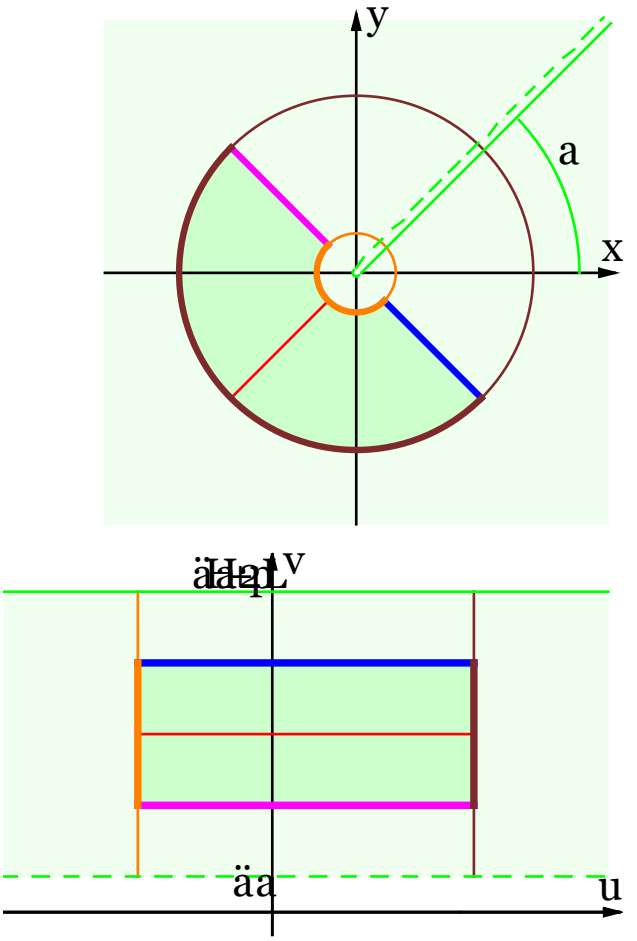

Figure 5.4 The branch $w = \log_\alpha(z)$ of the logarithm.

We can easily compute the derivative of any branch of the multivalued logarithm. For a particular branch $w = \log_\alpha(z)$ for $z = r\, e^{i\theta} \neq 0$, and $\alpha < \theta \leq \alpha + 2\pi$ (note the strict inequality for θ), we start with $z = \exp(w)$ in Equations (5-10) and differentiate both sides to get

$$\begin{aligned} 1 &= \frac{d}{dz} z \\ &= \frac{d}{dz} \exp(\log_\alpha(z)) \\ &= \exp(\log_\alpha(z)) \frac{d}{dz} \log_\alpha(z) \\ &= z \frac{d}{dz} \log_\alpha(z) \end{aligned}$$

Solving for $\frac{d}{dz} \log_\alpha(z)$ gives

$$\frac{d}{dz} \log_\alpha(z) = \frac{1}{z}, \quad \text{for} \quad \text{for } z = e^{i\theta} \neq 0, \text{ and } \alpha < \theta \leq \alpha + 2\pi.$$

The Riemann surface for the multivalued function $w = \log(z)$ is similar to the one we presented for the square root function. However, it requires infinitely many copies of the z plane cut along the negative x axis, which we label S_k for $k = \ldots, -n, \ldots, -1, 0, 1, \ldots, n, \ldots$. Now, we stack these cut planes directly on each other so that the corresponding points have the same position. We join the sheet S_k to S_{k+1} as follows. For each integer k, the edge of the sheet S_k in the upper half-plane is joined to the edge of the sheet S_{k+1} in the lower half-plane. The Riemann surface for the domain of $\log(z)$ looks like a spiral staircase that extends upward on the sheets $S_1, S_2, \ldots$ and downward on the sheets $S_{-1}, S_{-2}, \ldots$, as shown in Figure 5-5. We use polar coordinates for z on

each sheet. For S_k, we use

$$z = r\, e^{i\theta} = r\,(\cos\theta + i \sin\theta), \quad \text{where}$$

$$r = |z| \text{ and } 2\pi k - \pi < \theta \le \pi + 2\pi k.$$

Again, for S_k, the correct branch of `log(z)` on each sheet is

$$\log(z) = \ln r + i\,\theta, \quad \text{where}$$

$$r = |z| \text{ and } 2\pi k - \pi < \theta \le \pi + 2\pi k.$$

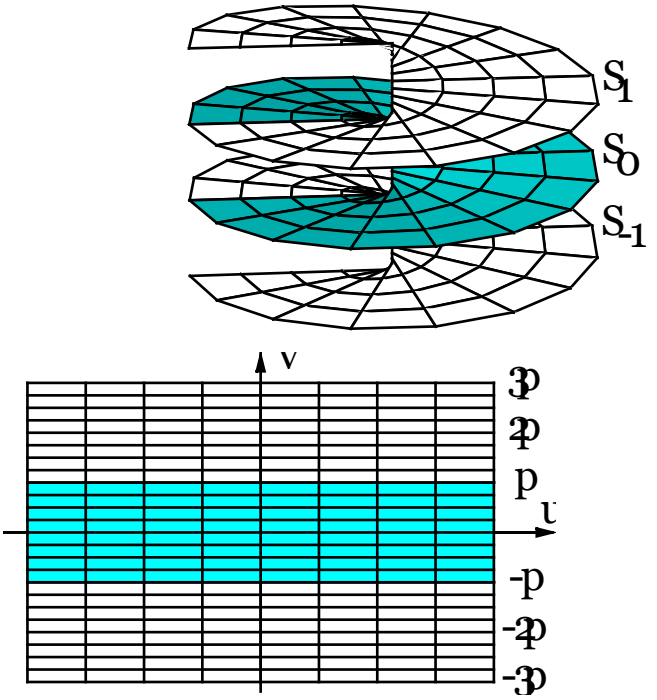

Figure 5.5 The Riemann surface for the mapping `w = log(z)`.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

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5.3 Complex Exponents

In Section 1.5 we indicated that it is possible to make sense out of expressions such as $\sqrt{1+i}$ or $(-1)^i$ without appealing to a number system beyond the framework of complex numbers. We now show how this is done by taking note of some rudimentary properties of the complex exponential and logarithm, and then using our imagination.

We begin by generalizing Identity (5-15). Equations (5-12) and (5-14) show that $\log(z)$ can be expressed as the set $\log(z) = \{\text{Log}(z) + i\,2n\pi : n \text{ is an integer}\}$. We can easily show (left as an exercise) that, for $z \neq 0$, $e^{\log_\alpha(z)} = z$, where $\log_\alpha(z)$ is any branch of the function $\log(z)$. But this means that for any $\zeta \in \log(z)$, the identity $e^\zeta = z$ holds true. Because $e^{\log(z)}$ denotes the set $\{e^\zeta : \zeta \in \log(z)\}$, we see that $e^{\log_\alpha(z)} = z$, for $z \neq 0$.

Next, we note that identity (5-17) gives us $\log(z^n) = n\log(z)$, where n is any natural number, so that $e^{\log(z^n)} = e^{n\log(z)} = z^n$ for $z \neq 0$. With these preliminaries out of the way, we can now come up with a definition of a complex number raised to a complex power.

Definition 5.4. Let c be a complex number. We define z^c as follows

(5-21) $$z^c = e^{c\log(z)}.$$

The right side of Equation (5-21) is a set. This definition makes sense because, if both z and c are real numbers with $z > 0$, Equation (5-21) gives the familiar (real) definition for z^c, as the following example illustrates.

Example 5.6. Use Equation (5-21) to evaluate $4^{1/2}$.

Solution. Calculating $4^{\frac{1}{2}} = \exp\left[\frac{1}{2}\log 4\right]$ gives

$$\frac{1}{2}\log 4 = \{\ln 2 + i\,n\pi : n \text{ is an integer}\}.$$

Thus $4^{\frac{1}{2}}$ is the set $\{\exp(\ln 2 + i\,n\pi) : n \text{ is an integer}\}$. The distinct values occur when $n = 0$ and 1; we get $\exp(\ln 2) = 2$ and $\exp(\ln 2 + i\pi) = \exp(\ln 2)\exp(i\pi) = -2$. In other

words, $4^{\frac{1}{2}} = \{-2, 2\}$.

Remark. The expression $4^{\frac{1}{2}}$ is different from $\sqrt{4}$, as the former represents the set $\{-2, 2\}$ and the latter gives only one value, $\sqrt{4} = 2$.

Explore Solution 5.6.

Because $\log(z)$ is multivalued, the function z^c will, in general, be multivalued. If we want to focus on a single value for z^c, we can do so via the function defined for $z \neq 0$ by

(5-22) $$f(z) = \exp(c \operatorname{Log}(z))$$

which is called the principal branch of the multivalued function z^c. Note that the principal branch of z^c is obtained from Equation (5-21) by replacing $\log(z)$ with the principal branch of the logarithm.

Example 5.7. Find the principal value of **(a)** $\sqrt{1+i}$, and **(b)** $(i)^i$.

Solution. From Example 5.3,

$$\operatorname{Log}(1+i) = \frac{\ln 2}{2} + i\frac{\pi}{4} = \ln 2^{\frac{1}{2}} + i\frac{\pi}{4}, \text{ and}$$

$$\operatorname{Log}(i) = i\frac{\pi}{2}.$$

Identity (5-22) yields the principal values of $\sqrt{1+i}$ and $(i)^i$:

$$\sqrt{1+i} = (1+i)^{\frac{1}{2}} = \exp\left[\frac{1}{2}\operatorname{Log}(1+i)\right]$$

$$= \exp\left[\frac{1}{2}\left(\ln 2^{\frac{1}{2}} + i\frac{\pi}{4}\right)\right] = \exp\left[\ln 2^{\frac{1}{4}} + i\frac{\pi}{8}\right]$$

$$= \exp\left[\ln 2^{\frac{1}{4}}\right]\exp\left[i\frac{\pi}{8}\right] = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

$$\approx 1.09868411 + 0.45508986\,i$$

and

$$\begin{aligned}(i)^i &= \exp[i\operatorname{Log}(i)] \\ &= \exp\left[i\left(i\frac{\pi}{2}\right)\right] \\ &= \exp\left[-\frac{\pi}{2}\right] \\ &\approx 0.20787957635\end{aligned}$$

Note that the result of raising a complex number to a complex power may be a real number in a nontrivial way.

Explore Solution 5.7 (a).

Explore Solution 5.7 (b).

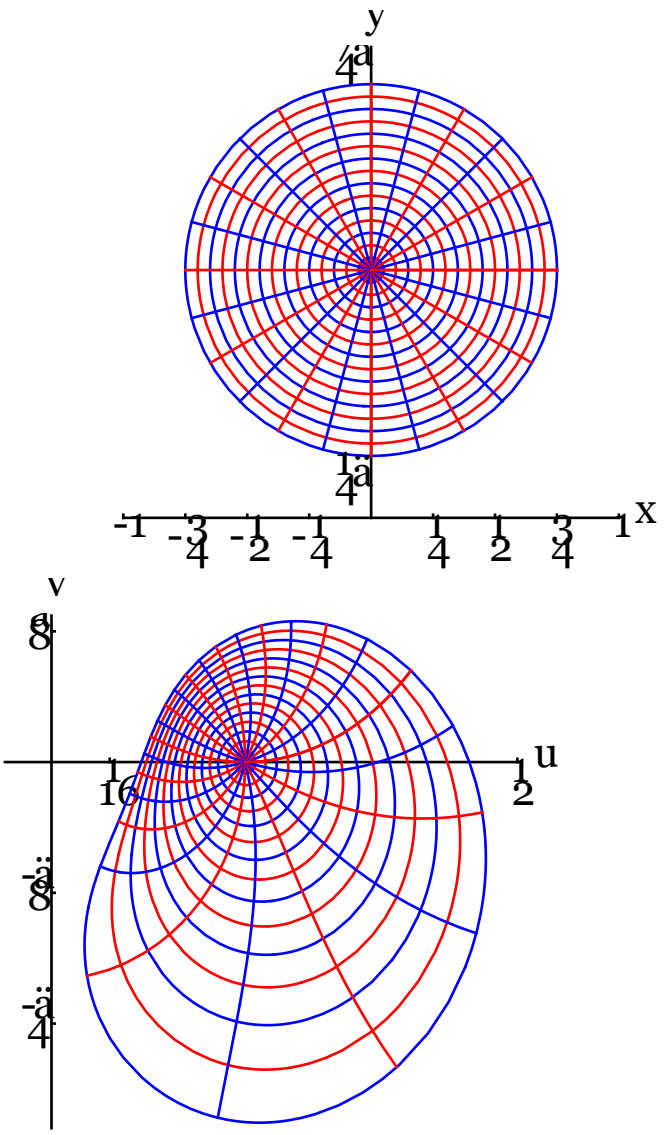

Figure 5.C The mapping $w = (z)^i$.

Complex Exponents

Let us now consider the various possibilities that may arise in the definition of z^c.

Case (i). Suppose $c = k$ where k an integer. Then, if $z = r\, e^{i\theta} \neq 0$,

$$k \log(z) = \{k \ln(r) + i\, k\, (\theta + 2\, n\, \pi) : \ n \text{ is an integer}\}.$$

Recalling that the complex exponential function has period $2\,\pi\,i$, we have

$$z^k = r^k\, (\cos k\,\theta + i \sin k\,\theta).$$

which is the single-valued kth power of z that we discussed in Section 1.5. This is easily verified by the computation

$$z^k = \exp[k \log z] = \exp[k \log (r\, e^{i\theta})]$$

$$= \exp[k \ln r + i\, k\, (\theta + 2\, n\, \pi)]$$

$$= \exp[\ln r^k + i\, k\, \theta + i\, 2\, k\, n\, \pi]$$

$$= \exp[\ln r^k] \exp[i\, k\, \theta] \exp[i\, 2\, k\, n\, \pi]$$

$$= r^k \exp(i\, k\, \theta)$$

$$= r^k (\cos k\, \theta + i \sin k\, \theta)$$

which is the single-valued kth power of z that we discussed in Section 1.5.

Case (ii). If $c = \frac{1}{k}$ where k is an integer and $z = r\, e^{i\theta} \neq 0$, then

(5-23) $$\frac{1}{k} \log(z) = \left\{ \frac{1}{k} \ln(r) + \frac{i\, (\theta + 2\, \pi\, n)}{k} : n \text{ is an integer} \right\}.$$

Hence Equation (5-21) becomes

$$z^{\frac{1}{k}} = r^{\frac{1}{k}} \left(\cos \frac{\theta + 2\, n\, \pi}{k} + i \sin \frac{\theta + 2\, n\, \pi}{k} \right) \text{ for } n = 0, 1, \ldots, k - 1.$$

When we again use the periodicity of the complex exponential function, Equation (5-23) gives k distinct values corresponding to $n = 0, 1, \ldots, k - 1$. Therefore, as Example 5.6 illustrated, the fractional power $z^{\frac{1}{k}}$ is the multivalued k^{th} root function. Equation (5-23) is easily verified by the computation

$$z^{\frac{1}{k}} = \exp\left[\frac{1}{k} \log(z)\right] = \exp\left[\frac{1}{k} \log(r\, e^{i\theta})\right]$$

$$= \exp\left[\frac{1}{k} (\ln r + i\, (\theta + 2\, n\, \pi))\right]$$

$$= \exp\left[\ln r^{\frac{1}{k}} + i \left(\frac{\theta + 2\, n\, \pi}{k}\right)\right]$$

$$= \exp\left[\ln r^{\frac{1}{k}}\right] \exp\left[i \left(\frac{\theta + 2\, n\, \pi}{k}\right)\right]$$

$$= r^{\frac{1}{k}} \left(\cos \frac{\theta + 2\, n\, \pi}{k} + i \sin \frac{\theta + 2\, n\, \pi}{k} \right)$$

Case (iii). If j and k are positive integers that have no common factors and $c = \frac{j}{k}$, then Equation (5-21) becomes

$$z^{\frac{j}{k}} = r^{\frac{j}{k}} \left(\cos \frac{(\theta + 2\, n\, \pi)\, j}{k} + i \sin \frac{(\theta + 2\, n\, \pi)\, j}{k} \right) \text{ for } n = 0, 1, \ldots, k - 1.$$

This is easy to establish. If $z = r\, e^{i\theta}$ then

$$z^{\frac{j}{k}} = \exp\left[\frac{j}{k}\log(z)\right] = \exp\left[\frac{j}{k}\log(r\,e^{i\theta})\right]$$

$$= \exp\left[\frac{j}{k}(\ln r + i(\theta + 2n\pi))\right]$$

$$= \exp\left[\ln r^{\frac{j}{k}} + i\frac{(\theta+2n\pi)j}{k}\right]$$

$$= \exp\left[\ln r^{\frac{j}{k}}\right]\exp\left[i\frac{(\theta+2n\pi)j}{k}\right]$$

$$= r^{\frac{j}{k}}\left(\cos\frac{(\theta+2n\pi)j}{k} + i\sin\frac{(\theta+2n\pi)j}{k}\right)$$

and again there are k distinct values corresponding to $n = 0, 1, \ldots, k-1$.

Case (iv). Suppose c is not a rational number, then there are infinitely many values for z^c, provided $z = r\,e^{i\theta} \neq 0$.

Example 5.8. The values of of $2^{\frac{1}{9}+\frac{i}{50}}$ are

$$2^{\frac{1}{9}+\frac{i}{50}} = \exp\left[\left(\frac{1}{9}+\frac{i}{50}\right)(\ln 2 + i\,2n\pi)\right]$$
$$= \exp\left[\frac{\ln 2}{9} - \frac{n\pi}{25} + i\left(\frac{\ln 2}{50} + \frac{2n\pi}{9}\right)\right]$$
$$= \exp\left[\frac{\ln 2}{9}\right]\exp\left[-\frac{n\pi}{25}\right]\exp\left[i\left(\frac{\ln 2}{50} + \frac{2n\pi}{9}\right)\right]$$
$$= 2^{\frac{1}{9}}\,e^{-\frac{n\pi}{25}}\left[\cos\left(\frac{\ln 2}{50} + \frac{2n\pi}{9}\right) + i\sin\left(\frac{\ln 2}{50} + \frac{2n\pi}{9}\right)\right]$$

where n is an integer. The principal value of $2^{\frac{1}{9}+\frac{i}{50}}$ is

$$2^{\frac{1}{9}+\frac{i}{50}} = 2^{\frac{1}{9}}\,e^{-\frac{n\pi}{25}}\left[\cos\left(\frac{\ln 2}{50}\right) + i\sin\left(\frac{\ln 2}{50}\right)\right]$$

$$\approx 1.07995595696 + 0.01497232767\,i$$

Figure 5.6 shows the terms for this multivalued expression corresponding to $n = -9, -8, \ldots, -1, 0, 1, \ldots, 8, 9$. They exhibit a spiral pattern that is often present in complex powers.

Explore Solution 5.8.

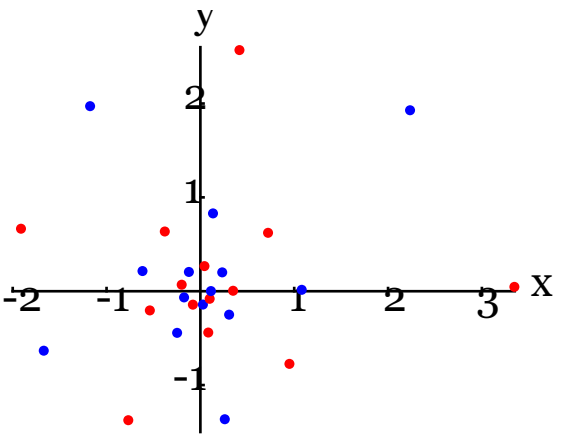

Figure 5.6 Some of the "spiral pattern" of values for $2^{1/9+i/50}$.

The Rules for exponents.

Some of the rules for exponents carry over from the real case. In the exercises we ask you to show that if c and d are complex numbers and $z \neq 0$, then

(5 – 24) $z^{-c} = \frac{1}{z^c}$,

(5 – 25) $z^c z^d = z^{c+d}$,

(5 – 26) $\frac{z^c}{z^d} = z^{c-d}$,

(5 – 27) $(z^c)^n = z^{cn}$,

where n is an integer.

The following example shows that Identity (5-21) does not hold if n is replaced with an arbitrary complex value.

Example 5.9. (a) $(i^2)^i = \exp[i \log(-1)] = \{e^{-(1+2n)\pi} : n \text{ is an integer}\}$,

and **(b)** $i^{2i} = \exp[2i \log i] = \{e^{-(1+4n)\pi} : n \text{ is an integer}\}$.

Since these sets of solutions are not equal, Identity (5-21) does not always hold.

Explore Solution 5.9.

We can compute the derivative of the principal branch of z^c, which is the function

$f(z) = \exp[c \operatorname{Log} z]$. By the chain rule,

(5-28) $f'(z) = \frac{c}{z} \exp[c \operatorname{Log} z]$.

If we restrict z^c to the principal branch, $z^c = \exp[c \operatorname{Log} z]$, then Equation (5-28) can be written in the familiar form that you learned in calculus. That is, for $z \neq 0$ and z not a negative real number,

$$\frac{d}{dz} z^c = z^c \frac{c}{z} = c\, z^{c-1}.$$

Exploration

We can use Identity (5-21) to define the exponential function with base b, where $b \neq 0$ is a complex number:

$$b^z = \exp[z \log(b)].$$

If we specify a branch of the logarithm, then b^z will be single-valued and we can use the rules of differentiation to show that the resulting branch of b^z is an analytic function. The derivative of b^z is then given by the familiar rule

(5-29) $$\frac{d}{dz} b^z = b^z \log_\alpha(b)$$

where $\log_\alpha(z)$ is any branch of the logarithm whose branch cut does not include the point $z = b$.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering
by
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5.4 Trigonometric and Hyperbolic Functions

Based on the success we had in using power series to define the complex exponential, we have reason to believe this approach will be fruitful for other elementary functions as well. The power series expansions for the real-valued sine and cosine functions are

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \text{and}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Thus, it is natural to make the following definitions.

Definition 5.5. The series for Sine and Cosine are

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \quad \text{and}$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Clearly, these definitions agree with their real counterparts when z is real. Additionally, it is easy to show that $\cos(z)$ and $\sin(z)$ are entire functions. (We leave the proof as an exercise.)

Exploration (i). Investigate the series $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$.

Exploration (i).

Exploration (ii). Investigate the series $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$.

Exploration (ii).

With these definitions in place, it is now easy to create the other complex trigonometric

functions, provided the denominators in the following expressions do not equal zero.

Definition 5.6. $\tan z = \frac{\sin z}{\cos z}$, $\cot z = \frac{\cos z}{\sin z}$, $\sec z = \frac{1}{\cos z}$, and $\csc z = \frac{1}{\sin z}$.

Exploration for Definition 5.6.

Since the series for the complex sine and cosine agree with the real sine and cosine when z is real, the remaining complex trigonometric functions likewise agree with their real counterparts. What additional properties are common? For starters, we have

Theorem 5.4. $\sin(z)$ and $\cos(z)$ are entire functions, with $\frac{d}{dz}\sin(z) = \cos(z)$ and $\frac{d}{dz}\cos(z) = -\sin(z)$.

Proof of Theorem 5.4.

Exploration for Theorem 5.4.

We now list several additional properties, providing proofs for some and leaving others as exercises. For all complex numbers z,

$$\sin(-z) = -\sin(z)$$

$$\cos(-z) = \cos(z)$$

$$(\cos z)^2 + (\sin z)^2 = 1$$

The verification that $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$ comes from substituting $-z$ for z in Definition 5.5. We leave verification of the identity $(\cos z)^2 + (\sin z)^2 = 1$ as an exercise (with hints).

Exploration for identities.

A series exploration (i) The derivative of sin(z) is cos(z).

A series exploration (i).

A series exploration (ii) The derivative of cos(z) is -sin(z).

A series exploration (ii).

For all complex numbers z for which the expressions are defined,

$$\frac{d}{dz}\tan z = (\sec z)^2,$$

$$\frac{d}{dz}\cot z = -(\csc z)^2,$$

$$\frac{d}{dz}\sec z = \sec z \tan z,$$

$$\frac{d}{dz}\csc z = -\cot z \, \text{Csc}\, z.$$

Proof.

Exploration for the identities.

To establish additional properties, it will be useful to express cos z and sin z in the Cartesian form u + i v. (Additionally, the applications in Chapters 10 and 11 will use these formulas.) We begin by observing that the argument given to prove part (iii) in Theorem 5.1 easily generalizes to the complex case with the aid of Definition 5.5. That is,

(5-30) $$e^{iz} = \cos z + i \sin z,$$

for all z, whether z is real or complex. Hence,

(5-31) $$e^{-iz} = \cos(-z) + i \sin(-z) = \cos z - i \sin z$$

Adding the above two expressions and solving for cos z gives

(5-32) $$\cos z = \frac{e^{iz} + e^{-iz}}{2},$$

and subtracting (5-32) from (5-31) and solving for sin z gives

(5-33) $$\sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

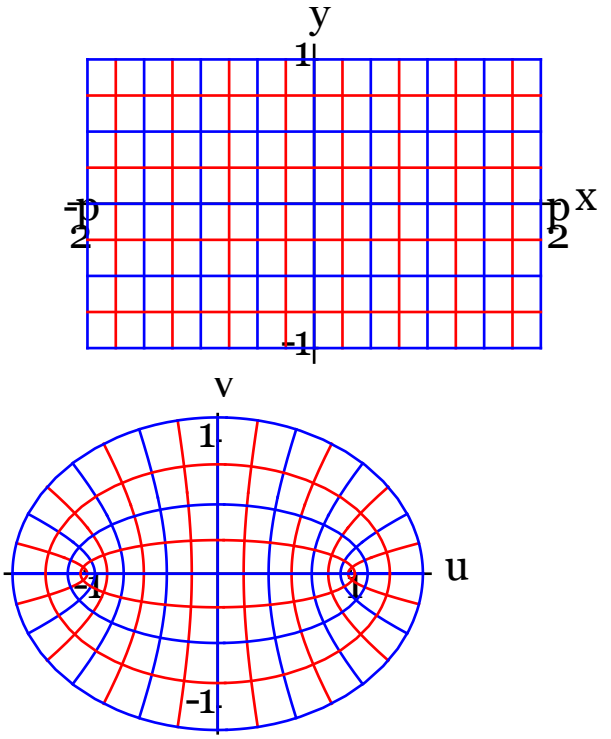

Figure 5.A The mapping w = u + i v = sin z.

These equations in turn are used to obtain the following important identities

(5-34) $$\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$

(5-35) $$\cos z = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

Proof.

Subtracting Equation (5-31) from Equation (5-30) and solving for sin z gives

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)$$

$$= \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right)$$

$$= \frac{1}{2i}\left(e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)\right)$$

$$= \sin x \frac{e^{y}+e^{-y}}{2} + i \cos x \frac{e^{y}-e^{-y}}{2}$$

$$= \sin x \cosh y + i \cos x \sinh y$$

where $\cosh y = \frac{e^{y}+e^{-y}}{2}$ and $\sinh y = \frac{e^{y}-e^{-y}}{2}$, respectively, are the hyperbolic cosine and hyperbolic sine functions that you studied in calculus.

Similarly

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)$$

$$= \frac{1}{2}\left(e^{-y+ix} + e^{y-ix}\right)$$

$$= \frac{1}{2}\left(e^{-y}(\cos x + i \sin x) + e^{y}(\cos x - i \sin x)\right)$$

$$= \cos x \frac{e^{y}+e^{-y}}{2} - i \sin x \frac{e^{y}-e^{-y}}{2}$$

$$= \cos x \operatorname{Cosh} y - i \sin x \sinh y$$

Exploration for the real and imaginary parts of Sin and Cos.

Equipped with Identities (5-32) - (5-35), we can now establish many other properties of the trigonometric functions. We begin with some periodic results. For all complex numbers $z = x + i y$,

$$\sin(z + 2\pi) = \sin z,$$

$$\cos(z + 2\pi) = \cos z,$$

$$\sin(z + \pi) = -\sin z,$$

$$\cos(z + \pi) = -\cos z,$$

$$\tan(z + \pi) = \tan z,$$

$$\cot(z + \pi) = \cot z.$$

Clearly, $\sin(z + 2\pi) = \sin((x + 2\pi) + i y)$. By Identity (5-34) this expression is

$$\sin(z + 2\pi) = \sin(x + 2\pi)\cosh y + i\cos(x + 2\pi)\sinh y$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$= \sin z$$

Again, the proofs for the other periodic results are left as exercises.

Exploration for trigonometric identities.

If z, z_1 and z_2 are any complex numbers, then

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$
$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2,$$

$$\sin(2z) = 2 \sin z \cos z,$$

$$\cos(2z) = (\cos z)^2 - (\sin z)^2,$$

$$\sin\left(\frac{\pi}{2} + z\right) = \sin\left(\frac{\pi}{2} - z\right) = \cos z.$$

We demonstrate that $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2$ by making use of Identities (5-32) - (5-35):

$$\cos z_1 \cos z_2 = \frac{1}{4}\left(e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}\right), \text{ and}$$

$$-\sin z_1 \sin z_2 = \frac{1}{4}\left(e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}\right)$$

Adding these expressions gives

$$\cos z_1 \cos z_2 - \sin z_1 \sin z_2 = \frac{1}{2}\left(e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}\right) = \cos(z_1 + z_2)$$

which is what we wanted.

Demonstration. Establish the identity $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$.

Exploration Method (i).

Exploration Method (ii).

A solution to the equation $f(z) = 0$ is called a zero of the given function $f(z)$. As we now show, the zeros of the sine and cosine function are exactly where you might expect them to be. We have $\sin z = 0$ iff $z = n\pi$, where n is any integer, and $\cos z = 0$ iff $z = \left(n + \frac{1}{2}\right)\pi$, where n is any integer.

We show the result for cos z and leave the result for sin z as an exercise. When we use Identity (5-35), cos z = 0 iff

$$0 = \cos x \cosh y - i \sin x \sinh y.$$

Equating the real and imaginary parts of this equation gives

$$0 = \cos x \cosh y \quad \text{and} \quad 0 = \sin x \sinh y.$$

The real-valued function cosh y is never zero, so the equation $0 = \cos x \cosh y$ implies that $0 = \cos x$, from which we obtain $x = \left(n + \frac{1}{2}\right) \pi$ for any integer n. Using the values for $z = x + i y = x = \left(n + \frac{1}{2}\right) \pi + i y$ in the equation $0 = \sin x \sinh y$ yields

$$0 = \sin\left(\left(n + \frac{1}{2}\right) \pi\right) \sinh y = (-1)^n \sinh y.$$

which implies that $y = 0$, so the only zeros for cos z are the values $z = \left(n + \frac{1}{2}\right) \pi$ for n an integer.

Exploration.

What does the mapping $w = \sin z$ look like? We can get a graph of the mapping $w = \sin z = \sin(x + i y) = \sin x \cosh y + i \cos x \sinh y$ by using parametric methods. Let's consider the vertical line segments in the z plane obtained by successfully setting $x = -\frac{\pi}{2} + \frac{k \pi}{12}$ for $k = 0, 1, \ldots, 12$, and for each x value and letting y vary continuously, $-3 \le y \le 3$. In the exercises we ask you to show that the images of these vertical segments are hyperbolas in the uv plane, as Figure 5.7 illustrates. In Section 10.4, we give a more detailed analysis of the mapping $w = \sin z$.

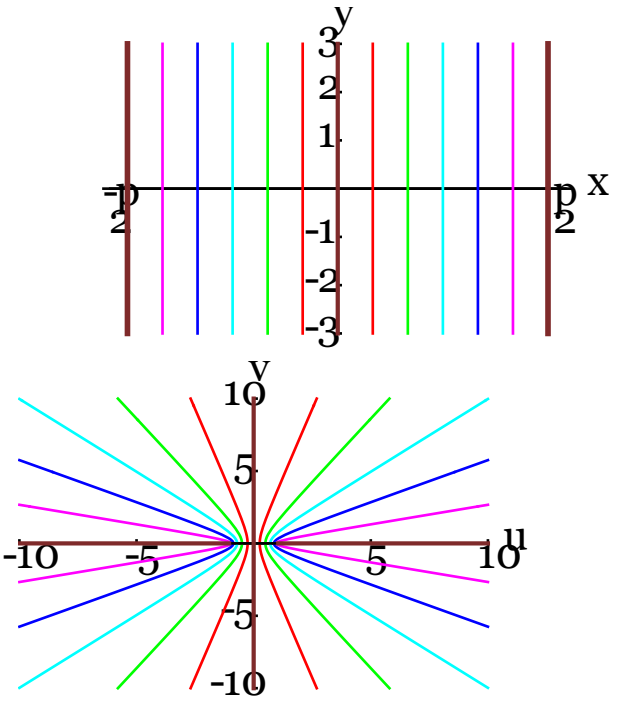

Figure 5.7 Vertical segments mapped onto hyperbolas by $w = \sin z$.

Exploration.

Figure 5.7 suggests one big difference between the real and complex sine functions. The real sine has the property that $|\sin x| \le 1$ for all real x. In Figure 5.7, however, the modulus of the complex sine appears to be unbounded, which is indeed the case. Using Identity (5-34) gives

$$|\sin z|^2 = |\sin(x + i y)|^2$$

$$= |\sin x \cosh y + i \cos x \sinh y|^2$$

$$= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x \left(\cosh^2 y - \sinh^2 y\right) + \sinh^2 y \left(\cos^2 x + \sin^2 x\right)$$

The identities $\cosh^2 y - \sinh^2 y = 1$ and $\cos^2 x + \sin^2 x = 1$ then yield

(5-36) $$|\sin z|^2 = \sin^2 x + \sinh^2 y.$$

A similar derivation produces

(5-37) $$|\cos z|^2 = \cos^2 x + \sinh^2 y.$$

If we set $z = x_0 + i y$ in Identity (5-36) and let $y \to \infty$, we get

$$\lim_{y\to\infty} |\sin(x_0 + i y)|^2 = \sin^2 x_0 + \lim_{y\to\infty} \sinh^2 y = \infty$$

As advertised, we have shown that $\sin z$ is not a bounded function; it is also evident from Identity (5-37) that $\cos z$ is unbounded.

Exploration.

The periodic character of the trigonometric functions makes apparent that any point in their ranges is actually the image of infinitely many points.

Example 5.10. Find all the values of z for which $\cos z = \cosh 2$.

Solution. Starting with Identity (5-35), we write

$$\cos z = \cos x \cosh y - i \sin x \sinh y = \cosh 2.$$

If we equate real and imaginary parts, then we get

$$\cos x \cosh y = \cosh 2 \quad \text{and} \quad \sin x \sinh y = 0.$$

The equation $\sin x \sinh y = 0$ implies either that $x = n\pi$, where n is an integer, or that $y = 0$. Using $y = 0$ in the equation $\cos x \cosh y = \cosh 2$ leads to the impossible situation $\cos x = \frac{\cosh 2}{\cosh 0} = \cosh 2 > 1$. Therefore $x = n\pi$, where n is an integer. Since $\cosh y \ge 1$ for all values of y, the term $\cos x$ in the equation $\cos x \cosh y = \cosh 2$ must also be positive. For this

reason we eliminate the odd values of n and get $x = 2k\pi$, where k is an integer.

Finally, we solve the equation $\cos(2k\pi)\cosh y = \cosh y = \cosh 2$ and use the fact that $\cosh y$ is an even function to conclude that $y = \pm 2$. Therefore the solutions to the equation $\cos z = \cosh 2$ are $z = 2\pi k \pm 2i$, where k is an integer.

Explore Solution 5.10.

The hyperbolic functions also have practical use in putting the tangent function into the Cartesian form $u + iv$.
Using Definition 5.6, and Equations (5-34) and (5-35), we have

(5-38) $$\tan z = \tan(x + iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}.$$

If we multiply each term on the right by the conjugate of the denominator, the simplified result is

$$\tan z = \frac{\cos x \sin x + i \cosh y \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

We leave it as an exercise to show that the identities $\cosh^2 y - \sinh^2 y = 1$ and $\sinh 2y = 2\cosh y \sinh y$ can be used in simplifying Equation (5-38) to get

(5-39) $$\tan z = \frac{\sin(2x)}{\cos(2x)+\cosh(2y)} + i\frac{\sinh(2y)}{\cos(2x)+\cosh(2y)}$$

Exploration.

As with $\sin z$, we obtain a graph of the mapping $w = \tan z$ parametrically. Consider the vertical line segments in the z plane obtained by successively setting $x = -\frac{\pi}{4} + \frac{k\pi}{16}$ for $k = 0, 1, \ldots, 8$, and for each z value letting y vary continuously, $-3 \le y \le 3$. In the exercises we ask you to show that the images of these vertical segments are circular arcs in the uv plane, as Figure 5.8 shows. In Section 10.4, we give a more detailed investigation of the mapping $w = \tan z$.

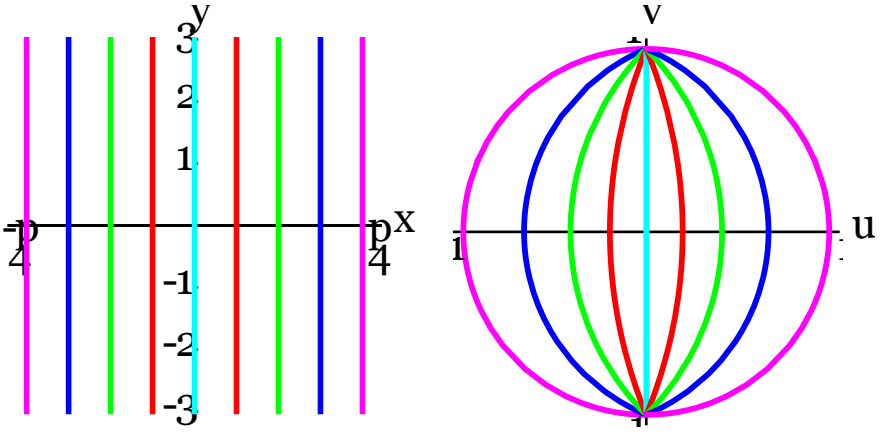

Figure 5.8 Vertical segments mapped onto circular arcs by $w = \tan z$.

Exploration.

How should we define the complex hyperbolic functions? We begin with

Definition 5.7. The hyperbolic cosine and hyperbolic sine functions are

$$\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$$

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)$$

With these definitions in place, we can now easily create the other complex hyperbolic trigonometric functions, provided the denominators in the following expressions are not zero.

Definition 5.8. Identities for the hyperbolic trigonometric functions are

$$\tanh z = \frac{\sinh z}{\cosh z},$$
$$\coth z = \frac{\cosh z}{\sinh z},$$

$$\operatorname{sech} z = \frac{1}{\cosh z},$$

$$\operatorname{csch} z = \frac{1}{\sinh z}.$$

As the series for the complex hyperbolic sine and cosine agree with the real hyperbolic sine and cosine when z is real, the remaining complex hyperbolic trigonometric functions likewise agree with their real counterparts. Many other properties are also shared. We state several results without proof, as they follow from the definitions we gave using standard operations, such as the quotient rule for derivatives. We ask you to establish some of these identities in the exercises.

The derivatives of the hyperbolic functions follow the same rules as in calculus:

$$\frac{d}{dz}\cosh z = \sinh z, \quad \text{and} \quad \frac{d}{dz}\operatorname{Sinh} z = \cosh z,$$

$$\frac{d}{dz}\operatorname{Tanh} z = \operatorname{sech}^2 z, \quad \text{and} \quad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z,$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \operatorname{Tanh} z, \quad \text{and} \quad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z.$$

The hyperbolic cosine and hyperbolic sine can be expressed as

$$\cosh z = \cosh(x + i\,y) = \cos y \cosh x + i \sinh x \sin y$$

$$\sinh z = \sinh(x + i\,y) = \sinh x \cos y + i \cosh x \sin y$$

Some of the important identities involving the hyperbolic functions are

$$\cosh^2 z - \sinh^2 z = 1,$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2,$$

$$\cosh(z + 2\pi i) = \cosh z,$$

$$\sinh(z + 2\pi i) = \sinh z,$$

$$\cosh(-z) = \cosh z,$$

$$\sinh(-z) = -\sinh z.$$

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering
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5.5 Inverse Trigonometric and Hyperbolic Functions

We expressed trigonometric and hyperbolic functions in Section 5.4 in terms of the exponential function. In this section we look at their inverses. When we solve equations such as $w = \sin z$ for z, we will obtain formulas that involve the logarithm. Because trigonometric and hyperbolic functions are all periodic, they are many-to-one; hence their inverses are necessarily multivalued. The formulas for the inverse trigonometric functions are

(5 – 45)
$$\arcsin z = -i \log\left(i z + (1 - z^2)^{1/2}\right),$$

$$\arccos z = -i \log\left(z + i (1 - z^2)^{1/2}\right),$$

$$\arctan z = \frac{i}{2} \log\left(\frac{i + z}{i - z}\right).$$

We derive the first equation and leave the others as exercises. If we take a particular branch of the multivalued function, $w = \arcsin z$, we have

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

which we can also write as

$$e^{iw} - 2iz - e^{-iw} = 0.$$

Multiplying both sides of this equation by e^{iw} gives

$$(e^{iw})^2 - 2iz e^{iw} - 1 = 0,$$

which is a quadratic equation in terms of e^{iw}. Using the quadratic equation to solve for e^{iw}, we obtain

$$e^{i w} = \frac{2 i z + \left(-4 z^2+4\right)^{1/2}}{2} = i z + (1 - z^2)^{1/2},$$

where the square root is a multivalued function. Taking the logarithm of both sides and obtain $i w = \log\left(i z + (1 - z^2)^{1/2}\right)$, then simplify this and obtain the desired result:

$$w = \arcsin z = -i \log\left(i z + (1 - z^2)^{1/2}\right),$$

where the multivalued logarithm is used. To construct a specific branch of `arcsin z`, we must first select a branch of the square root and then select a branch of the logarithm.

Remark. The function `arcsin z` is known to be an odd function and therefore `arcsin (-z) = arcsin z`. As an immediate consequence we have

$$-i \log\left(-i z + (1 - z^2)^{1/2}\right) = i \log\left(i z + (1 - z^2)^{1/2}\right).$$

We can now use the trigonometric identity $\arccos z = \frac{\pi}{2} - \arcsin z$ and obtain an alternate expression for `arccos z`:

$$\begin{aligned}
\arccos z &= \frac{\pi}{2} - \arcsin z \\
&= \frac{\pi}{2} + i \log\left(i z + (1 - z^2)^{1/2}\right) \\
&= \frac{\pi}{2} - i \log\left(-i z + (1 - z^2)^{1/2}\right) \\
&= -i \frac{\pi i}{2} - i \log\left(-i z + (1 - z^2)^{1/2}\right) \\
&= -i \log(i) - i \log\left(-i z + (1 - z^2)^{1/2}\right) \\
&= -i \left(\log(i) + \log\left(-i z + (1 - z^2)^{1/2}\right)\right) \\
&= -\log\left(i (-i) z + i (1 - z^2)^{1/2}\right) \\
&= -\log\left(z + i (1 - z^2)^{1/2}\right)
\end{aligned}$$

We mention this derivation because *Mathematica* uses the second line for determining the principal value, i.e.

$$\arccos z = \frac{\pi}{2} + i \log\left(i z + (1 - z^2)^{1/2}\right)$$

an we will see shortly that `ArcCos[z] = Pi/2 + I Log[I z + Sqrt[1 - z^2]]`.

The principal value used by *Mathematica* for the inverse trigonometric functions are

$$\text{ArcSin } z = -i \text{ Log}\left[i z + (1 - z^2)^{1/2}\right],$$

$$\text{ArcCos } z = \frac{\pi}{2} + i \text{ Log}\left[i z + (1 - z^2)^{1/2}\right],$$

$$\text{ArcTan } z = \frac{i}{2} \text{ Log}\left[\frac{i + z}{i - z}\right].$$

Exploration

When the principal value is used, $w = \text{Arcsin } z = -i \text{ Log}\left[i z + (1 - z^2)^{1/2}\right]$ maps the upper half-plane $\{z : \text{Im}(z) > 0\}$ onto a portion of the upper half-plane $\{z : \text{Im}(w) > 0\}$ that lies in the vertical strip $\left\{w : \frac{-\pi}{2} < \text{Re}(w) < \frac{\pi}{2}\right\}$. The image of a rectangular grid in the z plane is a "spider web" in the w plane, as Figure 5.10 shows.

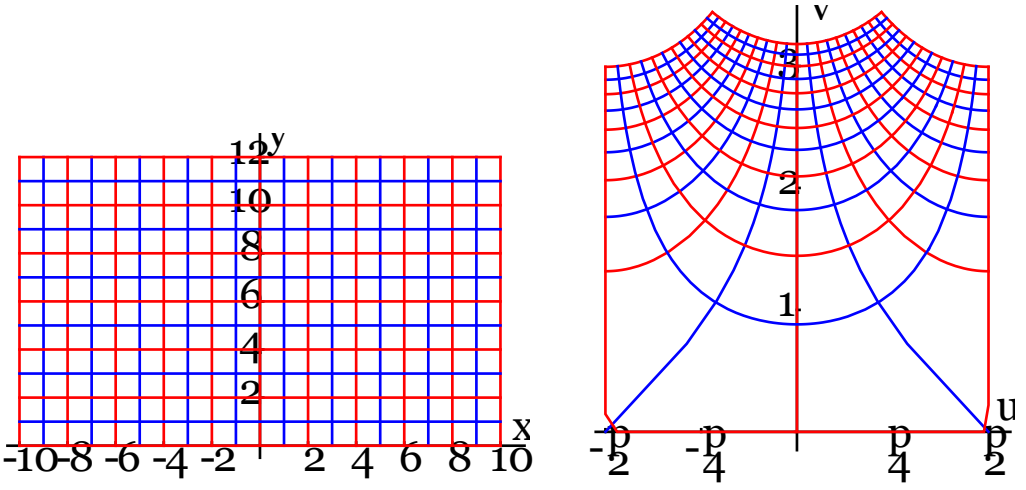

Figure 5.10 A rectangular grid is mapped onto a spider web by $w = \text{Arcsin } z$.

The derivatives of arcsin(z), arccos(z), arctan(z)

We can find the derivatives of any branch of these functions by using the chain rule:

(5 – 46)

$$\frac{d}{dz} \arcsin z = \frac{1}{\sqrt{1 - z^2}},$$

$$\frac{d}{dz} \arccos z = \frac{-1}{\sqrt{1 - z^2}},$$

$$\frac{d}{dz} \arctan z = \frac{1}{1 + z^2}.$$

Exploration.

We get the derivative of $w = \arcsin z$ by starting with the equation $\sin w = z$ and differentiating both sides, using the chain rule:

$$\frac{d}{dz}\sin w = \frac{d}{dz} z$$

$$\frac{d}{dw}\sin w \frac{dw}{dz} = 1$$

$$\cos w \frac{dw}{dz} = 1$$

$$\frac{dw}{dz} = \frac{1}{\cos w}$$

Then $\cos^2 w + \sin^2 w = 1$ is used to get the substitution $\cos w = \sqrt{1 - \sin^2 w} = \sqrt{1 - z^2}$ for the preceding equation, and we obtain the desired result:

$$\frac{d}{dz}\arcsin z = \frac{dw}{dz} = \frac{1}{\sqrt{1 - z^2}}.$$

Derivation of the other formulas are left as exercises for the reader.

The Inverse Sine arcsin(z). Verify that the formula

(i) $$\text{Arcsin } z = -i \text{ Log}\left[i z + \sqrt{1 - z^2}\right]$$

is correct. (At least for values of z in the upper half plane $\text{Im}(z) > 0$.)

Explore Formula (ii).

The Inverse Cosine arccos(z). Verify that the formula(s)

(ii a) $$\text{Arccos } z = -i \text{ Log}\left[z + i\sqrt{1 - z^2}\right],$$

(ii a) $$\text{Arccos } z = \frac{\pi}{2} + i \text{ Log}\left[i z + (1 - z^2)^{1/2}\right].$$

are correct. (At least for values of z in the upper half plane $\text{Im}(z) > 0$.)

Explore Formula (ii a).

Explore Formula (ii b).

The Inverse Tangent arctan(z). Verify that the formula

(iii) $$\text{Arctan } z = \frac{i}{2} \text{ Log}\left[\frac{i + z}{i - z}\right]$$

is correct. (At least for values of z in the upper half plane $\text{Im}(z) > 0$.)

Explore Formula (iii).

When the principal value is used, $w = \text{Arctan } z = \frac{i}{2} \text{ Log}\left[\frac{i + z}{i - z}\right]$ maps the upper half-plane $\{z : \text{Im}(z) > 0\}$ onto a portion of the upper half-plane $\{z : \text{Im}(w) > 0\}$ that lies in the

vertical strip $\left\{w : \frac{-\pi}{2} < \text{Re}(w) < \frac{\pi}{2}\right\}$.

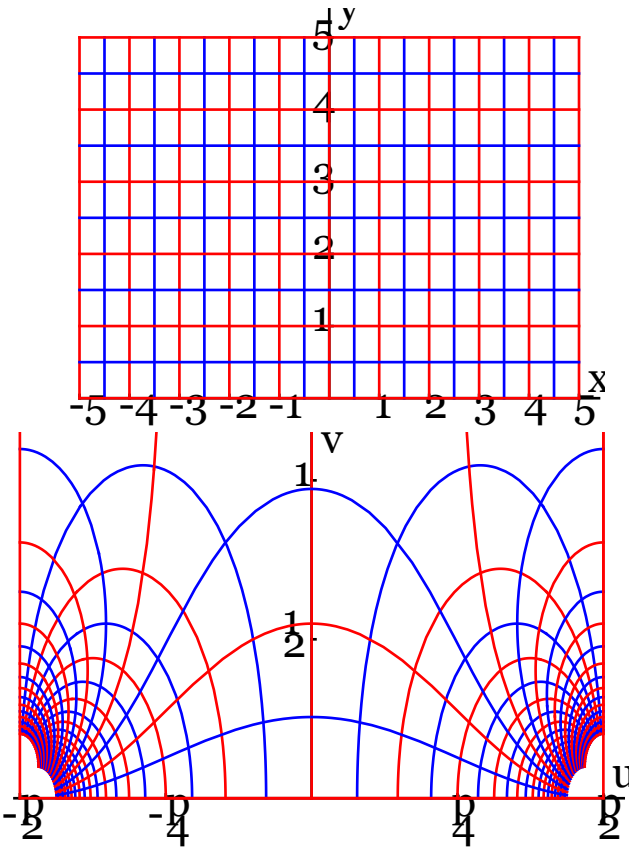

Figure 5.B The mapping $w = \text{Arctan}\, z$.

Example 5.11. The values of $\arcsin\left(\sqrt{2}\right)$ are given by

$$\arcsin\sqrt{2} = -i\log\left[i\sqrt{2} + \left(1-\left(\sqrt{2}\right)^2\right)^{1/2}\right] = -i\log\left[i\sqrt{2} \pm i\right].$$

Using straightforward techniques, we simplify this equation and obtain

(5 – 47)
$$\begin{aligned}\arcsin\sqrt{2} &= -i\log\left[i\left(\sqrt{2}\pm 1\right)\right]\\ &= -i\left[\ln\left(\sqrt{2}\pm 1\right) + i\left(\frac{\pi}{2}+2n\pi\right)\right]\\ &= \frac{\pi}{2}+2n\pi - i\ln\left(\sqrt{2}\pm 1\right),\end{aligned}$$

where n is an integer. We observe that

$$\ln\left(\sqrt{2}-1\right) = \ln\frac{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}{\left(\sqrt{2}+1\right)} = \ln\frac{1}{\left(\sqrt{2}+1\right)} = -\ln\left(\sqrt{2}+1\right)$$

and then write

$$\arcsin\sqrt{2} = \frac{\pi}{2}+2n\pi \pm i\ln\left(\sqrt{2}+1\right), \quad \text{where } n \text{ is an integer.}$$

Explore Solution 5.11.

Example 5.12. Suppose that we make specific choices in equation (5-47) by selecting $+i$ as the value of the square root $\left(1-\left(\sqrt{2}\right)^2\right)^{1/2}$ and using the principal value of the logarithm. With $f(z) = \text{Arcsin}\, z$, the result is

$$f\left(\sqrt{2}\right) = \text{Arcsin}\,\sqrt{2} = -i\,\text{Log}\left[i\sqrt{2}+i\right] = \frac{\pi}{2} - i\,\ln\left(\sqrt{2}+1\right),$$

and the corresponding value of the derivative is given by

$$f'\left(\sqrt{2}\right) = \frac{1}{\sqrt{1-\left(\sqrt{2}\right)^2}} = \frac{1}{i} = -i.$$

Explore Solution 5.12.

The Inverse Hyperbolic Functions

The inverse hyperbolic functions are

(5 – 48)

$$\text{arcsinh}\, z = \log\left[z+(z^2+1)^{1/2}\right],$$

$$\text{arccosh}\, z = \log\left[z+(z^2-1)^{1/2}\right],$$

$$\text{arctanh}\, z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

To establish the last identity given above, we start with $w = \text{arctanh}\, z$ and obtain

$$z = \tanh w = \frac{e^w - e^{-w}}{e^w + e^{-w}} = \frac{e^{2w}-1}{e^{2w}+1}$$

which we solve for e^{2w}, getting $e^{2w} = \frac{1+z}{1-z}$. Taking the logarithms of both sides gives

$$w = \text{arctanh}\, z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

which is what we wanted to show.

The Inverse Hyperbolic Sine arcsinh(z). Verify that the formula

(iv)

$$\text{Arcsinh}\, z = \text{Log}\left[z+\sqrt{z^2+1}\right]$$

is correct, we can verify this graphically. (At least for values of z in the upper half plane first quadrant.)

Explore Formula (iv) .

The Inverse Hyperbolic Cosine arccosh(z) . Verify that the formula

(v) $\text{Arccosh}\, z = \text{Log}\left[z + \sqrt{z^2 - 1}\right]$

is correct, we can verify this graphically. But it is correct only in quadrants I and IV.

Explore Formula (v) .

The Inverse Hyperbolic Tangent arctanh(z) . Verify that the formula

(vi) $\text{Arctanh}\, z = \frac{1}{2}\,\text{Log}\left[\frac{1+z}{1-z}\right]$

is correct, we can verify this graphically.

Explore Formula (vi) .

The derivatives of the inverse hyperbolic functions are

$$\frac{d}{dz}\,\text{arcsinh}\, z = \frac{1}{\sqrt{1+z^2}},$$

$$\frac{d}{dz}\,\text{arccosh}\, z = \frac{1}{\sqrt{z^2-1}},$$

$$\frac{d}{dz}\,\text{arctanh}\, z = \frac{1}{1-z^2}.$$

Exploration.

Example 5.13. Calculation reveals that

$$\begin{aligned}\text{arctanh}\,(1+2\,i) &= \frac{1}{2}\log\left(\frac{1+1+2\,i}{1-1-2\,i}\right)\\ &= \frac{1}{2}\log\left(\frac{2+2\,i}{-2\,i}\right)\\ &= \frac{1}{2}\log\,(-1+i)\\ &= \frac{1}{4}\ln 2 + i\left(\frac{3}{8}+n\right)\pi\end{aligned}$$

where n is an integer.

Explore Solution 5.13.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

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Chapter 6 Complex Integration

Overview

Of the two main topics studied in calculus - differentiation and integration - we have so far only studied derivatives of complex functions. We now turn to the problem of integrating complex functions. The theory you will learn is elegant, powerful, and a useful tool for physicists and engineers. It also connects widely with other branches of mathematics. For example, even though the ideas presented here belong to the general area of mathematics known as analysis, you will see as an application of them one of the simplest proofs of the fundamental theorem of algebra.

6.1 Complex Integrals

In Chapter 3 we saw how the derivative of a complex function is defined. We now turn our attention to the problem of integrating complex functions. We will find that integrals of analytic functions are well behaved and that many properties from calculus carry over to the complex case.

We introduce the integral of a complex function by defining the integral of a complex-valued function of a real variable

Definition 6.1 (Definite Integral of a Complex Integrand). Let $f(t) = u(t) + i\,v(t)$ where $u(t)$ and $v(t)$ are real-valued functions of the real variable t for $a \le t \le b$. Then

(6-1) $$\int_a^b f(t)\,dt = \int_a^b (u(t) + i\,v(t))\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

We generally evaluate integrals of this type by finding the antiderivatives of $u(t)$ and $v(t)$ and evaluating the definite integrals on the right side of Equation (6-1). That is, if $U'(t) = u(t)$ and $V'(t) = v(t)$, we have

(6-2) $$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt = U(b) - U(a) + i\,(V(b) - V(a)).$$

Example 6.1. Show that $\int_0^1 (t - i)^3\,dt = -\frac{5}{4}$.

Solution. We write the integrand in terms of its real and imaginary parts, i.e., $f(t) = (t - i)^3 = t^3 - 3t + i(-3t^2 + 1)$. Here, $u(t) = t^3 - 3t$ and $v(t) = -3t^2 + 1$. The integrals of $u(t)$ and $v(t)$ are

$$\int_0^1 u(t)\,dt = \int_0^1 (t^3 - 3t)\,dt = \left(\frac{t^4}{4} - \frac{3t^2}{2}\right)\Big|_{t=0}^{t=1} = -\frac{5}{4}, \text{ and}$$

$$\int_0^1 v(t)\,dt = \int_0^1 (-3t^2 + 1)\,dt = (-t^3 + t)\Big|_{t=0}^{t=1} = 0.$$

Hence, by Definition (6-1),

$$\int_0^1 (t - i)^3\,dt = \int_0^1 u(t)\,dt + i\int_0^1 v(t)\,dt$$

$$= -\frac{5}{4} + 0\,i = -\frac{5}{4}$$

Explore Solution 6.1.

Example 6.2. Show that $\int_0^{\pi/2} e^{t+it}\,dt = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right)$.

Solution. We use the method suggested by Definitions (6-1) and (6-2).

$$\int_0^{\pi/2} e^{t+it}\,dt = \int_0^{\pi/2} e^t e^{it}\,dt = \int_0^{\pi/2} e^t(\cos t + i \sin t)\,dt$$

$$= \int_0^{\pi/2} e^t \cos t\,dt + i\int_0^{\pi/2} e^t \sin t\,dt$$

We can evaluate each of the integrals via integration by parts. For example,

$$\int_0^{\pi/2} e^t \cos t\,dt = (e^t \sin t)\Big|_{t=0}^{t=\pi/2} - \int_0^{\pi/2} \sin t\, e^t\,dt$$

$$= \left(e^{\pi/2}\sin\frac{\pi}{2} - e^0 \sin 0\right) - \int_0^{\pi/2} \sin t\, e^t\,dt$$

$$= (e^{\pi/2} 1 - 1 * 0) - \int_0^{\pi/2} e^t \sin t\,dt$$

$$= e^{\pi/2} - \int_0^{\pi/2} e^t \sin t\,dt$$

$$= e^{\pi/2} - (e^t(-\cos t))\Big|_{t=0}^{t=\pi/2} + \int_0^{\pi/2} (-\cos t\, e^t)\,dt$$

$$= e^{\pi/2} + (e^t \cos t)\Big|_{t=0}^{t=\pi/2} - \int_0^{\pi/2} e^t \cos t\,dt$$

$$= e^{\pi/2} + e^{\pi/2} * 0 - 1 * 1 - \int_0^{\pi/2} e^t \cos t\,dt$$

$$= e^{\pi/2} - 1 - \int_0^{\pi/2} e^t \cos t\,dt$$

Adding $\int_0^{\pi/2} e^t \cos t \, dt$ to both sides of this equation and then dividing by 2 gives $\int_0^{\pi/2} e^t \cos t \, dt = \frac{1}{2}(e^{\pi/2} - 1)$. Likewise, $i \int_0^{\pi/2} e^t \sin t \, dt = i \frac{1}{2}(e^{\pi/2} + 1)$. Therefore,

$$\int_0^{\pi/2} e^{t+i\,t} \, dt = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right).$$

Explore Solution 6.2.

Complex integrals have properties that are similar to those of real integrals. We now trace through several commonalities. Let $f(t) = u(t) + i\,v(t)$ and $g(t) = p(t) + i\,q(t)$ be continuous on $a \le t \le b$.

Using Definition (6-1), we can easily show that the integral of their sum is the sum of their integrals, that is

(6-3) $$\int_a^b (f(t) + g(t)) \, dt = \int_a^b f(t) \, dt + \int_a^b g(t) \, dt.$$

If we divide the interval $a \le t \le b$ into $a \le t \le c$ and $c \le t \le b$ and integrate $f(t)$ over these subintervals by using (6-1), then we get

(6-4) $$\int_a^b f(t) \, dt = \int_a^c f(t) \, dt + \int_c^b f(t) \, dt.$$

Similarly, if $\alpha + i\,\beta$ denotes a complex constant, then

(6-5) $$\int_a^b (c + i\,d) f(t) \, dt = (c + i\,d) \int_a^b f(t) \, dt.$$

If the limits of integration are reversed, then

(6-6) $$\int_a^b f(t) \, dt = -\int_b^a f(t) \, dt.$$

The integral of the product $f(t)g(t)$ becomes

(6-7)
$$\int_a^b f(t)\, g(t) \, dt = \int_a^b (u(t) + i\,v(t))\,(p(t) + i\,q(t)) \, dt$$

$$= \int_a^b (u(t)\,p(t) - v(t)\,q(t)) \, dt + i \int_a^b (u(t)\,q(t) + v(t)\,p(t)) \, dt$$

Example 6.3. Let us verify property (6-5). We start by writing

$$(c + i\,d)\, f(t) = (c + i\,d)\,(u(t) + i\,v(t))$$

$$= c\,u(t) - d\,v(t) + i\,(c\,v(t) + d\,u(t))$$

Using Definition (6-1), we write the left side of Equation (6-5) as

$$\int_a^b (c + i\,d)\,(u(t) + i\,v(t))\,dt$$

$$= c\int_a^b u(t)\,dt - d\int_a^b v(t)\,dt + i\,c\int_a^b v(t)\,dt + i\,d\int_a^b u(t)\,dt$$

which is equivalent to

$$\int_a^b (c + i\,d)\,(u(t) + i\,v(t))\,dt$$

$$= c\int_a^b u(t)\,dt + i\,d\int_a^b u(t)\,dt + c\,i\int_a^b v(t)\,dt + i\,d\,i\int_a^b v(t)\,dt$$

$$= (c + i\,d)\int_a^b u(t)\,dt + (c + i\,d)\,i\int_a^b v(t)\,dt$$

$$= (c + i\,d)\left(\int_a^b u(t)\,dt + i\int_a^b v(t)\,dt\right)$$

Therefore, $\int_a^b (c + i\,d)\,f(t)\,dt = (c + i\,d)\int_a^b f(t)\,dt$.

Explore Solution 6.3.

It is worthwhile to point out the similarity between equation (6-2) and its counterpart in calculus. Suppose that U and V are differentiable on $a < t < b$ and $F(t) = U(t) + i\,V(t)$. Since $F'(t) = U'(t) + i\,V'(t) = u(t) + i\,v(t) = f(t)$, equation (6-2) takes on the familiar form

(6-8) $$\int_a^b f(t)\,dt = F(t)\Big|_{t=a}^{t=b} = F(b) - F(a).$$

where $F'(t) = f(t)$. We can view Equation (6-8) as an extension of the fundamental theorem of calculus. In Section 6.5 we show how to generalize this extension to analytic functions of a complex variable. For now, we simply note an important case of Equation (6-8):

(6-9) $$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Example 6.4. Use Equation (6-8) to show that $\int_0^{\pi/2} e^{t+i\,t}\,dt = \frac{1}{2}(e^{\pi/2} - 1) + \frac{i}{2}(e^{\pi/2} + 1)$.

Solution. We seek a function F with the property that $F'(t) = e^{(1+i)\,t}$. We note that $F(t) = \frac{1}{1+i}\,e^{(1+i)\,t}$ satisfies this requirement, so

$$\int_0^{\pi/2} e^{t+i\,t}\,dt = \left(\frac{1}{1+i}\,e^{(1+i)\,t}\right)\Big|_{t=0}^{t=\pi/2} = \frac{1}{1+i}\,e^{(1+i)\,\pi/2} - \frac{1}{1+i}\,e^{0}$$

$$= \frac{1}{1+i}\,e^{\pi/2}\,e^{i\,\pi/2} - \frac{1}{1+i} = \frac{1}{1+i}\,e^{\pi/2}\,i - \frac{1}{1+i}$$

$$= \frac{1}{1+i}\,(i\,e^{\pi/2} - 1) = \frac{1}{2}\,(1 - i)\,(-1 + i\,e^{\pi/2})$$

$$= \frac{1}{2}\,(e^{\pi/2} - 1) + \frac{i}{2}\,(e^{\pi/2} + 1)$$

which is the same result we obtained in Example 6.2, but with a lot less work.

Explore Solution 6.4.

Remark 6.1 Example 6.4 illustrates the potential computational advantage we have when we lift our sights to the complex domain. Using ordinary calculus techniques to evaluate $\int_0^{\pi/2} e^t \cos t \, dt$, for example, required a lengthy integration by parts procedure (Example 6.2). When we recognize this expression as the real part of $\int_0^{\pi/2} e^{t+it} \, dt$, however, the solution comes quickly. This is just one of the many reasons why good physicists and engineers, in addition to mathematicians, benefit from a thorough working knowledge of complex analysis.

Extra Example 1. Show that $\int_0^{\pi} e^{it} \, dt = 2i$.

Explore Solution for Extra Example 1.

Complex Analysis for Mathematics and Engineering

by

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6.2 Contours and Contour Integrals

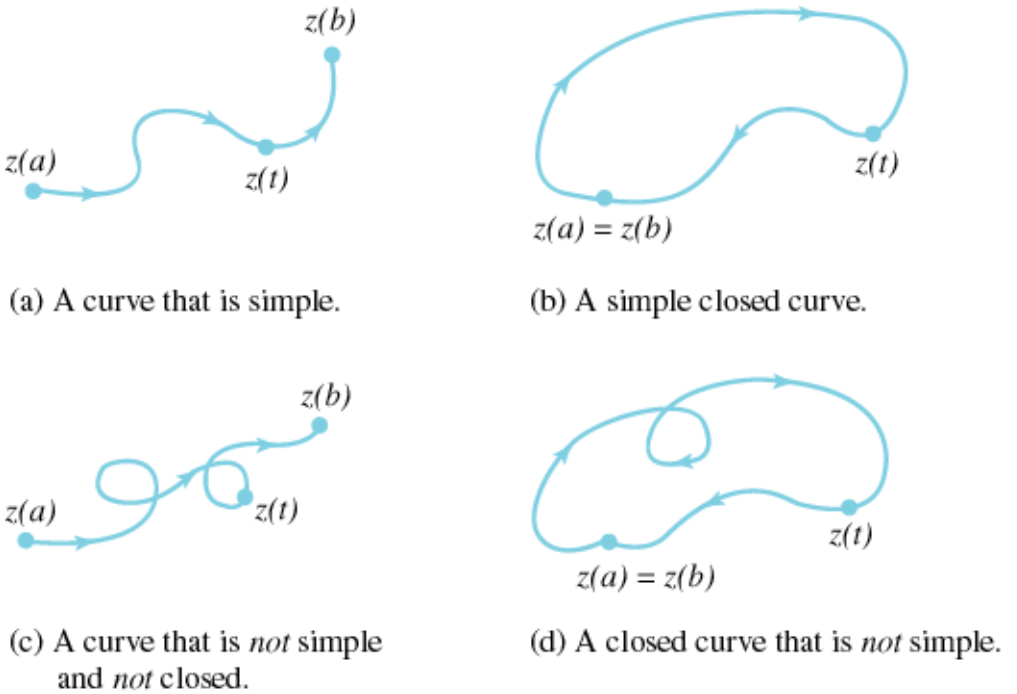
In Section 6.1 we learned how to evaluate integrals of the form $\int_a^b f(t)\,dt$, where $f(t)$ was complex-valued and $[a, b]$ was an interval on the real axis (so that t was real, with $t \in [a, b]$). In this section, we define and evaluate integrals of the form $\int_C f(z)\,dz$, where $f(t)$ is complex-valued and C is a contour in the plane (so that z is complex, with $z \in C$). Our main result is Theorem 6.1, which shows how to transform the latter type of integral into the kind we investigated in Section 6.1.

We will use concepts first introduced in Section 1.6. Recall that to represent a curve C we used the parametric notation

(6-10) $$C: z(t) = x(t) + i\,y(t) \quad \text{for } a \le t \le b,$$

where $x(t)$ and $y(t)$ are continuous functions. We now place a few more restrictions on the type of curve to be described. The following discussion leads to the concept of a contour, which is a type of curve that is adequate for the study of integration.

Recall that C is simple if it does not cross itself, which means that $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$, except possibly when $t_1 = a$ and $t_2 = b$. A curve C with the property that $z(a) = z(b)$ is a closed curve. If $z(a) = z(b)$ is the only point of intersection, then we say that C is a simple closed curve. As the parameter t increases from the value $t = a$ to the value $t = b$, the point $z(t)$ starts at the initial point $z(a)$, moves along the curve C, and ends up at the terminal point $z(b)$. If C is simple, then $z(t)$ moves continuously from $z(a)$ to $z(b)$ as t increases and the curve is given an orientation, which we indicate by drawing arrows along the curve. Figure 6.1 illustrates how the terms simple and closed describe a curve.



(a) A curve that is simple. (b) A simple closed curve.

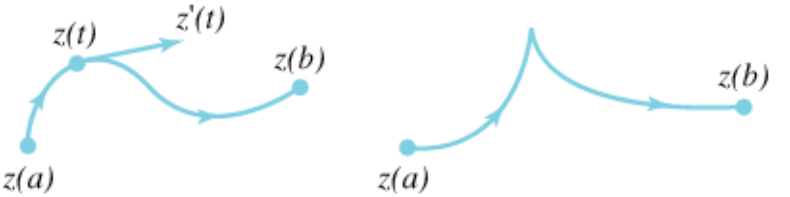
(c) A curve that is *not* simple and *not* closed. (d) A closed curve that is *not* simple.

Figure 6.1 The terms simple and closed used to describe curves.

The complex-valued function $z(t) = x(t) + i\,y(t)$ is said to be differentiable on $[a, b]$ if both $x(t)$ and $y(t)$ are differentiable for $a \le t \le b$. Here we require the one-sided derivatives of $x(t)$ and $y(t)$ to exist at the endpoints of the interval. As in Section 6.1, the derivative $z'(t)$ is

$$z'(t) = x'(t) + i\,y'(t) \quad \text{for} \quad a \le t \le b.$$

The curve C defined by Equation (6-10) is said to be a smooth curve if $z'(t)$ is continuous and nonzero on the interval. If C is a smooth curve, then C has a nonzero tangent vector at each point $z(t)$, which is given by the vector $z'(t)$. If $x'(t_0) = 0$, then the tangent vector $z'(t_0) = i\,y'(t_0)$ is vertical. If $x'(t_0) \neq 0$, then the slope $\frac{dy}{dx}$ of the tangent line to C at the point $z(t_0)$ is given by $\frac{y'(t_0)}{x'(t_0)}$. Hence for a smooth curve the angle of inclination $\theta(t)$ of its tangent vector $z'(t)$ is defined for all values of $t \in [a, b]$ and is continuous. Thus a smooth curve has no corners or cusps. Figure 6.2 illustrates this concept.



(a) A smooth curve. (b) A curve that is *not* smooth.

Figure 6.2 The term smooth used to describe curves.

If C is a smooth curve, then ds, the differential of arc length, is given by

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = |z'(t)|\,dt.$$

The function $s(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ is continuous, as $x'(t)$ and $y'(t)$ are continuous functions, so the length $L(C)$ of the curve C is

(6-11) $$L(C) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_a^b |z'(t)|\, dt.$$

Now consider C to be a curve with parameterization

$$C: z_1(t) = x(t) + i\, y(t) \quad \text{for} \quad a \le t \le b.$$

The opposite curve $-C$ traces out the same set of points in the plane, but in the reverse order, and has the parametrization

$$-C: z_2(t) = x(-t) + i\, y(-t) \quad \text{for} \quad a \le t \le b.$$

Since $z_2(t) = z_1(-t)$, $-C$ is merely C traversed in the opposite sense, as illustrated in Figure 6.3.

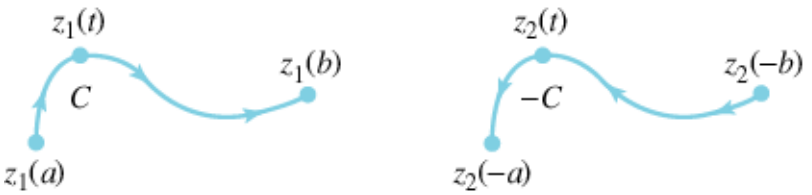


Figure 6.3 The curve C and its opposite curve -C.

A curve C that is constructed by joining finitely many smooth curves end to end is called a contour. Let $C_1, C_2, \ldots, C_n$ denote n smooth curves such that the terminal point of the curve C_k coincides with the initial point of C_{k+1} for $k = 1, 2, \ldots, n-1$. We express the contour C by the equation

$$C = C_1 + C_2 + \ldots + C_n.$$

A synonym for contour is path.

Example 6.5. Find a parameterization of the polygonal path $C = C_1 + C_2 + C_3$ from $-1 + i$ to $3 - i$ shown in Figure 6.4.

Here C_1 is the line from $-1 + i$ to -1, C_2 is the line from -1 to $1 + i$, and C_3 is the line from $1 + i$ to $3 - i$.

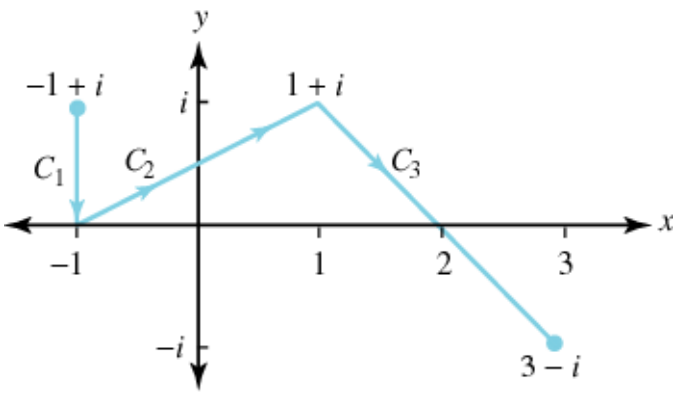


Figure 6.4 The polygonal path $C = C_1 + C_2 + C_3$ from $-1 + i$ to $3 - i$.

Solution. We express C as three smooth curves, or $C = C_1 + C_2 + C_3$. If we set $z_0 = -1 + i$ and $z_1 = -1$, we can use Equation (1-48) to get a formula for the straight-line segment joining two points:

$$C_1 : \; z_1(t) = z_0 + t(z_1 - z_0) = (-1 + i) + t(-1 - (-1 + i)),$$

for $0 \le t \le 1$. When simplified, this formula becomes

$$C_1 : \; z_1(t) = -1 + i(1 - t), \quad \text{for} \quad 0 \le t \le 1.$$

Similarly, the segments C_2 and C_3 are given by

$$C_2 : \; z_2(t) = -1 + 2t + it, \quad \text{for} \quad 0 \le t \le 1, \quad \text{and}$$

$$C_3 : \; z_3(t) = 1 + 2t + i(1 - 2t), \quad \text{for} \quad 0 \le t \le 1.$$

Explore Solution 6.5.

We are now ready to define the integral of a complex function along a contour C in the plane with initial point A and terminal point B. Our approach is to mimic what is done in calculus. We create a partition $P_n = \{z_0 = A, z_1, z_2, \ldots, z_n = B\}$ of points that proceed along C from A to B and form the differences $\Delta z_k = z_k - z_{k-1}$ for $k = 1, 2, \ldots, n$. Between each pair of partition points z_{k-1} and z_k we select a point c_k on C, as shown in Figure 6.5, and evaluate the function $f(c_k)$. These values are used to make a Riemann Sum for the partition:

(6-12) $$S(P_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\,\Delta z_k.$$

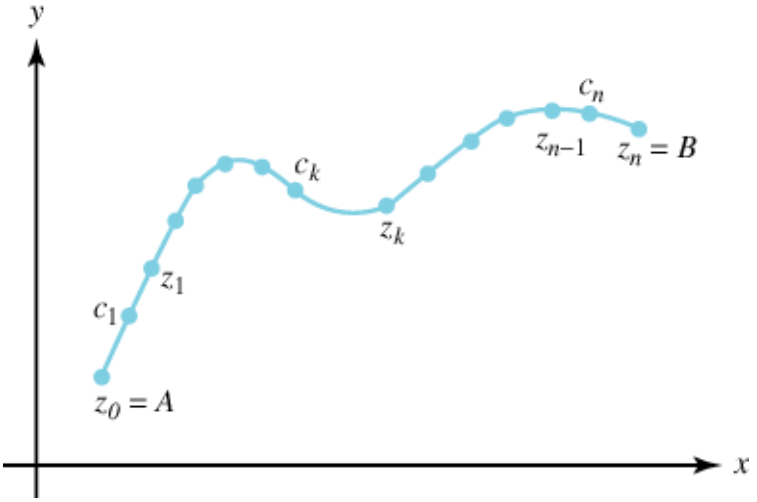


Figure 6.5 Partition points $\{z_k\}$ and function evaluation points $\{c_k\}$ for a Riemann sum along the contour C from $z = A$ to $z = B$.

Assume now that there exists a unique complex number L that is the limit of every sequence $\{S(P_n)\}$ of Riemann sums given in Equation (6-12), where the maximum of $|\Delta z_k|$ tends toward 0 for the sequence of partitions. We define the number L as the value of the integral of the function

$f(z)$ taken along the contour C.

Definition 6.2 (Complex Integral). Let C be a contour. Then

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta z_k,$$

provided that the limit exists in the sense previously discussed.

Note that in Definition 6.2, the value of the integral depends on the contour. In Section 6.3 the Cauchy-Goursat theorem will establish the remarkable fact that, if $f(z)$ is analytic, then $\int_C f(z)\,dz$ is independent of the contour.

Example 6.6. Use a Riemann sum to get an approximation for the integral $\int_C e^z\,dz$ where C is a the line segment joining the point $A = 0$ to $B = 2 + \frac{i\pi}{4}$.

Solution. Set $n=8$ in Equation (6-12) and form the partition $P_8 = \left\{z_k = \frac{k}{4} + i\,\frac{k\pi}{32},\ \text{for } k = 0, 1, 2, \ldots, 8\right\}$. For this situation, we have a uniform increment $\Delta z_k = \frac{1}{4} + i\,\frac{\pi}{32}$.

For convenience we select $c_k = \frac{z_{k-1}+z_k}{2} = \frac{2k-1}{8} + i\,\frac{\pi(2k-1)}{64}$ for $k = 0, 1, 2, \ldots, 8$. Figure 6.6 shows the points $\{z_k\}$ and $\{c_k\}$.

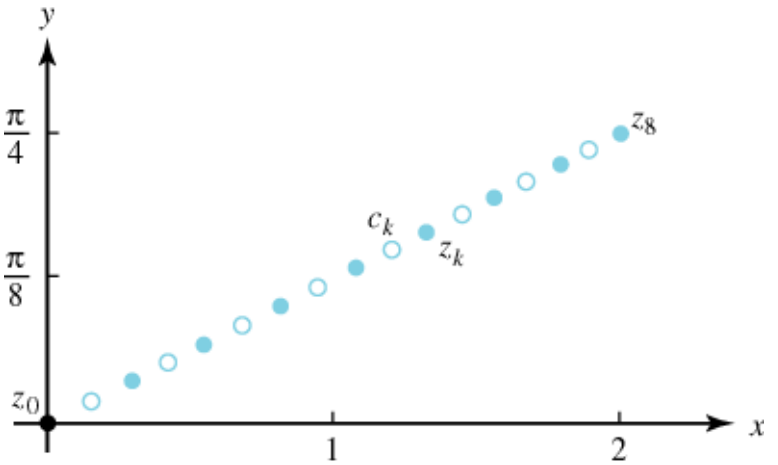


Figure 6.6 Partition and evaluation points for the Riemann sum $S(P_8)$.

One possible Riemann sum, then, is

$$S(P_8) = \sum_{k=1}^{8} f(c_k)\,\Delta z_k = \sum_{k=1}^{8} \exp\left(\frac{2k-1}{8} + i\,\frac{\pi(2k-1)}{64}\right)\left(\frac{1}{4} + i\,\frac{\pi}{32}\right)$$

By rounding the terms in this Riemann sum to five decimal digits, we obtain an approximation for the integral:

$$\begin{aligned} S(P_8) \approx\ & 0.27749 + 0.12501\,i + 0.33885 + 0.19467\,i + 0.40850 + 0.29140\,i \\ & + 0.48532 + 0.42378\,i + 0.56683 + 0.60261\,i + 0.64848 + 0.84138\,i \\ & + 0.72276 + 1.15676\,i + 0.77798 + 1.56912\,i \end{aligned}$$

$$S(P_8) \approx 4.226201 + 5.20473\,i$$

This result compares favorably with the precise value of the integral, which you will soon see equals

$$\exp\left(2+\frac{i\pi}{4}\right)-1 = -1+e^2\frac{\sqrt{2}}{2}+i\,e^2\frac{\sqrt{2}}{2} = 4.22485+5.22485\,i.$$

Explore Solution 6.6.

In general, obtaining an exact value for an integral given by Definition 6.2 is a daunting task. Fortunately, there is a beautiful theory that allows for an easy computation of many contour integrals. Suppose that we have a parametrization of the contour C given by the function z(t) for $a \le t \le b$. That is, C is the range of the function z(t) over the interval [a, b], as Figure 6.7 shows.

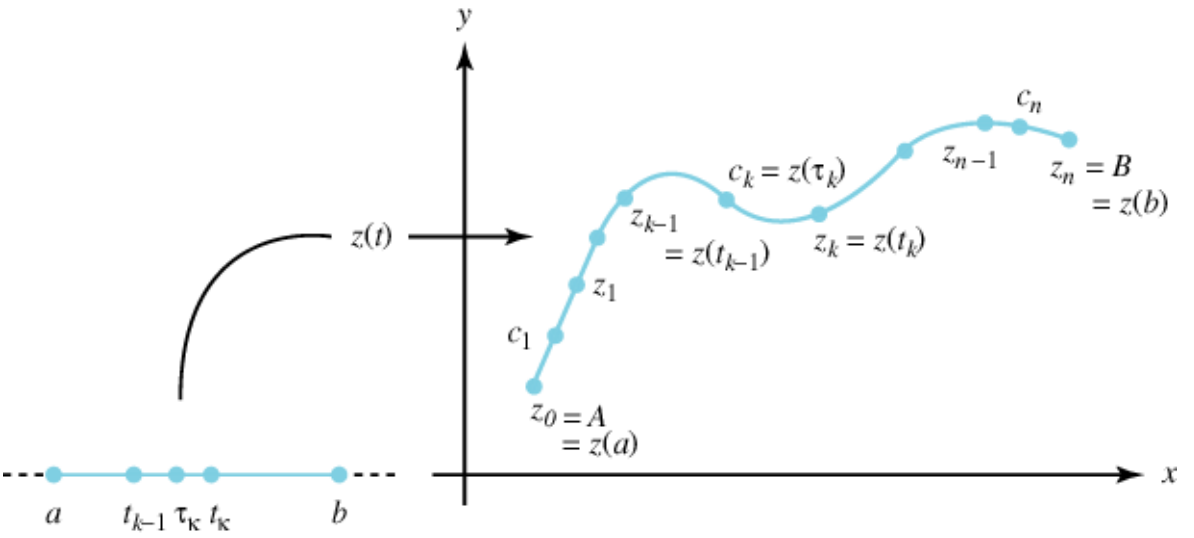


Figure 6.7 A parametrization of the contour C by z(t) for $a \le t \le b$.

It follows that

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,\Delta z_k = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,(z_k - z_{k-1})$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f(z(\tau_k))\,(z(t_k) - z(t_{k-1}))$$

where τ_k and t_k are the points contained in the interval [a, b] with the property that $c_k = z(\tau_k)$ and $z_k = z(t_k)$, as is also shown in Figure 6.7. If for all k we multiply the kth term in the last sum by $\frac{t_k - t_{k-1}}{t_k - t_{k-1}}$, then we get

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\,\Delta z_k = \lim_{n\to\infty}\sum_{k=1}^{n} f(z(\tau_k))\left(\frac{z(t_k)-z(t_{k-1})}{t_k - t_{k-1}}\right)(t_k - t_{k-1})$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f(z(\tau_k))\left(\frac{z(t_k)-z(t_{k-1})}{t_k - t_{k-1}}\right)\Delta t_k$$

The quotient inside the last summation looks suspiciously like a derivative, and the entire quantity looks like a Riemann sum. Assuming no difficulties, this last expression should equal

$\int_a^b f(z(t))\, z'(t)\, dt$, as defined in Section 6.1. Of course, if we're to have any hope of this happening, we would have to get the same limit regardless of how we parametrize the contour C. As Theorem 6.1 states, this is indeed the case.

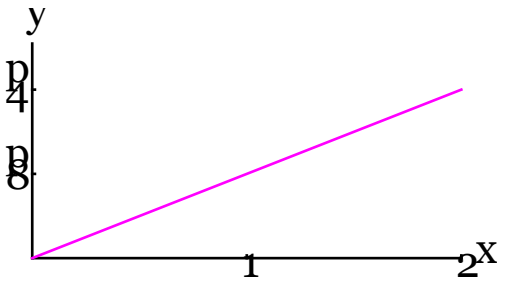
Theorem 6.1. Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \le t \le b$. Then

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

Proof.

Two important facets of Theorem 6.1 are worth mentioning. First, Theorem 6.1 makes the problem of evaluating complex-valued functions along contours easy, as it reduces the task to the evaluation of complex - valued functions over real intervals - a procedure that you studied in Section 6.1. Second, according to Theorem 6.1, this transformation yields the same answer regardless of the parametrization we choose for C.

Example 6.7. Give an exact calculation of the integral in Example 6.6: $\int_C e^z\, dz$ where C is a the line segment joining the point $A = 0$ to $B = 2 + \frac{i\pi}{4}$.



Solution. We must compute $\int_C e^z\, dz = \int_C \exp(z)\, dz$, where C is the line segment joining $A = 0$ to $B = 2 + \frac{i\pi}{4}$. According to Equation (1-48), we can parametrize C by $z(t) = \left(2 + i\frac{\pi}{4}\right) t$, for $0 \le t \le 1$. As $z'(t) = \left(2 + i\frac{\pi}{4}\right)$, Theorem 6.1 guarantees that

$$\int_C e^z\, dz = \int_C \exp(z)\, dz = \int_0^1 \text{Exp}[z[t]]\, z'[t]\, dt$$

$$= \int_0^1 \text{Exp}\left[\left(2 + i\frac{\pi}{4}\right) t\right] \left(2 + i\frac{\pi}{4}\right) dt$$

$$= \left(2 + i\frac{\pi}{4}\right) \int_0^1 \text{Exp}\left[\left(2 + i\frac{\pi}{4}\right) t\right] dt$$

$$= \left(2 + i\frac{\pi}{4}\right) \int_0^1 e^{2t}\, e^{\frac{i\pi}{4} t}\, dt$$

$$= \left(2 + i\frac{\pi}{4}\right) \int_0^1 e^{2t} \left(\text{Cos}\left[\frac{\pi t}{4}\right] + i\, \text{Sin}\left[\frac{\pi t}{4}\right]\right) dt$$

$$= \left(2 + i\frac{\pi}{4}\right) \left(\int_0^1 e^{2t}\, \text{Cos}\left[\frac{\pi t}{4}\right] dt + i \int_0^1 e^{2t}\, \text{Sin}\left[\frac{\pi t}{4}\right] dt\right)$$

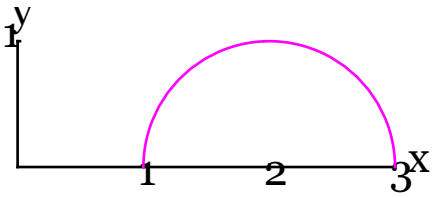
$$= \left(2 + i\frac{\pi}{4}\right) \left(\frac{2\sqrt{2}\, e^2 (\pi + 8)}{\pi^2 + 64} - \frac{32}{\pi^2 + 64} + i \left(\frac{-2\sqrt{2}\, e^2 (\pi - 8)}{\pi^2 + 64} + \frac{4\pi}{\pi^2 + 64}\right)\right)$$

$$= 4.2248516741 + 5.2248516741\, i$$

Each integral in the last expression can be done using integration by parts. (There is a simpler way-see Remark 6.1.) We leave as an exercise to show that the final answer simplifies to $\exp\left(2+i\,\frac{\pi}{4}\right)-1 = 4.2248516741+5.2248516741\,i$, as we claimed in Example 6.6.

Explore Solution 6.7.

Example 6.8. Evaluate the contour integral $\int_C \frac{1}{z-2}\,dz$ where C is a the upper semicircle with radius 1 centered at $x = 2$.



Solution. The function $z(t) = 2+e^{i\,t}$, for $0 \le t \le \pi$ is a parametrization for C. We apply Theorem 6.1 with $f(z) = \frac{1}{z-2}$. (Note: $f(z(t)) = \frac{1}{z(t)-2}$), and $z'(t) = i\,e^{i\,t}$.) Hence

$$\int_C \frac{1}{z-2}\,dz = \int_0^{\pi} \frac{1}{z(t)-2}\,z'(t)\,dt = \int_0^{\pi} \frac{1}{2+e^{i\,t}-2}\,i\,e^{i\,t}\,dt = \int_0^{\pi} i\,dt = i\,\pi.$$

Explore Solution 6.8.

To help convince yourself that the value of the integral is independent of the parametrization chosen for the given contour, try working through Example 6.8 with $z(t) = 2+e^{i\pi t}$, for $0 \le t \le 1$.

A convenient bookkeeping device can help you remember how to apply Theorem 6.1.

Because $\int_C \frac{1}{z-2}\,dz = \int_a^b \frac{1}{z(t)-2}\,z'(t)\,dt$, you can symbolically equate z with $z(t)$ and dz with $z'(t)\,dt$.

These identities should be easy to remember because z is supposed to be a point on the contour C parametrized by $z(t)$, and $\frac{dz}{dt} = z'(t)$, according to the Leibniz notation for the derivative.

If $z(t) = x(t)+i\,y(t)$, then by the preceding paragraph we have

(6-13) $$dz = z'(t)\,dt = (x'(t)+i\,y'(t))\,dt = dx + i\,dy$$

where dx and dy are the differentials for $x(t)$ and $y(t)$, respectively (i.e., dx is equated with $x'(t)\,dt$, etc.). The expression dz is often called the complex differential of z. Just as dx and dy are intuitively considered to be small segments along the x and y axes in real variables, we can think of dz as representing a tiny piece of the contour C. Moreover, if we write

$$|dz| = |(x'(t)+i\,y'(t))\,dt| = |(x'(t)+i\,y'(t))|\,dt = \sqrt{[x'(t)]^2+[y'(t)]^2}\,dt$$

we can put Equation (6-11) into the form

(6-14) $$L(C) = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_C |dz|.$$

so we can think of $|dz|$ as representing the length of dz.

Suppose $f(z) = u(z) + i\,v(t)$, and $z(t) = x(t) + i\,y(t)$ is a parametrization for the contour C. Then

(6-15) $$\begin{aligned}\int_C f(z)\,dz &= \int_a^b f(z(t))\,z'(t)\,dt \\ &= \int_a^b (u(z(t)) + i\,v(z(t)))\,(x'(t) + i\,y'(t))\,dt \\ &= \int_a^b [u(z(t))\,x'(t) - v(z(t))\,y'(t)]\,dt \\ &\quad + i\int_a^b [v(z(t))\,x'(t) + u(z(t))\,y'(t)]\,dt \\ &= \int_a^b (u\,x' - v\,y')\,dt + i\int_a^b (v\,x' + u\,y')\,dt\end{aligned}$$

where we are equating $u(t)$ with $u(z(t))$, x' with $x'(t)$, and so on.

If we use the differentials given in Equation (6-13), then we can write Equation (6-15) in terms of line integrals of the real-valued functions $u(t)$ and $v(t)$, giving

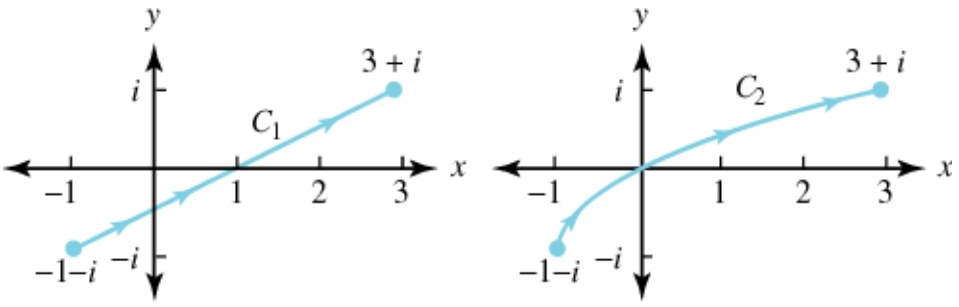
(6-16) $$\int_C f(z)\,dz = \int_a^b u\,dx - v\,dy + i\int_a^b v\,dx + u\,dy,$$

which is easy to remember if we recall that symbolically

$$\begin{aligned}f(z)\,dz &= (u + i\,v)\,(dx + i\,dy) \\ &= u\,dx - v\,dy + i\,(v\,dx + u\,dy)\end{aligned}.$$

We emphasize that Equation (6-16) is merely a notational device for applying Theorem 6.1. You should carefully apply Theorem 6.1as illustrated in Examples 6.7 and 6.8 before using any shortcuts suggested by the latter.

Example 6.9. Show that $\int_{C_1} z\,dz = \int_{C_2} z\,dz = 4 + 2\,i$,
where C_1 is the line segment from $-1 - i$ to $3 + i$, and C_2 is the portion of the parabola $x = y^2 + 2\,y$ joining $-1 - i$ to $3 + i$, as indicated in Figure 6.8.



(a) The line segment. (b) The portion of the parabola.

Figure 6.8 The two contours C_1 and C_2 joining $-1-i$ to $3+i$.

Solution. The line segment joining $(-1,-1)$ to $(3,1)$ is given by the slope intercept formula $y = \frac{1}{2}x - \frac{1}{2}$, which can be written as $x = 2y + 1$.

If we choose the parametrization $y = t$ and $x = 2t + 1$, we can write segment C_1 as

$$C_1 : \; z(t) = 2t + 1 + it \quad \text{and} \quad dz = (2 + i)\,dt$$

for $-1 \le t \le 1$.

Along C_1 we have $f(z(t)) = 2t + 1 + it$. Applying Theorem 6.1 gives

$$\int_{C_1} z\,dz = \int_a^b \frac{1}{z(t) - 2}\, z'(t)\,dt = \quad = \int_{-1}^{1} (2t + 1 + it)(2 + i)\,dt.$$

We now multiply out the integrand and put it into its real and imaginary parts:

$$\begin{aligned}
\int_{C_1} z\,dz &= \int_{-1}^{1} (2t + 1 + it)(2 + i)\,dt \\
&= \int_{-1}^{1} (3t + 2 + i(4t + 1))\,dt \\
&= \int_{-1}^{1} (3t + 2)\,dt + i\int_{-1}^{1} (4t + 1)\,dt \\
&= 4 + 2i
\end{aligned}$$

Similarly, we can parametrize the portion of the parabola $x = y^2 + 2y$ joining $(-1,-1)$ to $(3,1)$ by $y = t$ and $x = 2t + 1$ and $x = t^2 + 2t$ so that

$$C_2 : \; z(t) = t^2 + 2t + it \quad \text{and} \quad dz = (2t + 2 + i)\,dt$$

for $-1 \le t \le 1$.

Along C_2 we have $f(z(t)) = t^2 + 2t + it$. Theorem 6.1 now gives

$$\int_{C_2} z\,dz = \int_{-1}^{1} (t^2 + 2t + i\,t)\,(2t + 2 + i)\,dt$$

$$= \int_{-1}^{1} (2t^3 + 6t^2 + 3t + i\,(3t^2 + 4t))\,dt$$

$$= \int_{-1}^{1} (2t^3 + 6t^2 + 3t)\,dt + i \int_{-1}^{1} (3t^2 + 4t)\,dt$$

$$= 4 + 2i$$

Explore Solution 6.9 (a)

Explore Solution 6.9 (b)

Extra Example 1. Evaluate the contour integrals of $f(z) = z^2$ starting at the points $z_1 = -1 - i$ and $z_2 = 3 + i$.

(a) Use the line segment joining the points.

(b) Use a portion of a parabola joining the points.

Remark. The example in the text used the function $f(z) = z$ which is difficult for hand computations but it is not a challenge for *Mathematica*, hence we choose to use the function $f(z) = z^2$.

Explore Solution for Extra Example 1 (a)

Explore Solution for Extra Example 1 (b)

Extra Example 2. Evaluate the contour integrals of $f(z) = \overline{z}$ starting at the points $z_1 = -1 - i$ and $z_2 = 3 + i$.

(a) Use the line segment joining the points.

(b) Use a portion of a parabola joining the points.

Remark. This example illustrates the situation when f(z) is not analytic.

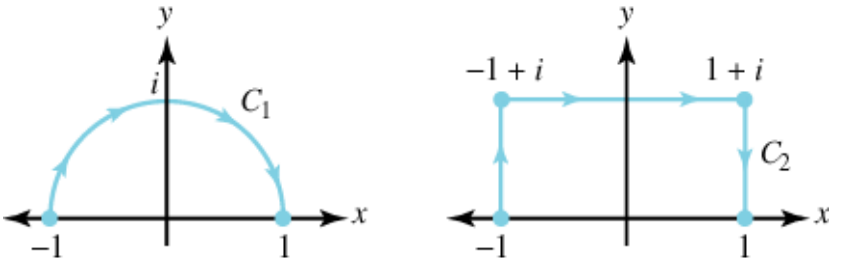
Explore Solution for Extra Example 2 (a)

Explore Solution for Extra Example 2 (b)

In Example 6.9, the value of the two integrals is the same. This outcome doesn't hold in general, as Example 6.10 shows.

Example 6.10. (a) Show that $\int_{C_1} \overline{z}\,dz = -\pi i$ but that $\int_{C_2} \overline{z}\,dz = -4i$,

where C_1 is the semicircular path from -1 to 1, in the upper half plane, and C_2 is the polygonal path from -1 to $-1 + i$ to $1 + i$ to 1, respectively, shown in Figure 6.9.



(a) The semicircular path. (b) The polygonal path.

Figure 6.9 The two contours C_1 and C_2 joining -1 to 1.

Solution. We parametrize the semicircle C_1 as

$$C_1 : z(t) = -\cos t + i \sin t \quad \text{and} \quad dz = (\sin t + i \cos t)\, dt,$$

for $0 \le t \le \pi$.

Applying Theorem 6.1, we have $f(z) = \overline{z}$, so

$$f(z(t)) = \overline{z(t)} = \overline{-\cos t + i \sin t}$$

$$= -\cos t - i \sin t$$

and

$$\int_{C_1} \overline{z}\, dz = \int_0^{\pi} (-\cos t - i \sin t)(\sin t + i \cos t)\, dt$$

$$= \int_0^{\pi} \left(-i\left(\cos^2 t + \sin^2 t\right)\right) dt$$

$$= -i \int_0^{\pi} (1)\, dt = -\pi i$$

We parametrize C_2 in three parts, one for each line segment:

$$\begin{aligned} z_1(t) &= -1 + i t, & dz &= i\, dt, & \text{and} \quad f(z_1(t)) &= -1 - i t; \\ z_2(t) &= -1 + 2t + i, & dz &= 2\, dt, & \text{and} \quad f(z_2(t)) &= -1 + 2t - i; \\ z_3(t) &= 1 + i(1 - t), & dz &= -i\, dt, & \text{and} \quad f(z_3(t)) &= 1 - i(1 - t), \end{aligned}$$

where $0 \le t \le 1$ in each case. Integrating over these three line segments we obtain

$$\int_0^1 z_1(t)\, z_1'(t)\, dt = \int_0^1 (-1 - i t)(i)\, dt = \int_0^1 (t - i)\, dt = \frac{1}{2} - i,$$

$$\int_0^1 z_2(t)\, z_2'(t)\, dt = \int_0^1 (-1 + 2t - i)(2)\, dt = \int_0^1 (-2 + 4t - 2i)\, dt = -2i,$$

$$\int_0^1 z_3(t)\, z_3'(t)\, dt = \int_0^1 (1 - i(1-t))(-i)\, dt = \int_0^1 (-1 + t - i)\, dt = -\frac{1}{2} - i.$$

We get our answer by adding the three integrals along the three segments:

$$\int_{C_2} \bar{z}\, dz = \int_0^1 (t - i)\, dt + \int_0^1 (-2 + 4t - 2i)\, dt + \int_0^1 (-1 + t - i)\, dt$$

$$= \frac{1}{2} - i - 2i - \frac{1}{2} - i = -4i$$

Note that the value of the contour integral along C_1 isn't the same as the value of the contour integral along C_2, although both integrals have the same initial and terminal points.

Explore Solution 6.10 (a)

Explore Solution 6.10 (b)

Contour integrals have properties that are similar to those of integrals of a complex function of a real variable, which you studied in Section 6.1. If C is given by Equation (6-10), then the integral for the opposite contour $-C$ is

$$\int_{-C} f(z)\, dz = \int_{-b}^{-a} f(z(-\tau))(-z'(-\tau))\, d\tau$$

Using the change of variable $t = -\tau$ in this last equation and the property that $\int_a^b f(t)\, dt = -\int_b^a f(t)\, dt$, we obtain

(6-17) $$\int_{-C} f(z)\, dz = -\int_C f(z)\, dz.$$

If two functions f and g can be integrated over the same path of integration C, then their sum can be integrated over C, and we have the familiar result

$$\int_C (f(z) + g(z))\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz.$$

Constant multiples also behave as we would expect:

$$\int_C (a + ib) f(z)\, dz = (a + ib) \int_C f(z)\, dz.$$

If two contours C_1 and C_2 are placed end to end so that the terminal point of C_1 coincides with the initial point of C_2, then the contour $C = C_1 + C_2$ is a continuation of C_1, and

(6-18) $$\int_{C_1+C_2} f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz.$$

If the contour C has two parametrizations

$$C: z_1(t) = x_1(t) + i\, y_1(t) \quad \text{for} \quad a \le t \le b, \quad \text{and}$$

$$C: z_2(\tau) = x_2(\tau) + i\, y_2(\tau) \quad \text{for} \quad c \le \tau \le d,$$

and there exists a differentiable function $\tau = \phi(t)$ such that

(6-19) $$c = \phi(a), \quad d = \phi(b), \quad \text{and} \quad \phi'(t) > 0 \quad \text{for} \quad a \le t \le b,$$

then we say that $z_2(\tau)$ is a reparametrization of the contour C. If f is continuous on C, then we have

(6-20) $$\int_a^b f(z_1(t))\, z_1'(t)\, dt = \int_c^d f(z_2(\tau))\, z_2'(\tau)\, d\tau.$$

Equation (6-20) shows that the value of a contour integral is invariant under a change in the parametric representation of its contour if the reparametrization satisfies Equations (6-19).

We now give two important inequalities relating to complex integrals.

Theorem 6.2 (Absolute Value Inequality). If $f(t) = u(t) + i\, v(t)$ is a continuous function of the real parameter t, then

(6-21) $$\left| \int_a^b f(t)\, dt \right| \le \int_a^b |f(t)|\, dt.$$

Proof.

Theorem 6.3 (ML - Inequality). If $f(z) = u(x, y) + i\, v(x, y)$ is continuous on the contour C, then

(6-23) $$\int_C |f(z)|\, dz \le ML.$$

Proof.

Example 6.11. Use Inequality (6-23) to show that $\left| \int_C \frac{1}{z^2+1}\, dz \right| \le \frac{1}{2\sqrt{5}}$, where C is the straight-line segment from 2 to $2 + i$.

Solution. Here $|z^2 + 1| = |z - i|\,|z + i|$, and the terms $|z - i|$ and $|z + i|$ represent the distance from the point z to the points i and $-i$, respectively. Referring to Figure 6.10 and using a geometric argument, we get

$$|z - i| \ge 2 \quad \text{and} \quad |z + i| \ge \sqrt{5}, \quad \text{for } z \text{ on } C.$$

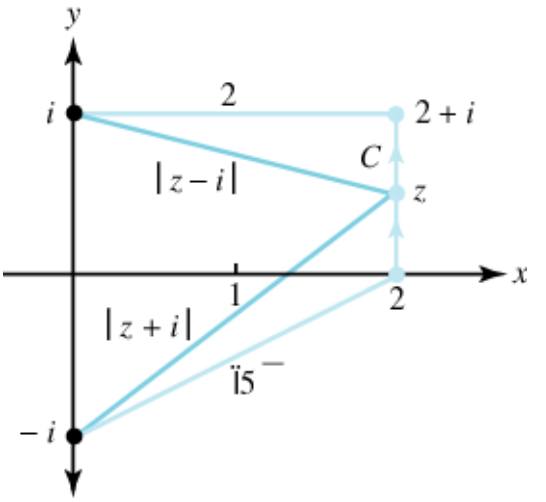


Figure 6.10 The distances $|z - i|$ and $|z + i|$ for z on C.

Thus we have

$$\left| f(z) \right| = \left| \frac{1}{z^2+1} \right| = \frac{1}{|z-i|\,|z+i|} \leq \frac{1}{2\sqrt{5}} = M.$$

Because L, the length of C, equals 1, Inequality (6-23) implies that

$$\left| \int_C \frac{1}{z^2+1}\, dz \right| \leq ML \leq \frac{1}{2\sqrt{5}} \times 1 = \frac{1}{2\sqrt{5}}.$$

Complex Analysis for Mathematics and Engineering
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6.3 The Cauchy-Goursat Theorem

The Cauchy-Goursat theorem states that within certain domains the integral of an analytic function over a simple closed contour is zero. An extension of this theorem allows us to replace integrals over certain complicated contours with integrals over contours that are easy to evaluate. We demonstrate how to use the technique of partial fractions with the Cauchy-Goursat theorem to evaluate certain integrals. In Section 6.4 we will see that the Cauchy-Goursat theorem implies that an analytic function has an antiderivative. To begin, we need to introduce some new concepts.

Recall from Section 1.6 that each simple closed contour C divides the plane into two domains. One domain is bounded and is called the interior of C; the other domain is unbounded and is called the exterior of C. Figure 6.15 illustrates this concept, which is known as the Jordan curve theorem.

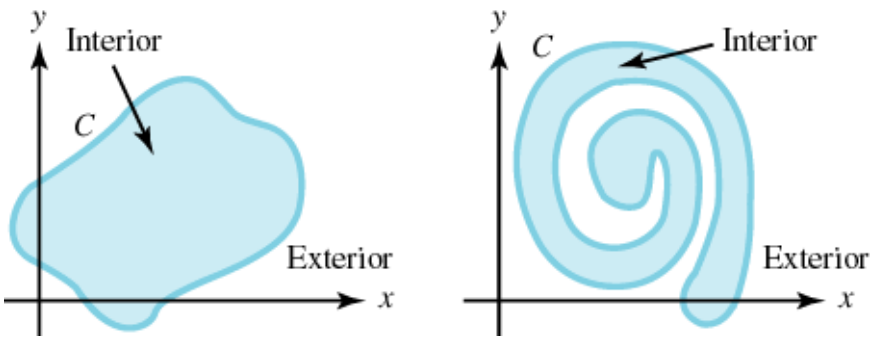
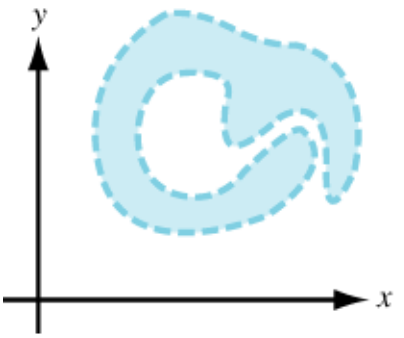
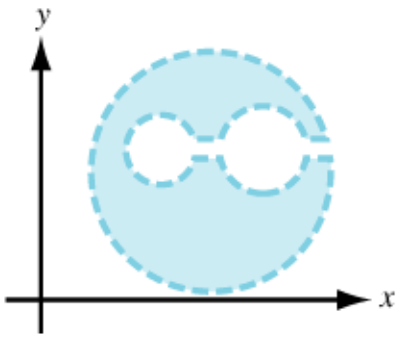


Figure 6.15 The interior and exterior of simple closed contours.

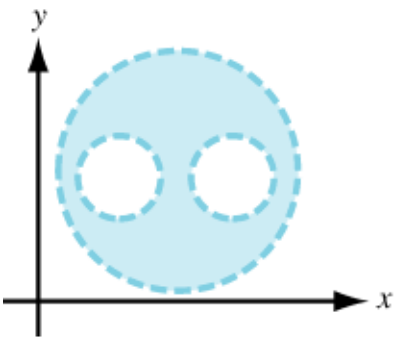
Recall also that a domain D is a connected open set. In particular, if z_1 and z_2 are any pair of points in D, then they can be joined by a curve that lies entirely in D. A domain D is said to be a simply connected domain if the interior of any simple closed contour C contained in D is contained in D. In other words, there are no "holes" in a simply connected domain. A domain that is not simply connected is said to be a multiply connected domain. Figure 6.16 illustrates uses of the terms simply connected and multiply connected.



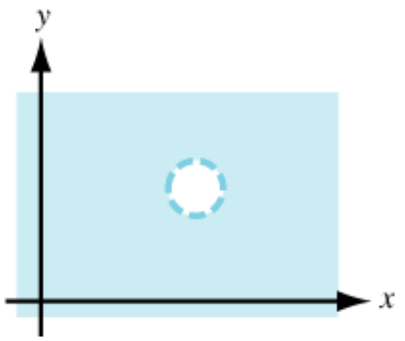
(a) A simply connected domain



(b) A simply connected domain



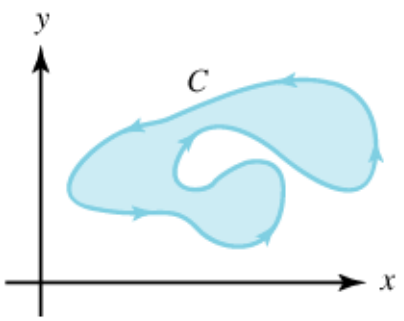
(c) A multiply connected domain



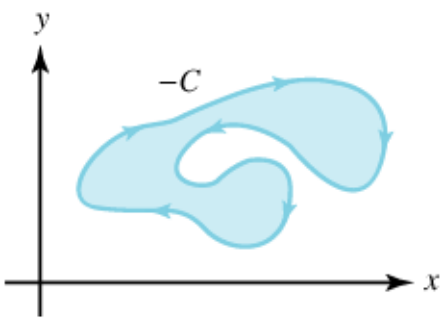
(d) A multiply connected domain

Figure 6.16 Simply connected and multiply connected domains.

Let the simple closed contour C have the parametrization `C : z(t) = x(t) + i y(t)` for `a ≤ t ≤ b`. Recall that if C is parametrized so that the interior of C is kept on the left as `z(t)` moves around C, then we say that C is oriented positively (counterclockwise); otherwise, C is oriented negatively (clockwise). If C is positively oriented, then `−C` is negatively oriented. Figure 6.17 illustrates the concept of positive and negative orientation.



(a) A positively oriented contour.



(b) A negatively oriented contour.

Figure 6.17 Simple closed contours that are positively and negatively oriented.

Green's theorem is an important result from the calculus of real variables. It tells you how to evaluate the line integral of real-valued functions.

Theorem 6.4 (Greens Theorem). Let C be a simple closed contour with positive orientation and let R be the domain that forms the interior of C. If P and Q are continuous and have continuous partial derivatives P_x, P_y, Q_x, and Q_y at all points on C and R, then

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \iint_R (Q_x(x, y) - P_y(x, y))\, dx\, dy.$$

Proof.

We are now ready to state the main result of this section.

Theorem 6.5 (Cauchy-Goursat Theorem). Let $f(z)$ be analytic in a simply connected domain D. If C is a simple closed contour that lies in D, then

$$\int_C f(z)\, dz = 0.$$

Proof.

Example 6.12. Let us recall that e^z, $\cos(z)$, and z^n (where n is a positive integer) are all entire functions and have continuous derivatives. The Cauchy-Goursat theorem implies that, for any simple closed contour,

(a) $\int_C e^z\, dz = 0$,

(b) $\int_C \cos(z)\, dz = 0$, and

(c) $\int_C z^n\, dz = 0$.

Explore Solution 6.12 (a)

Explore Solution 6.12 (b)

Explore Solution 6.12 (c)

Example 6.13. If C is a simple closed contour such that the origin does not lie interior to C, then there is a simply connected domain D that contains C in which $f(z) = \frac{1}{z^n}$ is analytic, as is indicated in Figure 6.22. The Cauchy-Goursat theorem implies that $\int_C \frac{1}{z^n}\, dz = 0$.

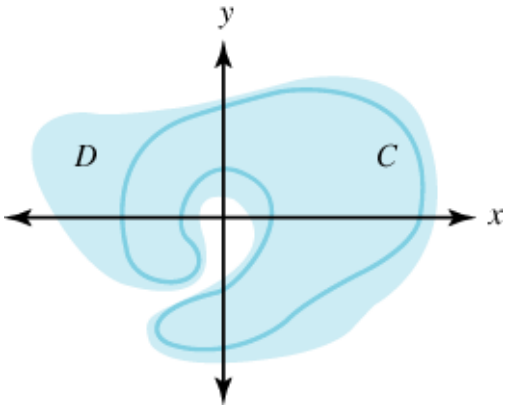


Figure 6.22 A simple connected domain D containing the simple closed contour C that does not contain the origin.

We want to be able to replace integrals over certain complicated contours with integrals that are easy to evaluate. If C_1 is a simple closed contour that can be "continuously deformed" into another simple closed contour C_2 without passing through a point where f is not analytic, then the

value of the contour integral of f over C_1 is the same as the value of the integral of f over C_2. To be precise, we state the following result.

Theorem 6.6 (Deformation of Contour). Let C_1 and C_2 be two simple closed positively oriented contours such that C_1 lies interior to C_2. If $f(z)$ is analytic in a domain D that contains both C_1 and C_2 and the region between them, as shown in Figure 6.23, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

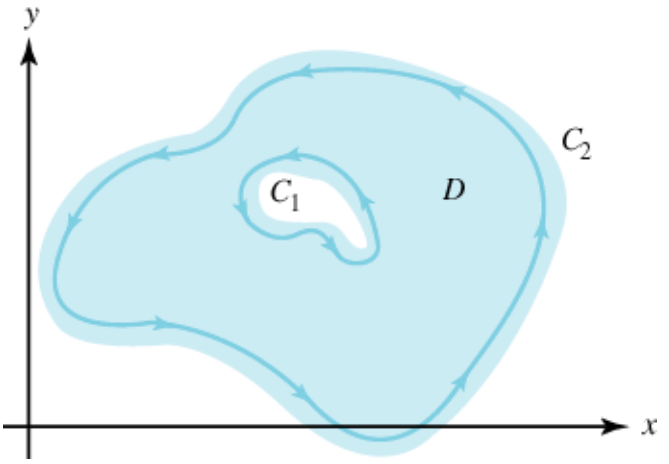


Figure 6.23 The domain D that contains the simple closed contours C_1 and C_2 and the region between them.

Proof.

We now state as a corollary an important result that is implied by the deformation of contour theorem. This result occurs several times in the theory to be developed and is an important tool for computations. You may want to compare the proof of Corollary 6.1 with your solution to Exercise 23 from Section 6.2.

Corollary 6.1. Let z_0 denote a fixed complex value. If C is a simple closed contour with positive orientation such that z_0 lies interior to C, then

(i) $$\int_C \frac{1}{z - z_0}\,dz = 2\pi i, \text{ and}$$

(ii) $$\int_C \frac{1}{(z - z_0)^n}\,dz = 0, \text{ where } n \text{ is any integer except } n = 1.$$

Proof.

Demonstration for (i).

Demonstration for (ii).

The deformation of contour theorem is an extension of the Cauchy-Goursat theorem to a doubly connected domain in the following sense. Let D be a domain that contains C_1 and C_2 and the region between them, as shown in Figure 6.23. Then the contour $C = C_2 - C_1$ is a parametrization of the boundary of the region R that lies between C_1 and C_2 so that the points of R lie to the left of C as a point $z(t)$ moves around C. Hence C is a positive orientation of the boundary of R, and Theorem 6.6 implies that $\int_C f(z)\,dz = 0$.

We can extend Theorem 6.6 to multiply connected domains with more than one "hole." The proof, which is left for the reader, involves the introduction of several cuts and is similar to the proof of Theorem 6.6.

Theorem 6.7 (Extended Cauchy-Goursat Theorem). Let $C, C_1, C_2, \ldots, C_n$ be simple closed positively oriented contours with the property that C_k lies interior to C for $k = 1, 2, \ldots, n$ and the set of interior to C_k has no points in common with the set interior to C_j if $k \neq j$. Let $f(z)$ be analytic on a domain D that contains all the contours and the region between C and $C_1 + C_2 + \ldots + C_n$, as shown in Figure 6.26. Then

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_k} f(z)\,dz.$$

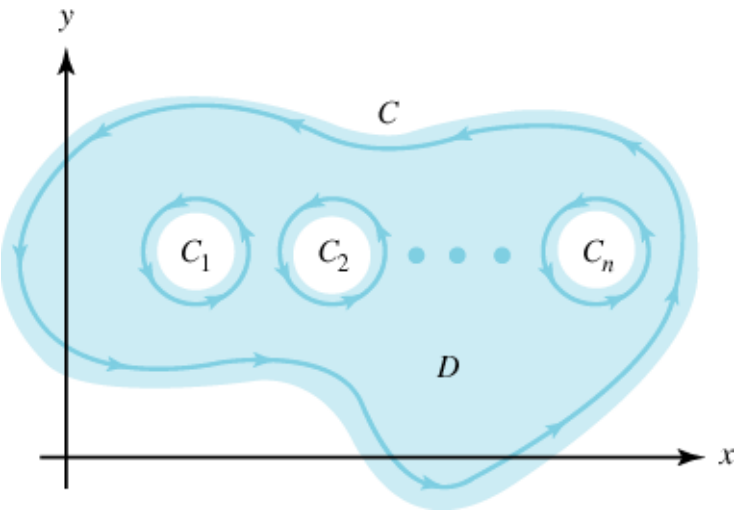
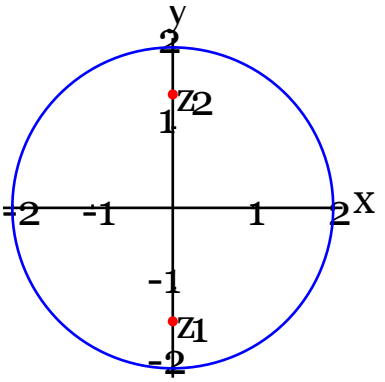


Figure 6.26 The multiply connected domain D and the contours $C, C_1, C_2, \ldots, C_n$ in the statement of the extended Cauchy-Goursat theorem.

Proof.

Example 6.14. Show that $\int_C \frac{2z}{z^2+2}\,dz = 4\pi i$, where C is the circle $C: |z| = 2$ taken with positive orientation.



Solution. Using partial fraction decomposition gives

$$\frac{2z}{z^2+2} = \frac{2z}{\left(z+i\sqrt{2}\right)\left(z-i\sqrt{2}\right)} = \frac{1}{z+i\sqrt{2}} + \frac{1}{z-i\sqrt{2}}, \text{ so}$$

(6-38)
$$\int_C \frac{2z}{z^2+2}\,dz = \int_C \frac{1}{z+i\sqrt{2}}\,dz + \int_C \frac{1}{z-i\sqrt{2}}\,dz.$$

The points $z = \pm i\sqrt{2}$ lie interior to C, so Corollary 6.1 implies that

$$\int_C \frac{1}{z \pm i\sqrt{2}}\, dz = 2\pi i.$$

Substituting these values into Equation (6-38) yields

$$\begin{aligned}\int_C \frac{2z}{z^2+2}\, dz &= \int_C \frac{1}{z+i\sqrt{2}}\, dz + \int_C \frac{1}{z-i\sqrt{2}}\, dz \\ &= 2\pi i + 2\pi i \\ &= 4\pi i\end{aligned}$$

Explore Solution 6.14.

Example 6.15. Show that $\int_C \frac{2z}{z^2+2}\, dz = 2\pi i$, where C is the circle $C: |z - i| = 1$ taken with positive orientation.

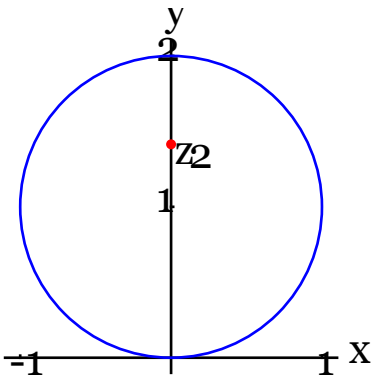


Figure 6.27 The circle $C: |z - i| = 1$ and the points $z = \pm i\sqrt{2}$.

Solution. Using partial fractions again, we have

$$\int_C \frac{2z}{z^2+2}\, dz = \int_C \frac{1}{z+i\sqrt{2}}\, dz + \int_C \frac{1}{z-i\sqrt{2}}\, dz$$

In this case, $z = i\sqrt{2}$ lies interior to C but $z = -i\sqrt{2}$ does not, as shown in Figure 6.27. By Corollary 6.1, the second integral on the right side of the above equation has the value $2\pi i$. The first integral equals zero by the Cauchy-Goursat theorem because the function $f(z) = \frac{1}{z+i\sqrt{2}}$ is analytic on a simply connected domain that contains C. Thus

$$\begin{aligned}\int_C \frac{2z}{z^2+2}\, dz &= \int_C \frac{1}{z+i\sqrt{2}}\, dz + \int_C \frac{1}{z-i\sqrt{2}}\, dz \\ &= 0 + 2\pi i \\ &= 2\pi i\end{aligned}$$

Explore Solution 6.15.

Example 6.16. Show that $\int_C \frac{z-2}{z^2-z}\,dz = -6\pi i$, where C is the "figure eight" contour shown in Figure 6.28.

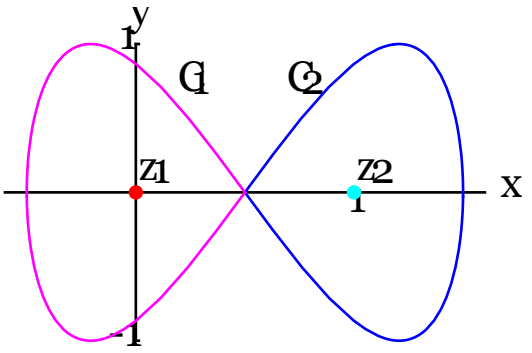


Figure 6.28 The contour $C = C_1 + C_2$.

Solution. Again, we use partial fractions to express the integral:

(6-39) $$\int_C \frac{z-2}{z^2-z}\,dz = \int_C \left(\frac{2}{z} - \frac{1}{z-1}\right) dz = 2\int_C \frac{1}{z}\,dz - \int_C \frac{1}{z-1}\,dz.$$

Using the Cauchy-Goursat theorem, Property (6-17), and Corollary 6.1 (with $z_0 = 0$), we compute the value of the first integral on the right side of Equation (6-39):

$$\begin{aligned} 2\int_C \frac{1}{z}\,dz &= 2\int_{C_1} \frac{1}{z}\,dz + 2\int_{C_2} \frac{1}{z}\,dz \\ &= -2\int_{-C_1} \frac{1}{z}\,dz + 0 \\ &= -2(2\pi i) \\ &= -4\pi i \end{aligned}$$

Similarly, we find that

$$\begin{aligned} -\int_C \frac{1}{z-1}\,dz &= -\int_{C_1} \frac{1}{z-1}\,dz + -\int_{C_2} \frac{1}{z-1}\,dz \\ &= 0 - 2\pi i \\ &= -2\pi i \end{aligned}$$

If we substitute the results of the last two equations into Equation (6-39) we get

$$\begin{aligned} \int_C \frac{z-2}{z^2-z}\,dz &= 2\int_C \frac{1}{z}\,dz - \int_C \frac{1}{z-1}\,dz \\ &= -4\pi i - 2\pi i \\ &= -6\pi i \end{aligned}$$

Explore Solution 6.16.

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6.4 The Fundamental Theorems of Integration

Let f be analytic in the simply connected domain D. The theorems in this section show that an antiderivative F can be constructed by contour integration. A consequence will be the fact that in a simply connected domain, the integral of an analytic function f along any contour joining z_1 to z_2 is the same, and its value is given by $F(z_2) - F(z_1)$. As a result, we can use the antiderivative formulas from calculus to compute the value of definite integrals. The next two theorems are generalizations of the Fundamental Theorems of Calculus.

Theorem 6.8 (Indefinite Integrals or Antiderivatives). Let $f(z)$ be analytic in the simply connected domain D. If z_0 is a fixed value in D and if C is any contour in D with initial point z_0 and terminal point z, then the function

$$F(z) = \int_C f(z)\,dz = \int_{z_0}^{z} f(\zeta)\,d\zeta$$

is well-defined and analytic in D, with its derivative given by $F'(z) = f(z)$.

Proof.

Remark 6.2. It is important to stress that the line integral of an analytic function is independent of path. In Example 6.9 we showed that $\int_{C_1} z\,dz = \int_{C_2} z\,dz = 4 + 2i$, where C_1 and C_2 were different contours joining $-1 - i$ to $3 + i$. Because the integrand $f(z) = z$ is an analytic function, Theorem 6.8 lets us know ahead of time that the value of the two integrals is the same; hence one calculation would have sufficed. If you ever have to compute a line integral of an analytic function over a difficult contour, change the contour to something easier. You are guaranteed to get the same answer. Of course, you must be sure that the function you're dealing with is analytic in a simply connected domain containing your original and new contours.

Exploration.

If we set $z = z_1$ in Theorem 6.8, then we obtain the following familiar result for evaluating a

definite integral of an analytic function.

Theorem 6.9 (Definite Integrals). Let $f(z)$ be analytic in a simply connected domain D. If z_0 and z_1 are two points in D joined by a contour C, then

$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0),$$

where $F(z)$ is any antiderivative of $f(z)$ in D.

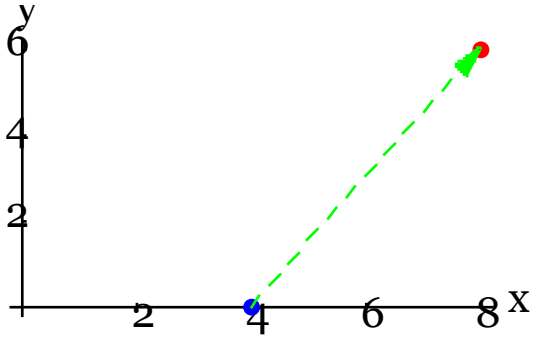
Proof.

Proof of Theorem 6.9 is in the book.

Theorem 6.9 gives an important method for evaluating definite integrals when the integrand is an analytic function in a simply connected domain. In essence, it permits you to use all the rules of integration that you learned in calculus. When the conditions of Theorem 6.9 are met, applying it is generally much easier than parametrizing a contour.

Example 6.17. Show that $\int_C \frac{1}{2z^{1/2}}\,dz = 1 + i$ where $z^{1/2}$ is the principal branch of the square root function and C is the line segment joining 4 to $8 + 6i$.

Remark. Sometimes we write this as $\int_4^{8+6i} \frac{z^{-1/2}}{2}\,dz = 1 + i$.

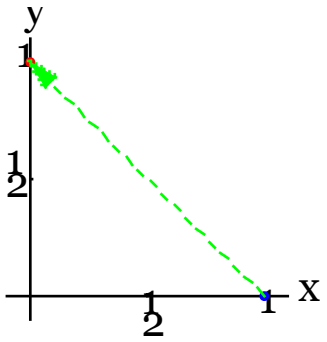


Solution. We showed in Chapter 3 that if $F(z) = z^{\frac{1}{2}}$, then $F'(z) = \frac{1}{2z^{1/2}}$, where the principal branch of the square root function is used in both the formulas for $F(z)$ and $F'(z)$. We note that C is contained in the simply connected domain $D_4(6 + 3i)$, which is the open disk of radius 4 centered at the midpoint of the segment C. Since $f(z) = \frac{1}{2z^{1/2}}$ is analytic in the domain $D_4(6 + 3i)$ and $F(z)$ is an anti-derivative of $f(z)$, Theorem 6.9 guarantees that

$$\int_C \frac{1}{2z^{1/2}}\,dz = \int_4^{8+6i} \frac{z^{-1/2}}{2}\,dz = F(8 + 6i) - F(4)$$
$$= (8 + 6i)^{\frac{1}{2}} - (4)^{\frac{1}{2}} = 3 + i - 2$$
$$= 1 + i$$

Explore Solution 6.17.

Example 6.18. Show that $\int_C \cos(z)\,dz = -\sin 1 + i \sinh 1$, where C is the line segment between 1 and i.



Solution. An antiderivative of $f(z) = \cos z$ is $F(z) = \sin z$. Because $F(z)$ is entire, we use Theorem 6.9 to conclude that

$$\int_C \cos(z)\,dz = \int_1^i \cos(z)\,dz$$

$$= \int_1^i f(z)\,dz = F(i) - F(1)$$

$$= \sin i - \sin 1$$

$$= i \sinh 1 - \sin 1$$

$$= -\sin 1 + i \sinh 1$$

$$= -0.841471 + 1.1752\,i$$

Explore Solution 6.18.

Example 6.19. We let $D = \{z = r e^{i\theta} : r > 0 \text{ and } -\pi < \theta < \pi\}$ be the simply connected domain which is the z-plane slit along the negative x-axis, shown in Figure 6.32. We know that $f(z) = \frac{1}{z}$ is analytic in D, and has an antiderivative $f(z) = \text{Log}(z)$ for all z in D. If C is a contour in D that joins the point z_1 to the point z_2, then Theorem 6.9 implies that

$$\int_{z_1}^{z_2} \frac{1}{z}\,dz = \text{Log}\, z_2 - \text{Log}\, z_1.$$

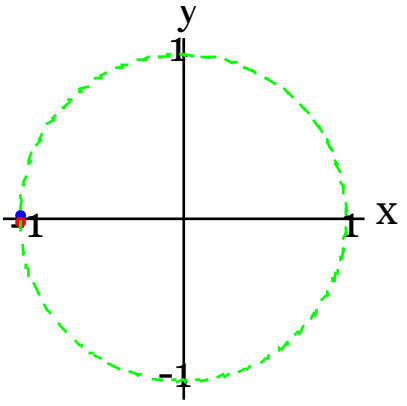


Figure 6.32 The simply connected domain D shown in Examples 6.19 and 6.20.

Example 6.20. Show that $\int_C \frac{1}{z}\,dz = 2\pi i$, where C is the unit circle $C: |z| = 1$, taken with

positive orientation.

Solution. We let C be that circle with the point -1 omitted, as shown in Figure 6.32(b). The contour C is contained in the simply connected domain D of Example 6.19. We know that $f(z) = \frac{1}{z}$ is analytic in D, and has an antiderivative $F(z) = \text{Log}(z)$, for all $z \in D$. Therefore, if we let z_2 approach -1 on C through the upper half-plane and z_1 approach -1 on C through the lower half-plane,

$$\int_C \frac{1}{z}\,dz = \lim_{\substack{z_2 \to -1 \,(z_2 \in C,\, \text{Im}\, z_2 > 0) \\ z_1 \to -1 \,(z_1 \in C,\, \text{Im}\, z_1 < 0)}} \int_{z_1}^{z_2} \frac{1}{z}\,dz$$

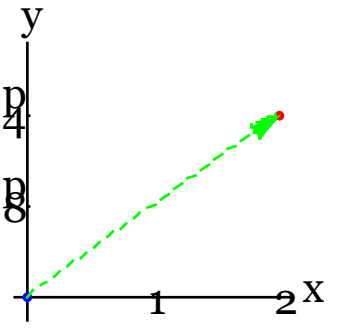
$$= \lim_{z_2 \to -1 \,(z_2 \in C,\, \text{Im}\, z_2 > 0)} \text{Log}(z_2) - \lim_{z_1 \to -1 \,(z_1 \in C,\, \text{Im}\, z_1 < 0)} \text{Log}(z_1)$$

$$= i\pi - (-i\pi)$$

$$= 2\pi i$$

Explore Solution 6.20.

Extra Example 1. Show that $\int_0^{2 + \frac{i\pi}{4}} e^z\,dz = \frac{(1+i)\,e^2}{\sqrt{2}} - 1$.



Explore Solution for Extra Example 1.

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6.5 Integral Representations for Analytic Functions

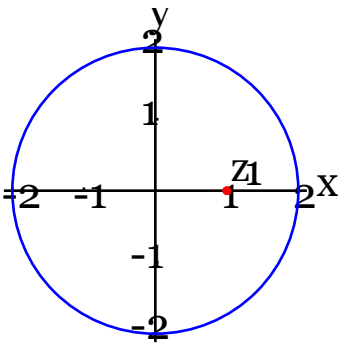
We now present some major results in the theory of functions of a complex variable. The first result is known as Cauchy's integral formula and shows that the value of an analytic function $f(z)$ can be represented by a certain contour integral. The n^{th} derivative, $f^{(n)}(z)$, will have a similar representation. In Chapter 7, we use the Cauchy integral formulas to prove Taylor's theorem and also establish the power series representation for analytic functions. The Cauchy integral formulas are a convenient tool for evaluating certain contour integrals.

Theorem 6.10 (Cauchy Integral Formula). Let $f(z)$ be analytic in the simply connected domain D, and let C be a simple closed positively oriented contour that lies in D. If z_0 is a point that lies interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz.$$

Proof.

Example 6.21. Show that $\int_C \frac{e^z}{z-1}\,dz = i\,2\pi e$, where C is the circle $C: |z| = 2$ with positive orientation.



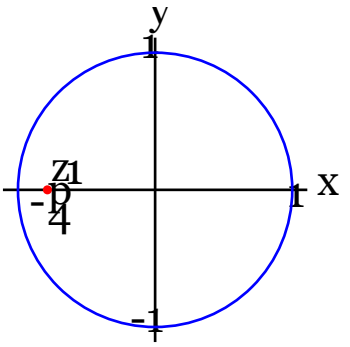
Solution. We have $f(z) = e^z$ and $f(1) = e$. The point $z_0 = 1$ lies interior to the circle, so Cauchy's integral formula implies that

$$e = f(1) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi i}\int_C \frac{e^z}{z-1}\,dz,$$

and multiplication by $2\pi i$ establishes the desired result.

Explore Solution 6.21.

Example 6.22. Show that $\int_C \frac{\sin(z)}{4z+\pi}\,dz = -i\frac{\pi\sqrt{2}}{4}$, where C is the circle $C: |z| = 1$ with positive orientation.



Solution. Here we have $f(z) = \sin(z)$. We manipulate the integral and use Cauchy's integral formula to obtain

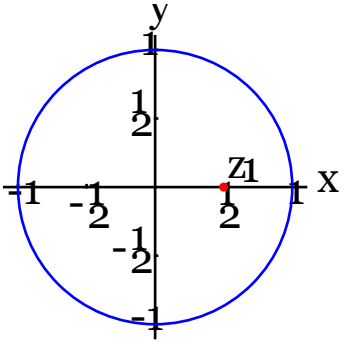
$$\int_C \frac{\sin(z)}{4z+\pi}\,dz = \frac{1}{4}\int_C \frac{\sin(z)}{z+\frac{\pi}{4}}\,dz = \frac{1}{4}\int_C \frac{\sin(z)}{z-\left(-\frac{\pi}{4}\right)}\,dz$$

$$= \frac{1}{4}(2\pi i) f\left(-\frac{\pi}{4}\right) = \frac{1}{4}(2\pi i)\sin\left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi i}{2}\left(-\frac{\sqrt{2}}{2}\right) = -i\frac{\pi\sqrt{2}}{4}$$

Explore Solution 6.22.

Example 6.23. Show that $\int_C \frac{e^{i\pi z}}{2z^2 - 5z + 2}\,dz = \frac{2\pi}{3}$, where C is the circle $C: |z| = 1$ with positive orientation.



Solution. We see that $2z^2 - 5z + 2 = (2z-1)(z-2) = 2\left(z-\frac{1}{2}\right)(z-2)$. The only zero of this expression that lies in the interior of C is $z_0 = \frac{1}{2}$.

We set $f(z) = \frac{e^{i\pi z}}{(z-2)}$ and use Theorem 6.10 to conclude that

$$\int_C \frac{e^{i\pi z}}{2z^2-5z+2}\,dz = \int_C \frac{e^{i\pi z}}{2\left(z-\frac{1}{2}\right)(z-2)}\,dz = \frac{1}{2}\int_C \frac{e^{i\pi z}}{(z-2)}\,\frac{1}{\left(z-\frac{1}{2}\right)}\,dz$$

$$= \frac{1}{2}\int_C \frac{f(z)}{\left(z-\frac{1}{2}\right)}\,dz = \frac{1}{2}(2\pi i)\,f\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}(2\pi i)\,\frac{e^{i\pi/2}}{\left(\frac{1}{2}-2\right)} = (\pi i)\,\frac{i}{\left(-\frac{3}{2}\right)}$$

$$= \frac{2\pi}{3}$$

Explore Solution 6.23.

Theorem 6.11 (Leibniz's Rule). Let G be an open set, and let $I: a \le t \le b$ be an interval of real numbers. Let $g(z, t)$ and its partial derivative $g_z(z, t)$ with respect to z be continuous functions for all z in G and all t in I. Then

$$F(z) = \int_a^b g(z, t)\,dt \quad \text{is analytic for z in G, and}$$

$$F'(z) = \int_a^b g_z(z, t)\,dt.$$

Proof.

Demonstration for Theorem 6.11.

We now generalize Theorem 6.10 to give an integral representation for the n^{th} derivative, $f^{(n)}(z)$. We use Leibniz's rule in the proof and note that this method of proof is a mnemonic device for remembering Theorem 6.12.

Theorem 6.12 (Cauchy's Integral Formulae for Derivatives). Let $f(z)$ be analytic in the simply connected domain D, and let C be a simple closed positively oriented contour that lies in D. If z is a point that lies interior to C, then for any integer $n \ge 0$, we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta.$$

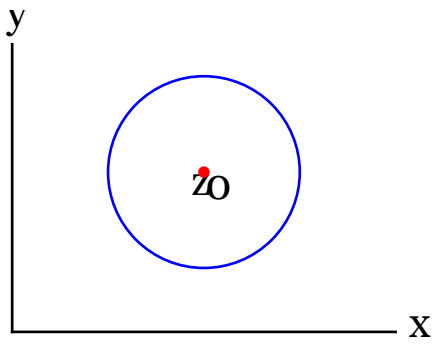
Proof.

Example 6.24. Let z_0 denote a fixed complex value. Show that, if C is a simple closed positively oriented contour such that z_0 lies interior to C, then

$$\int_C \frac{1}{z-z_0}\,dz = 2\pi i, \quad \text{and}$$

(6-50)

$$\int_C \frac{1}{(z-z_0)^{n+1}}\,dz = 0, \quad \text{for any integer } n \ge 1.$$



Solution. We let $f(z) = 1$. Then $f^{(n)}(z) = 0$ for $n \geq 1$. Theorem 6.10 implies that the value of the first integral in Equations (6-50) is

$$\int_C \frac{1}{z - z_0}\, dz = 2\pi i\, f(z_0) = 2\pi i,$$

and Theorem 6.12 further implies that

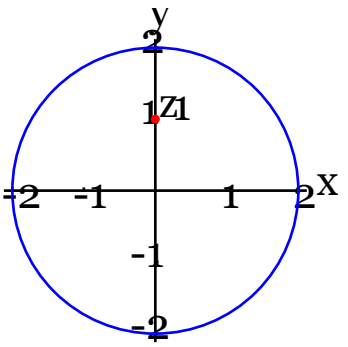
$$\int_C \frac{1}{(z - z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0) = 0.$$

This result is the same as that proven earlier in Corollary 6.1. Obviously, though, the technique of using Theorems 6.10 and 6.12 is easier.

Explore Solution 6.24 (a).

Explore Solution 6.24 (b).

Example 6.25. Show that $\int_C \frac{e^{z^2}}{(z - i)^4}\, dz = \frac{-4\pi}{3e}$, where C is the circle $C: |z| = 2$ with positive orientation.



Solution. If we set $f(z) = e^{z^2}$, then a straightforward calculation shows that $f^{(3)}(z) = (12z + 8z^3)e^{z^2}$. Using Cauchy's integral formulas with $n = 3$, we conclude that

$$\int_C \frac{e^{z^2}}{(z - i)^4}\, dz = \frac{2\pi i}{3!} f^{(3)}(i) = \frac{2\pi i}{3!}(12i + 8i^3)e^{i^2}$$

$$= \frac{2\pi i}{3!}(12i - 8i)e^{-1} = \frac{2\pi i}{3!}\frac{4i}{e}$$

$$= -\frac{4\pi}{3e}$$

Explore Solution 6.25.

We now state two important corollaries of Theorem 6.12.

Corollary 6.2. If $f(z)$ is analytic in the domain D, then all derivatives $f'(z), f''(z), \ldots, f^{(n)}(z), \ldots$ exists for $z \in D$ (and therefore are analytic in D).

Proof.

Remark 6.3. This result is interesting, as it illustrates a big difference between real and complex functions. A real function $f(z)$ can have the property that $f'(z)$ exists everywhere in a domain D, but $f''(z)$ exists nowhere. Corollary 6.2 states that if a complex function $f(z)$ has the property that $f'(z)$ exists everywhere in a domain D, then, remarkably, all derivatives of $f(z)$ exist in D.

Corollary 6.3. If $u(x, y)$ is a harmonic function at each point (x, y) in the domain D, then all partial derivatives $u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y), \ldots$ exists and are harmonic functions.

Proof.

Extra Example 1. Show that the partial derivatives of $u(x, y) = x^7 - 21 x^5 y^2 + 35 x^3 y^4 - 7 x y^6$ are harmonic functions.

Explore Extra Solution 1.

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6.6 The Theorems of Morera and Liouville and Extensions

In this section we investigate some of the qualitative properties of analytic and harmonic functions. Our first result shows that the existence of an antiderivative for a continuous function is equivalent to the statement that the integral of $f(z)$ is independent of the path of integration. This result stated in a form that will serve as a converse to the Cauchy-Goursat theorem.

Theorem 6.13 (Morera's Theorem). Let $f(z)$ be a continuous function in a simply connected domain D. If $\int_C f(z)\,dz = 0$ for every closed contour in D, then $f(z)$ is analytic in D.

Proof.

Cauchy's integral formula show how the value $f(z_0)$ can be represented by a certain contour integral. If we choose the contour of integration C to be a circle with center z_0, then we can show that the value $f(z_0)$ is the integral average of the values of $f(z)$ at points z on the circle C.

Theorem 6.14 (Gauss's Mean Value Theorem). If $f(z)$ is analytic in a simply connected domain D that contains the circle $C: |z - z_0| = R$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R e^{i\theta})\,d\theta.$$

Proof.

We now prove an important result concerning the modulus of an analytic function.

Theorem 6.15 (Maximum Modulus Principle). Let $f(z)$ be analytic and nonconstant in the bounded domain D. Then $|f(z)|$ does not attain a maximum value at any point z_0 in D.

Proof.

We sometimes state the maximum modulus principle in the following form.

Theorem 6.16 (Maximum Modulus Principle). Let $f(z)$ be analytic and nonconstant in the bounded domain D. If $f(z)$ is continuous on the closed region R that consists of D and all of its boundary points B, then $|f(z)|$ assumes its maximum value, and does so only at point(s) z_0 on

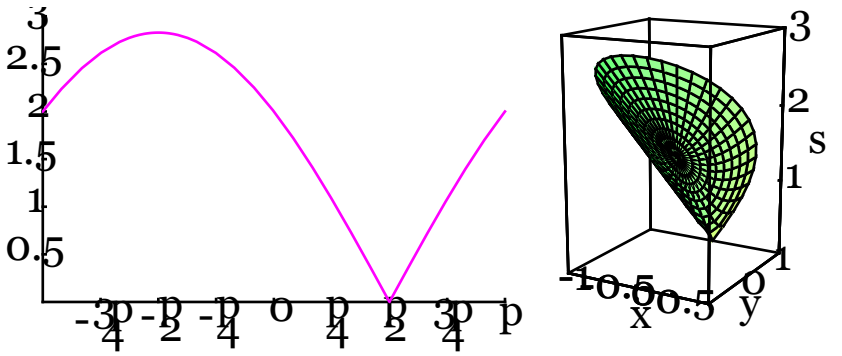
the boundary B.

Proof.

Example 6.26. Let $f(z) = a z + b$. If we set our domain D to be $D: |z| < 1$, then $f(z)$ is continuous on the closed region $R: |z| \le 1$. Prove that

$$\max_{|z| \le 1} |f(z)| = |a| + |b|,$$

and this value is assumed by $f(z)$ at a point $z_0 = e^{i\theta_0}$ on the boundary of D.



Solution. From the triangle inequality and the fact that $z \le 1$ in D, it follows that

(6-58) $$|f(z)| = |az + b| \le |az| + |b| \le |a| + |b|.$$

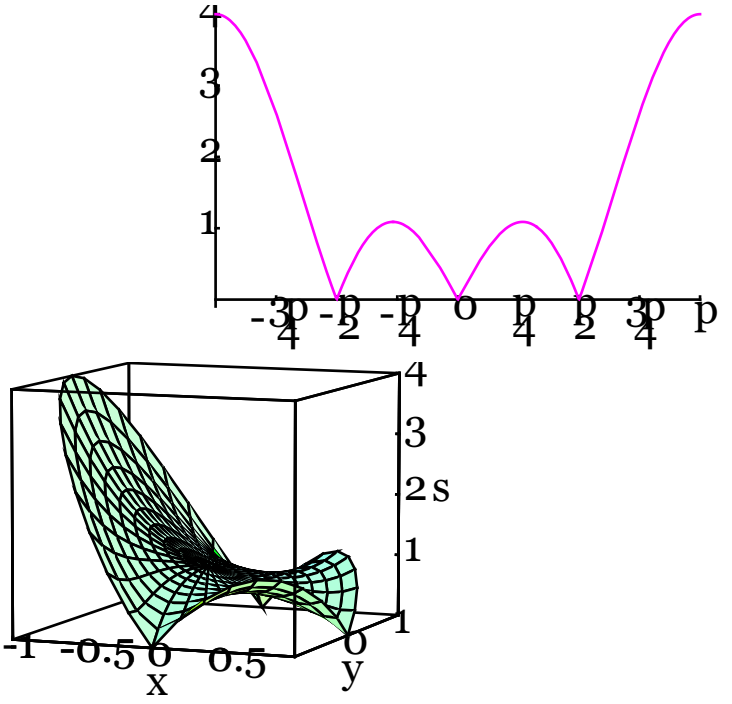
If we choose $z_0 = e^{i\theta_0}$, where $\theta_0 = \text{Arg}(b) - \text{Arg}(a)$, then

$$\begin{aligned} \arg(a z_0) &= \arg(a) + \arg(z_0) = \arg(a) + \arg(e^{i\theta_0}) \\ &= \arg(a) + \arg(e^{i(\text{Arg}(b) - \text{Arg}(a))}) \\ &= \arg(a) + \arg(b) - \arg(a) \\ &= \arg(b) \end{aligned}$$

so the vectors $a z_0$ and b lie on the same ray through the origin. This is the requirement for the Inequality (6-58) to be an equality (see Exercise 19 in Section 1.3).
Hence $|az_0 + b| = |az_0| + |b| \le |a| + |b|$, and the result is established.

Explore Solution 6.26.

Extra Example 1. Let $f(z) = z^3 - z^2 + z - 1$. If we set our domain D to be $D: |z| < 1$, then $f(z)$ is continuous on the closed region $R: |z| \le 1$. Show that $\max_{|z| \le 1} |f(z)| = 4$, and this value is assumed by $f(z)$ at a point $z_0 = e^{i\pi} = -1$ on the boundary of D.



Explore Solution for Extra Example 1.

Extra Example 2. Let $f(z) = z^5 - z^4 + z^3 - z^2 + z - 1$. If we set our domain D to be $D: |z| < 1$, then $f(z)$ is continuous on the closed region $R: |z| \le 1$. Show that $\max_{|z| \le 1} |f(z)| = 6$, and this value is assumed by $f(z)$ at a point $z_0 = e^{i\pi} = -1$ on the boundary of D.

Explore Solution for Extra Example 2.

Theorem 6.17 (Cauchy's Inequalities). Let $f(z)$ be analytic in the simply connected domain D that contains the circle $C_R(z_0) = \{z: |z - z_0| = R\}$. If $|f(z)| \le M$ holds for all points $z \in C_R(z_0)$, then

$$\left| f^{(n)}(z_0) \right| \le \frac{n!\,M}{R^n} \quad \text{for} \quad n = 1, 2, \ldots.$$

Proof.

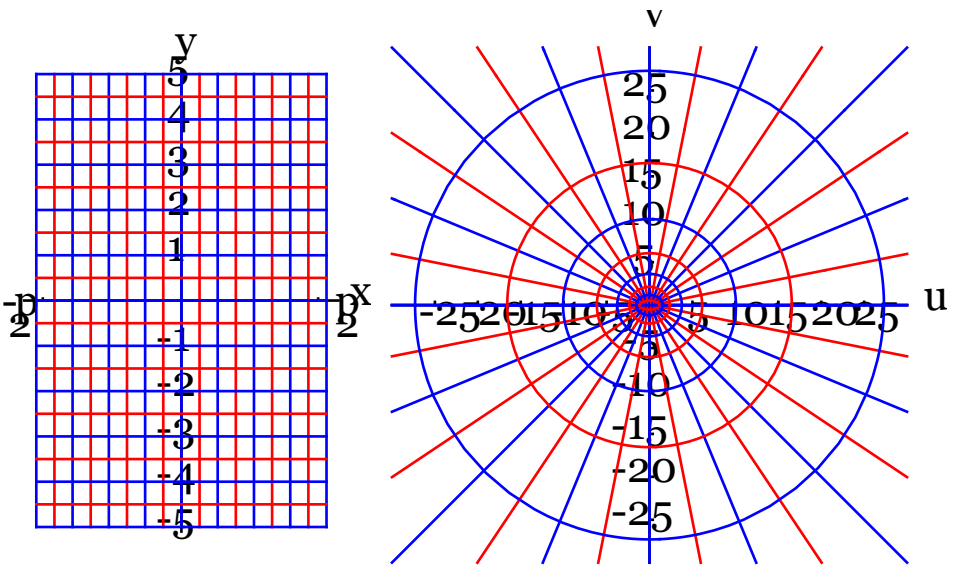
Theorem 6.18 shows that a nonconstant entire function cannot be a bounded function.

Theorem 6.18 (Liouville's Theorem). If $f(z)$ is an entire function and is bounded for all values of z in the complex plane, then $f(z)$ is constant.

Proof.

Example 6.27. Show that the function sin(z) is not a bounded function.

Solution. We established this characteristic with a somewhat tedious argument in Section 5.4. All we need do now is observe that $f(z)$ is not constant, and hence it is not bounded.



Explore Solution 6.27.

We can use Liouville's theorem to establish an important theorem of algebra.

Theorem 6.19 (Fundamental Theorem of Algebra). If $P(z)$ is a polynomial of degree $n \geq 1$, then $P(z)$ has at least one zero.

Proof.

Corollary 6.4. Let $P(z)$ be a polynomial of degree $n \geq 1$. Then $P(z)$ can be expressed as the product of linear factors. That is,

$$P(z) = A(z - z_1)(z - z_2) \ldots (z - z_n)$$

where $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$ counted according to multiplicity an A is a constant.

Proof.

Extra Example 3. Find the n zeros of the equation $z^n - 1 = 0$.

Explore Solution for Extra Example 3.

Extra Example 4. Find the n zeros of the equation $z^n + z^{n-1} + \ldots + z^2 + z + 1 = 0$.

Explore Solution for Extra Example 4.

Extra Example 5. Find the roots of the n^{th} Chebyshev polynomial.

Explore Solution for Extra Example 5.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

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Chapter 7 Taylor and Laurent Series

Overview

Throughout this book we have compared and contrasted properties of complex functions with functions whose domain and range lie entirely within the real numbers. There are many similarities, such as the standard differentiation formulas. However, there are also some surprises, and in this chapter you will encounter one of the hallmarks that distinguishes complex functions from their real counterparts: It is possible for a function defined on the real numbers to be differentiable everywhere and yet not be expressible as a power series (see Exercise 20, Section 7.2). For a complex function, however, things are much simpler! You will soon learn that if a complex function is analytic in the disk $D_r(\alpha)$, its Taylor series about $z = \alpha$ converges to the function at every point in this disk. Thus, analytic functions are locally nothing more than glorified polynomials.

7.1 Uniform Convergence

Complex functions are the key to unlocking many of the mysteries encountered when power series are first introduced in a calculus course. We begin by discussing an important property associated with power series-uniform convergence.

Recall that, for a function $f(z)$ defined on a set T, the sequence of functions $\{S_n(z)\}$ converges to the function $f(z)$ at the point $z_0 \in T$ provided that $\lim_{n\to\infty} S(z_0) = f(z_0)$. Thus, for the particular point z_0, we know that for each $\epsilon > 0$, there exists a positive integer N_{ϵ,z_0} (which depends on both ϵ and z_0) such that

(7-1) if $n \geq N_{\epsilon,z_0}$, then $|S(z_0) - f(z_0)| < \epsilon$.

If $S_n(z)$ is the n^{th} partial sum of the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$, Statement (7-1) becomes

if $n \geq N_{\epsilon,z_0}$, then $\left|\sum_{k=0}^{n-1} c_k (z_0-\alpha)^k - f(z_0)\right| < \epsilon$.

For a given value of ϵ, the integer N_{ϵ,z_0} needed to satisfy Statement (7-1) often depends on our choice of z_0. This is not the case if the sequence $\{S_n(z)\}$ converges uniformly. For a

uniformly convergent sequence, it is possible to find an integer N_ϵ (depending only on ϵ) that guarantees Statement (7-1) no matter what value for $z_0 \in T$ we pick. In other words, if n is large enough, the function $S_n(z)$ is uniformly close to the function $f(z)$ for all $z \in T$. Formally, we have the following definition.

Definition 7.1 (Uniform Convergence),. The sequence $\{S_n(z)\}$ converges uniformly to f(z) on the set T if for every $\epsilon > 0$, there exists a positive integer N_ϵ (depending only on ϵ) such that

(7-2) if $n \geq N$, then $|S_n(z) - f(z)| < \epsilon$ for all $z \in T$.

If $S_n(z)$ is the n^{th} partial sum of the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$, we say that the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ converges uniformly to f(z) on the set T.

Example 7.1. The sequence $\{S_n(z)\} = \left\{e^z + \frac{1}{n}\right\}$ converges uniformly to the function $f(z) = e^z$ on the entire complex plane because for any $\epsilon > 0$, statement (7-2) is satisfied for all z for $n \geq N_\epsilon$, where N_ϵ is any integer greater than $\frac{1}{\epsilon}$. We leave the details of showing this result as an exercise.

A good example of a sequence of functions that does not converge uniformly is the sequence of partial sums comprising the geometric series. Recall that the geometric series has $S_n(z) = \sum_{k=0}^{n} z^k$ converging to $f(z) = \frac{1}{1-z}$ for all $z \in D_1(0)$. Because the real numbers are a subset of the complex numbers, we can show statement (7-2) is not satisfied by demonstrating it does not hold when we restrict our attention to the real numbers. In that context, $D_1(0)$ becomes the open interval (-1,1), and the inequality, $|S_n(z) - f(z)| < \epsilon$, becomes $|S_n(x) - f(x)| < \epsilon$, which for real variables is equivalent to $f(x) - \epsilon < S_n(x) < f(x) + \epsilon$. If Statement (7-2) were to be satisfied, then given $\epsilon > 0$, $S_n(x)$ would be within an ϵ-bandwidth of $f(x)$ for all x in the interval $(-1, 1)$ provided n were large enough. Figure 7.1 illustrates that there is an such that, no matter how large n is, we can find $x_0 \in (-1, 1)$ such that $S_n(x_0)$ lies outside this bandwidth. In other words, Figure 7.1 illustrates the negation of Statement (7-2), which in technical terms we state as:

(7-3) There exists $\epsilon > 0$, such that for all positive integers N,
there is some $n \geq N$ and some $z_0 \in T$
such that $|S_n(z_0) - f(z_0)| \geq \epsilon$.

In the exercises, we ask you to use Statement (7-3) to show that the partial sums of the geometric series do not converge uniformly to $f(z) = \frac{1}{1-z}$ for $z \in D_1(0)$.

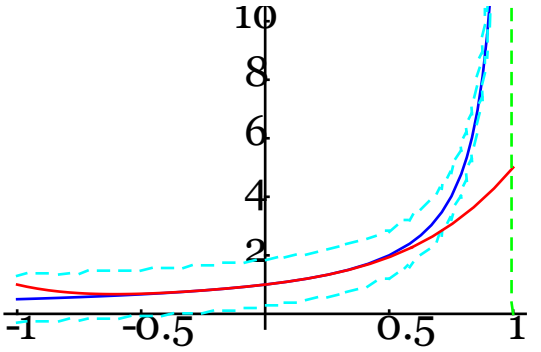


Figure 7.1 The geometric series does not converge uniformly on (-1,1).

Exploration.

A useful procedure known as the Weierstrass M-test can help determine whether an infinite series is uniformly convergent.

Theorem 7.1. (Weierstrass M-Test) Suppose the infinite series $\sum_{k=0}^{\infty} u_k(z)$ has the property that for each k, we have $|u_k(z)| \le M_k$ for all $z \in T$. If $\sum_{k=0}^{\infty} M_k$ converges, then $\sum_{k=0}^{\infty} u_k(z)$ converges uniformly on T.

Proof.

Theorem 7.2 gives an interesting application of the Weierstrass M-test.

Theorem 7.2. Suppose the power series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ has radius of convergence $\rho > 0$. Then for each r, (where $0 < r < \rho$), the series converges uniformly on the closed disk $\overline{D_r}(\alpha) = \{z : |z-\alpha| \le r\}$.

Proof.

An immediate consequence of Theorem 7.2 is Corollary 7.1.

Corollary 7.1. For each r, (where $0 < r < 1$), the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly on the closed disk $\overline{D_r}(0) = \{z : |z| \le r\}$.

Proof.

Theorem 7.3 gives important properties of uniformly convergent sequences.

Theorem 7.3. Suppose $\{S_k(z)\}$ is a sequence of continuous functions defined on a set T containing the contour C.
If $\{S_k(z)\}$ converges uniformly to $f(z)$ on the set T, then

(i) $f(z)$ is continuous on T, and

(ii) $\lim_{k\to\infty} \int_C S_k(z)\,dz = \int_C \lim_{k\to\infty} S_k(z)\,dz = \int_C f(z)\,dz.$

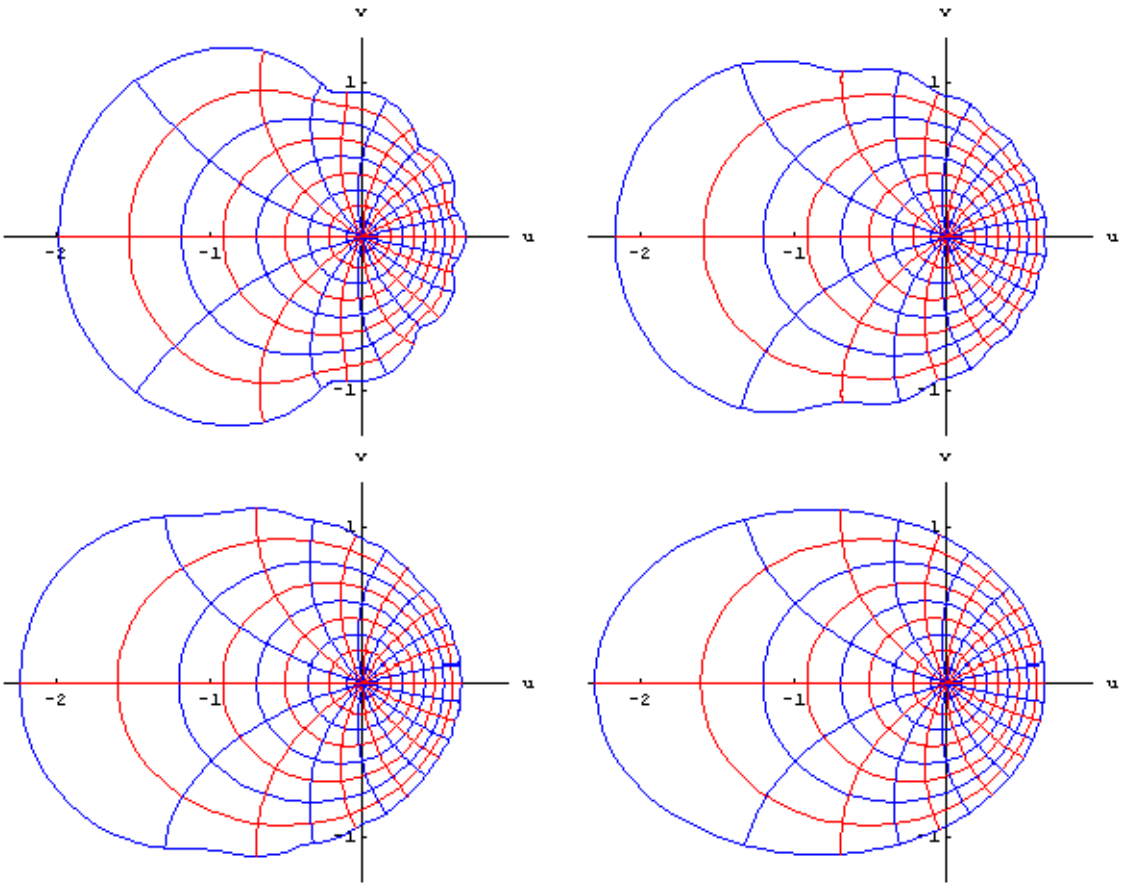
Proof.

Corollary 7.2. If the series $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ converges uniformly to $f(z)$ on the set T, and C is a contour contained in T, then

$$\sum_{n=0}^{\infty} \int_C c_n (z-\alpha)^n\,dz = \int_C \sum_{n=0}^{\infty} c_n (z-\alpha)^n\,dz = \int_C f(z)\,dz.$$

Proof.

Example 7.2. Show that $-\mathrm{Log}(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ for all $z \in D_1(0) = \{z : |z| < 1\}$.



Solution. For $z_0 \in D_1(0)$, we choose r and R so that $0 \le |z_0| < r < R < 1$, thus ensuring that $z_0 \in \overline{D_r}(0)$ and that $\overline{D_r}(0) \subset D_R(0)$. By Corollary 7.1, the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly to $f(z) = \frac{1}{1-z}$ on $\overline{D_r}(0)$. If C is any contour contained in $\overline{D_r}(0)$, Corollary 7.2 gives

(7-4) $$\int_C \frac{1}{1-z}\, dz = \sum_{n=0}^{\infty} \int_C z^n\, dz.$$

Clearly, the function $f(z) = \frac{1}{1-z}$ is analytic in the simply connected domain $D_R(0)$, and $F(z) = -\mathrm{Log}(1-z)$ is an antiderivative of f(z) for all $z \in D_R(0)$, where Log is the principal branch of the logarithm. Likewise, $g(z) = z^n$ is analytic in the simply connected domain $D_R(0)$, and $G(z) = \frac{1}{n+1} z^{n+1}$ is an antiderivative of g(z) for all $z \in D_R(0)$. Hence, if C is the straight-line segment joining 0 to z_0, we can apply Theorem 6.9 to Equation (7-4) to get

$$-\mathrm{Log}(1-z)\Big|_{z=0}^{z=z_0} = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} z^{n+1} \Big|_{z=0}^{z=z_0} \right),$$

which becomes

$$-\mathrm{Log}(1-z_0) + \mathrm{Log}(1) = \sum_{n=0}^{\infty} \frac{1}{n+1} z_0^{n+1},$$

which can be written as

$$-\mathrm{Log}(1-z_0) = \sum_{n=1}^{\infty} \frac{1}{n} z_0^n.$$

The point $z_0 \in D_1(0)$ was arbitrary, so we are done.

Explore Solution 7.2.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

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7.2 Taylor Series Representations

In Section 4.4 we showed that functions defined by power series have derivatives of all orders (Theorem 4.16). In Section 6.5 we demonstrated that analytic functions also have derivatives of all orders (Corollary 6.2). It seems natural, therefore, that there would be some connection between analytic functions and power series. As you might guess, the connection exists via the Taylor and Maclaurin series of analytic functions.

Definition 7.2 (Taylor Series). If $f(z)$ is analytic at $z = \alpha$, then the series

$$f(\alpha) + f'(\alpha)(z-\alpha) + \frac{f^{(2)}(\alpha)}{2!}(z-\alpha)^2 + \frac{f^{(2)}(\alpha)}{3!}(z-\alpha)^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k$$

is called the Taylor series for $f(z)$ centered at $z = \alpha$. When the center is $\alpha = 0$, the series is called the Maclaurin series for $f(z)$.

To investigate when these series converge we will need the following lemma.

Lemma 7.1. If z, z_0 and α are complex numbers with $z \neq z_0$, and $z \neq \alpha$, then

$$\frac{1}{z-z_0} = \frac{1}{z-\alpha} + \frac{z_0-\alpha}{(z-\alpha)^2} + \frac{(z_0-\alpha)^2}{(z-\alpha)^3} + \cdots + \frac{(z_0-\alpha)^n}{(z-\alpha)^{n+1}} + \frac{1}{z-z_0}\frac{(z_0-\alpha)^{n+1}}{(z-\alpha)^{n+1}}$$

where n is a positive integer.

Proof.

We are now ready for the main result of this section.

Theorem 7.4 (Taylor's Theorem). Suppose $f(z)$ is analytic in a domain G, and that $D_R(\alpha) = \{z : |z-\alpha| < R\}$ is any disk contained in G. Then the Taylor series for $f(z)$ converges to $f(z)$ for all z in $D_R(\alpha)$; that is,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (z-\alpha)^k \quad \text{for all} \quad z \in D_R(\alpha).$$

Furthermore, for any r, 0<r<R, the convergence is uniform on the closed subdisk $\overline{D_r}(\alpha) = \{z : |z-\alpha| \le r\}$ for $0 < r < R$.

Proof.

A singular point of a function is a point at which the function fails to be analytic. You will see in Section 7.4 that singular points of a function can be classified according to how badly the function behaves at those points. Loosely speaking, a nonremovable singular point of a function has the property that it is impossible to redefine the value of the function at that point so as to make it analytic there. For example, the function $f(z) = \frac{1}{1-z}$ has a nonremovable singularity at $z = 1$. We give a formal definition of this concept in Section 7.4, but with this language we can nuance Taylor's theorem a bit.

Corollary 7.3. Suppose that $f(z)$ is analytic in the domain G that contains the point $z = \alpha$. Let z_0 be a nonremovable singular point of minimum distance to the point $z = \alpha$. If $|z_0 - \alpha| = R$, then

(i) the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z-\alpha)^n$ converges to $f(z)$ on all of $D_R(\alpha)$,

and

(ii) if $|z_1 - \alpha| = S > R$, the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z_1-\alpha)^n$ does not converges to $f(z_1)$.

Proof.

Example 7.3. Show that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n$ is valid for all $z \in D_1(0) = \{z : |z| < 1\}$.

Solution. In Example 4.24 we established this identity with the use of Theorem 4.17. We now do so via Theorem 7.4. If $f(z) = \frac{1}{(1-z)^2}$, then a standard induction argument (which we leave as an exercise) will show that $f^{(n)}(z) = \frac{(n+1)!}{(1-z)^{n+2}}$ for $z \in D_1(0)$. Thus $f^{(n)}(0) = (n+1)!$, and Taylor's theorem gives

$$f(z) = \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} z^n = \sum_{n=0}^{\infty} (n+1) z^n,$$

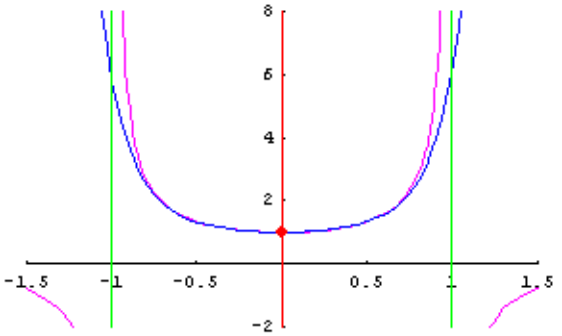
and since $f(z)$ is analytic in $D_1(0)$, this series expansion is valid for all $z \in D_1(0)$.

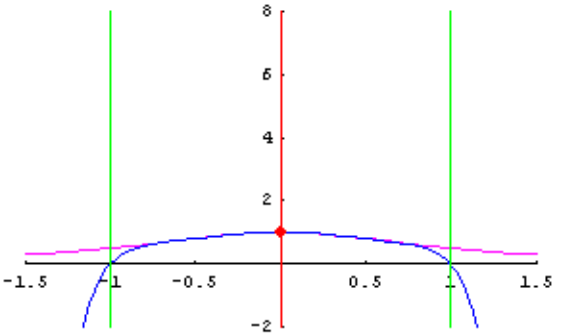
Explore Solution 7.3.

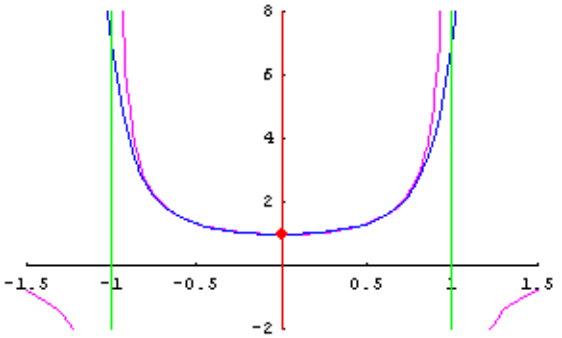
Example 7.4. Show that, for $z \in D_1(0)$,

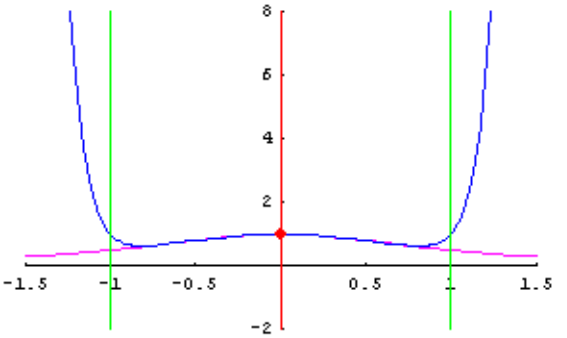
(7-12) $$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n} \quad \text{and} \quad \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

Remark. For the real variable case we have the following series approximations that were studied in calculus. Observe that their interval of convergence is the open interval $(-1, 1)$.









$\frac{1}{1-x^2} \approx \sum_{n=0}^{5} x^{2n}, \sum_{n=0}^{6} x^{2n}$ $\qquad$ $\frac{1}{1+x^2} \approx \sum_{n=0}^{5} (-1)^n x^{2n}, \sum_{n=0}^{6} (-1)^n x^{2n}$

Solution. For $z \in D_1(0)$,

(7-13) $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

If we let z^2 take the role of z in (7-13), we get that

$$\frac{1}{1-z^2} = \frac{1}{1-(z^2)} = \sum_{n=0}^{\infty} (z^2)^n = \sum_{n=0}^{\infty} z^{2n},$$

for $z^2 \in D_1(0)$. But $z^2 \in D_1(0)$ iff $z \in D_1(0)$, thus we have proven that $\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}$ for $z \in D_1(0)$.

Next, let $-z^2$ take the role of z in Equation (7-13), we get that

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

gives the second part of Equations (7-12).

Explore Real Solution 7.4 (a).

Explore Complex Solution 7.4 (a).

Explore Real Solution 7.4 (b).

Explore Complex Solution 7.4 (b).

Remark 7.1 Corollary 7.3 clears up what often seems to be a mystery when series are first introduced in calculus. The calculus analog of Equations (7-12) is

(7-14) $\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ and $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $x \in (-1, 1)$.

For many students, it makes sense that the first series in Equations (7-14) converges only on the interval $(-1, 1)$ because $\frac{1}{1-x^2}$ is undefined at the points $x = \pm 1$. It seems unclear as to why this should also be the case for the series representing $\frac{1}{1+x^2}$, since the real-valued function $f(x) = \frac{1}{1+x^2}$ is defined everywhere. The explanation, of course, comes from the complex domain. The complex function $f(z) = \frac{1}{1+z^2}$ is not defined everywhere. In fact, the singularities of $f(z)$ are at the points $z = \pm i$, and the distance between them and the point $z = \alpha = 0$ equals 1.According to Corollary 7.3, therefore, Equations (7-14) are valid only for $z \in D_1(0)$, and thus Equations (7-14) are valid only for the real numbers $x \in (-1, 1)$.

Alas, there is a potential fly in this ointment: Corollary 7.3 applies to Taylor series. To form the Taylor series of a function, we must compute its derivatives. We didn't get the series in Equations (7-12) by computing derivatives, so how do we know that they are indeed the Taylor series centered at $z = \alpha = 0$? Perhaps the Taylor series would give completely different expressions from those given by Equations (7-12). Fortunately, Theorem 7.5 removes this possibility.

Theorem 7.5 (Uniqueness of Power Series). Suppose that in some disk $D_r(\alpha)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n = \sum_{n=0}^{\infty} b_n (z-\alpha)^n.$$

Then $a_n = b_n$ for $n = 0, 1, 2, \ldots$.

Proof.

Example 7.5. Find the Maclaurin series for $f(z) = \sin^3 z$.

Solution. Computing derivatives for $f(z)$ would be an onerous task. Fortunately, we can make use of the trigonometric identity

$$\sin^3 z = \frac{3}{4}\sin z - \frac{1}{4}\sin 3z.$$

Recall that the series for sin z (valid for all z) is $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$. Using the identity for $\sin^3 z$, we obtain

$$\begin{aligned}
\sin^3 z &= \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (3z)^{2n+1} \\
&= \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} - \frac{3}{4}\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n+1)!} z^{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{3(1-9^n)}{4(2n+1)!} z^{2n+1}
\end{aligned}$$

By the uniqueness of power series, this last expression is the Maclaurin series for $\sin^3 z$.

Explore Solution 7.5.

In the preceding argument we used some obvious results of power series representations that we haven't yet formally stated. The requisite results are part of Theorem 7.6.

Theorem 7.6. Let $f(z)$ and $g(z)$ have the power series representations

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \quad \text{for } z \in D_{r_1}(\alpha),$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n (z-\alpha)^n \quad \text{for } z \in D_{r_2}(\alpha).$$

If $r = \min\{r_1, r_2\}$ and β is any complex constant, then

(7-15) $$\beta f(z) = \sum_{n=0}^{\infty} \beta a_n (z-\alpha)^n \quad \text{for } z \in D_{r_1}(\alpha),$$

(7-16) $$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z-\alpha)^n \quad \text{for } z \in D_r(\alpha), \text{ and}$$

(7-17) $$f(z) * g(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \quad \text{for } z \in D_r(\alpha), \text{ where}$$

(7-18) $$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Identity (7-17) is known as the Cauchy product of the series for $f(z)$ and $g(z)$.

Proof.

Example 7.6. Use the Cauchy product of series to show that

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \quad \text{for} \quad z \in D_1(0).$$

Solution. We let $f(z) = g(z) = \frac{1}{1-z}$, for $z \in D_1(0)$. In terms of Theorem 7.6, we have $a_n = b_n = 1$, for all n, and thus Equation (7-17) gives

$$\frac{1}{(1-z)^2} = h(z) = f(z)\, g(z)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} 1\right) z^n$$

$$= \sum_{n=0}^{\infty} (n+1) z^n$$

Explore Solution 7.6.

Extra Example 1. Use the Cauchy product of series to show that

$$\frac{z}{(z-1)^4} = \sum_{n=0}^{\infty} \frac{n(1+n)(2+n)}{6} z^n \quad \text{for} \quad z \in D_1(0).$$

Explore Solution for Extra Example 1.

Extra Example 2. Show that $\frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$ for $z \in D_1(0)$.

Solution. Use the result of Example 7.6 and (7-16) and obtain

$$\frac{z}{(1-z)^2} = \frac{z-1+1}{(1-z)^2}$$

$$= \frac{1}{(1-z)^2} + \frac{-1}{(1-z)}$$

$$= \sum_{n=0}^{\infty} (n+1)\, z^n - \sum_{n=0}^{\infty} z^n$$

$$= \sum_{n=0}^{\infty} n\, z^n$$

$$= \sum_{n=0}^{\infty} n\, z^n$$

Explore Solution for Extra Example 2.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

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7.3 Laurent Series Representations

Suppose $f(z)$ is not analytic in $D_R(\alpha)$, but is analytic in $D_R^*(\alpha) = \{z : 0 < |z-\alpha| < R\}$. For example, the function $f(z) = \frac{1}{z^3} e^z$ is not analytic when $z = 0$ but is analytic for $|z| > 0$. Clearly, this function does not have a Maclaurin series representation. If we use the Maclaurin series for $g(z) = e^z$, however, and formally divide each term in that series by z^3, we obtain the representation

$$f(z) = \frac{1}{z^3} e^z = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!\,z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \frac{z^3}{6!} + \cdots$$

that is valid for all z such that $|z| > 0$.

Extra Example 1. Use *Mathematica* to find the series for $f(z) = \frac{e^z}{z^3}$.

Explore Solution for Extra Example 1.

This example raises the question as to whether it might be possible to generalize the Taylor series method to functions analytic in an annulus

$$A(\alpha, r, R) = \{z : r \le |z-\alpha| \le R\}.$$

Perhaps we can represent these functions with a series that employs negative powers of z in some way as we did with $f(z) = \frac{1}{z^3} e^z$. As you will see shortly, we can indeed. We begin by defining a series that allows for negative powers of z.

Definition 7.3 (Laurent Series). Let c_n be a complex number for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. The doubly infinite series $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$, called a Laurent series, is defined by

(7-21) $$\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n = \sum_{n=1}^{\infty} c_{-n} (z-\alpha)^{-n} + \sum_{n=0}^{\infty} c_n (z-\alpha)^n,$$

provided the series on the right-hand side of this equation converge.

Remark 7.2. Recall that $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ is a simplified expression for the sum $c_{n0} + \sum_{n=1}^{\infty} c_n (z-\alpha)^n$. At times it will be convenient to write $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ as

$$\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n = \sum_{n=-\infty}^{-1} c_n (z-\alpha)^n + \sum_{n=0}^{\infty} c_n (z-\alpha)^n,$$

rather than using the expression given in Equation (7-21).

Definition 7.4. Given $0 \le r < R$, we define the annulus centered at α with radii r and R by

$$A(\alpha, r, R) = \{z : r < |z-\alpha| < R\}.$$

The closed annulus centered at with radii r and R is denoted by

$$\overline{A}(\alpha, r, R) = \{z : r \le |z-\alpha| \le R\}.$$

Figure 7.3 illustrates these terms.

Figure 7.3 The closed annulus $\overline{A}(\alpha, r, R)$. The shaded portion is the open annulus $A(\alpha, r, R)$.

Theorem 7.7. Suppose that the Laurent series $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ converges on an annulus $A(\alpha, r, R) = \{z : r < |z-\alpha| < R\}$. Then the series converges uniformly on any closed subannulus $\overline{A}(\alpha, s, t) = \{z : s \le |z-\alpha| \le t\}$ where $r < s < t < R$.

Proof.

The main result of this section specifies how functions analytic in an annulus can be expanded in a Laurent series. In it, we will use symbols of the form $C_\rho^+(\alpha)$, which - we remind you - designate the positively oriented circle with radius ρ and center α. That is, $C_\rho^+(\alpha) = \{z : |z-\alpha| < \rho\}$, oriented counterclockwise.

Theorem 7.8 (Laurent's Theorem). Suppose $0 \le r < R$, and that $f(z)$ is analytic in the annulus $A = A(\alpha, r, R) = \{z : r < |z-\alpha| < R\}$ shown in Figure 7.3. If ρ is any number such that $r < \rho < R$, then for all $z_0 \in A(\alpha, r, R)$ the function value $f(z_0)$ has the Laurent series representation

(7-22) $$f(z_0) = \sum_{n=-\infty}^{\infty} c_n (z_0-\alpha)^n = \sum_{n=1}^{\infty} c_{-n} (z_0-\alpha)^{-n} + \sum_{n=0}^{\infty} c_n (z_0-\alpha)^n,$$

where for $n = 0, 1, 2, \ldots$, the coefficients c_{-n} and c_n are given by

(7-23) $$c_{-n} = \frac{1}{2\pi i} \int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{-n+1}} dz \quad \text{and} \quad c_n = \frac{1}{2\pi i} \int_{C_\rho^+(\alpha)} \frac{f(z)}{(z-\alpha)^{n+1}} dz.$$

Proof.

Remark. What happens to the Laurent series if $f(z)$ is analytic in the disk $D_R(\alpha)$? If we look at equation (7-23), we see that the coefficient c_n for the positive power $(z_0-\alpha)^n$ equals $\frac{f^{(n)}(z_0)}{n!}$ by using Cauchy's integral formula for derivatives. Hence, the series in equation (7-22) involving the positive powers of $(z_0-\alpha)$ is actually the Taylor series for $f(z)$. The Cauchy-Goursat theorem shows us that the coefficients for the negative powers of $(z_0-\alpha)$ equal zero. In this case, therefore, there are no negative powers involved,

and the Laurent series reduces to the Taylor series.

Theorem 7.9 delineates two important aspects of the Laurent series.

Theorem 7.9 (Uniqueness and differentiation of Laurent Series). Suppose that $f(z)$ is analytic in the annulus $A(\alpha, r, R)$, and has the Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n \text{ for all } z \in A(\alpha, r, R).$$

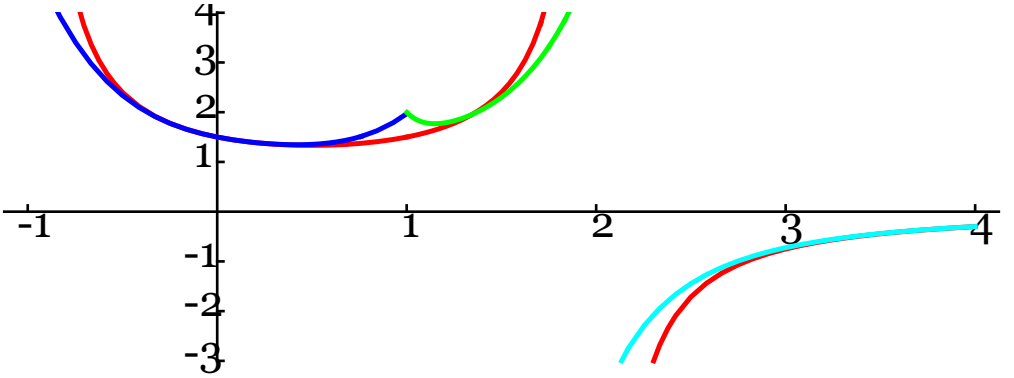
(i) If $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-\alpha)^n$ for all $z \in A(\alpha, r, R)$, then $b_n = c_n$ for all n. (In other words, the Laurent series for $f(z)$ in a given annulus is unique.)

(ii) For all $z \in A(\alpha, r, R)$, the derivatives for $f(z)$ may be obtained by termwise differentiation of its Laurent series.

Proof.

The uniqueness of the Laurent series is an important property because the coefficients in the Laurent expansion of a function are seldom found by using Equation (7-23). The following examples illustrate some methods for finding Laurent series coefficients.

Example 7.7. Find three different Laurent series representations for the function $f(z) = \frac{3}{2+z-z^2}$ involving powers of z.



Solution. The function $f(z)$ has singularities at $z = -1, 2$ and is analytic in the disk $D: |z| < 1$, in the annulus $A: 1 < |z| < 2$, and in the region $R: |z| > 2$. We want to find a different Laurent series for $f(z)$ in each of the three domains D, A, and R. We start by writing $f(z)$ in its partial fraction form:

(7-30) $$f(z) = \frac{3}{(1+z)(2-z)} = \frac{1}{1+z} + \frac{1}{2-z} = \frac{1}{1+z} + \frac{1}{2}\,\frac{1}{1-\frac{z}{2}}.$$

We use Theorem 4.12 and Corollary 4.1 to obtain the following representations for the terms on the right side of Equation (7-30):

(7 - 31) $$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{valid for} \quad |z| < 1,$$

(7 - 32) $$\frac{1}{1+z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} \quad \text{valid for} \quad |z| > 1,$$

(7 - 33) $$\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad \text{valid for} \quad |z| < 2,$$

(7 - 34) $$\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = \sum_{n=1}^{\infty} \frac{-2^{n-1}}{z^n} \quad \text{valid for} \quad |z| > 2.$$

Representations (7-31) and (7-33) are both valid in the disk $D: |z| < 1$, and thus we have

$$f(z) = \sum_{n=0}^{\infty} (-1)^n z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{2^{n+1}} \right) z^n \quad \text{valid for } |z| < 1,$$

which is a Laurent series that reduces to a Maclaurin series.

In the annulus $A: 1 < |z| < 2$, representations (7-32) and (7-33) are valid; hence we get

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad \text{valid for } 1 < |z| < 2.$$

Finally, in the region $R: |z| > 2$ we use Representations (7-32) and (7-34) to obtain

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{z^n} + \sum_{n=1}^{\infty} \frac{-2^{n-1}}{z^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 2^{n-1}}{z^n} \quad \text{valid for } |z| > 2.$$

Explore Solution 7.7.

Example 7.8. Find the Laurent series representation for $f(z) = \frac{\cos(z) - 1}{z^4}$ that involves powers of z.

Solution. We know that $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$, and hence the Maclaurin series for $\cos z - 1$ is

$$\cos z - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots,$$

then we can write

$$f(z) = \frac{1}{z^4} \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n z^{2n-4}}{(2n)!}$$

or in another way we can write

$$f(z) = \frac{\cos z - 1}{z^4} = \frac{-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots}{z^4}$$

We formally divide each term by z^4 to obtain the Laurent series

$$f(z) = -\frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \frac{z^4}{8!} - \frac{z^6}{10!} + \cdots$$

$$= -\frac{1}{2z^2} + \frac{1}{24} - \frac{z^2}{720} + \frac{z^4}{40\,320} - \frac{z^6}{3\,628\,800} + \cdots$$

Explore Solution 7.8.

Example 7.9. Find the Laurent series for $f(z) = \exp\left(\frac{-1}{z^2}\right)$ centered at $\alpha = 0$.

Solution. The Maclaurin series for $\exp z$ is $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, which is valid for all z. We let $\frac{-1}{z^2}$ take the role of z in this equation to get

$$\exp\left(\frac{-1}{z^2}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-1}{z^2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n}{(z^2)^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(z^2)^n},$$

which is valid for $|z| > 0$.

Explore Solution 7.9.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

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7.4 Singularities, Zeros, and Poles

Recall that the point $z = \alpha$ is called a singular point, or singularity of the complex function $f(z)$ if $f(z)$ is not analytic at $z = \alpha$, but every neighborhood $D_R(\alpha)$ of α contains at least one point at which $f(z)$ is analytic. For example, the function $f(z) = \frac{1}{1-z}$ is not analytic at $z = 1$, but is analytic for all other values of z. Thus the point $z = 1$ is a singular point of $f(z)$. As another example, consider $g(z) = \text{Log}(z)$. We saw in Section 5.2 that $g(z)$ is analytic for all z except at the origin and at all points on the negative real-axis. Thus, the origin and each point on the negative real axis is a singularity of $g(z) = \text{Log}(z)$.

The point α is called a isolated singularity of the complex function $f(z)$ if f is not analytic at $z = \alpha$, but there exists a real number $R > 0$ such that $f(z)$ is analytic everywhere in the punctured disk $D_R^*(\alpha)$. The function $f(z) = \frac{1}{1-z}$ has an isolated singularity at $z = 1$.

The function $g(z) = \text{Log}(z)$, however, the singularity at $z = 0$ (or at any point of the negative real axis) that is not isolated, because any neighborhood of contains points on the negative real axis, and $g(z) = \text{Log}(z)$ is not analytic at those points. Functions with isolated singularities have a Laurent series because the punctured disk $D_R^*(\alpha)$ is the same as the annulus $A(\alpha, 0, R)$. The logarithm function $g(z) = \text{Log}(z)$ does not have a Laurent series at any point $z = -a$ on the negative real-axis. We now look at this special case of Laurent's theorem in order to classify three types of isolated singularities.

Definition 7.5 (Removable Singularity, Pole of order k, Essential Singularity). Let $f(z)$ have an isolated singularity at α with Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

Then we distinguish the following types of singularities at α.

(i) If $c_n = 0$ for $n = -1, -2, -3, \ldots$, then we say that $f(z)$ has a removable singularity at α.

(ii) If k is a positive integer such that $c_{-k} \neq 0$ but $c_n = 0$ for $n = -k-1, -k-2, -k-3, \ldots$, then we say that $f(z)$ has a pole of order k at α.

(iii) If $c_n \neq 0$ for infinitely many negative integers n, then we say that $f(z)$ has an essential singularity at $z = \alpha$.

Let's investigate some examples of these three cases.

(i). If $f(z)$ has a removable singularity at $z = \alpha$, then it has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

Theorem 4.17 implies that the power series for $f(z)$ defines an analytic function in the disk $D_R(\alpha)$.

If we use this series to define $f(\alpha) = c_0$, then the function $f(z)$ becomes analytic at $z = \alpha$, removing the singularity.

For example, consider the function $f(z) = \frac{\sin(z)}{z}$. It is undefined at $z = 0$ and has an isolated singularity at $z = 0$, as the Laurent series for $f(z)$ is

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \ldots\right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \frac{z^{10}}{11!} + \ldots$$

valid for $|z| > 0$.

We can remove this singularity if we define $f(0) = 1$, for then $f(z)$ will be analytic at $z = 0$ in accordance with Theorem 4.17.

Exploration 1.

Another example is $g(z) = \frac{\cos(z) - 1}{z^2}$, which has an isolated singularity at the point $z = 0$, as the Laurent series for $g(z)$ is

$$g(z) = \frac{1}{z^2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots\right)$$

$$= -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \frac{z^6}{8!} - \frac{z^8}{10!} + \ldots$$

valid for $|z| > 0$. If we define $f(0) = -\frac{1}{2}$, then $g(z)$ will be analytic for all z.

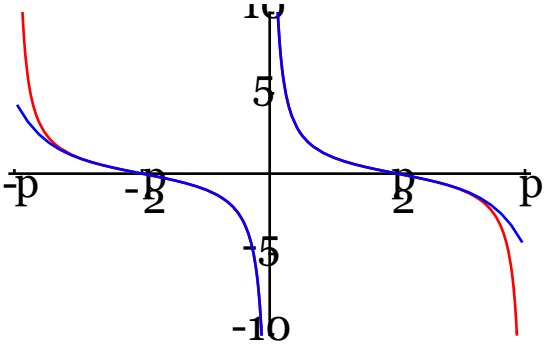
Exploration 2.

(ii). If $f(z)$ has a pole of order k at $z = \alpha$, the Laurent series for $f(z)$ is

$$f(z) = \sum_{n=-k}^{\infty} c_n (z-\alpha)^n \quad \text{valid for} \quad z \in A(\alpha, 0, R).$$

where $c_{-k} \neq 0$.

Extra Example 1. The following example will help this concept. Consider the function $f(z) = \cot z$. The leading term in the Laurent series expansion $S(z)$ is $\frac{1}{z}$ and $S(z)$ goes to ∞ as $z \to 0$ in the same manner as $\cot z$.



Explore Solution Extra Example 1.

For example,

$$f(z) = \frac{\sin(z)}{z^3}$$

$$= \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \ldots\right)$$

$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \frac{z^6}{9!} - \frac{z^{10}}{11!} + \ldots$$

has a pole of order $k = 2$ at $z = 0$.

Exploration 3.

If $f(z)$ has a pole of order 1 at $z = \alpha$, we say that $f(z)$ has a simple pole at $z = \alpha$.

For example,

$$g(z) = \frac{1}{z}e^z = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{n!}z^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}z^{n-1}$$

$$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \frac{z^4}{5!} + \frac{z^5}{6!} + \ldots$$

has a simple pole at $z = 0$.

Exploration 4.

(iii). If infinitely many negative powers of $(z - \alpha)$ occur in the Laurent series, then $f(z)$ has an essential singularity at $z = \alpha$. For example,

$$f(z) = z^2 \sin\left(\frac{1}{z}\right)$$

$$= z^2\left(\frac{1}{z} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \frac{\left(\frac{1}{z}\right)^7}{7!} + \frac{\left(\frac{1}{z}\right)^9}{9!} - \frac{\left(\frac{1}{z}\right)^{11}}{11!} + \ldots\right)$$

$$= z - \frac{1}{3!}z^{-1} + \frac{1}{5!}z^3 - \frac{1}{7!}z^{-5} + \frac{1}{9!}z^{-7} - \frac{1}{11!}z^{-9} + \ldots$$

has an essential singularity at the origin.

Exploration 5.

Definition 7.6 (Zero of order k). A function $f(z)$ analytic in $D_r(\alpha)$ has a zero of order k at the point $z = \alpha$ if and only if

$$f^{(n)}(\alpha) = 0 \text{ for } n = 0, 1, 2, \ldots, k-1, \text{ and } f^{(k)}(\alpha) \neq 0.$$

A zero of order one is sometimes called a simple zero.

Theorem 7.10. A function $f(z)$ analytic in $D_R(\alpha)$ has a zero of order k at the point $z = \alpha$ iff its Taylor series given by $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has

$$c_0 = c_1 = \ldots = c_{k-1} = 0 \text{ and } c_k \neq 0.$$

Proof.

Example 7.10. From Theorem 7.10 we see that the function

$$f(z) = z \sin(z^2) = z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} - \frac{z^{15}}{7!} + \ldots$$

has a zero of order $k = 3$ at $z = 0$. Definition 7.6 confirms this fact because

$$f'(z) = 2z^2 \operatorname{Cos} z^2 + \operatorname{Sin} z^2$$

$$f''(z) = 6z \operatorname{Cos} z^2 - 4z^3 \operatorname{Sin} z^2$$

$$f'''(z) = 6 \operatorname{Cos} z^2 - 8z^4 \operatorname{Cos} z^2 - 24z^2 \operatorname{Sin} z^2$$

Then, $f(0) = f''(0) = f''(0) = 0$, but $f'''(0) = 6 \neq 0$.

Explore Solution 7.10.

Theorem 7.11. Suppose $f(z)$ is analytic in $D_R(\alpha)$. Then $f(z)$ has a zero of order k at the point $z = \alpha$ if and only if it can be expressed in the form

(7-35) $$f(z) = (z-\alpha)^k g(z),$$

where $g(z)$ is analytic at $z = \alpha$ and $g(\alpha) \neq 0$.

Proof.

Proof of Theorem 7.11 is in the book.

An immediate consequence of Theorem 7.11 is Corollary 7.4. The proof is left as an exercise.

Corollary 7.4. If $f(z)$ and $g(z)$ are analytic at $z = \alpha$, and have zeros of orders m and n, respectively at $z = \alpha$, then their product $h(z) = f(z)\, g(z)$ has a zero of order $m + n$ at $z = \alpha$.

Proof.

Example 7.11. Let $f(z) = z^3 \sin z$. Then $f(z)$ can be factored as the product of z^3 and $\sin z$, which have zeros of orders $m = 3$ and $n = 1$, respectively, at $z = 0$.
Hence $z = 0$ is a zero of order 4 of $f(z)$.

Explore Solution 7.11.

Theorem 7.12 gives a useful way to characterize a pole.

Theorem 7.12. A function $f(z)$ analytic in the punctured disk $D_R^*(\alpha)$ has a pole of order k at $z = \alpha$ if and only if it can be expressed in the form

(7-37) $$f(z) = \frac{h(z)}{(z-\alpha)^k},$$

where the function $h(z)$ is analytic at the point $z = \alpha$ and $h(\alpha) \neq 0$.

Proof.

Corollaries 7.5-7.8 are useful in determining the order of a zero or a pole. The proofs follow easily from Theorems 7.10 and 7.12 and are left as exercises.

Corollary 7.5. If $f(z)$ is analytic and has a zero of order k at the point $z = \alpha$, then $g(z) = \frac{1}{f(z)}$ has a pole of order k at $z = \alpha$.

Proof.

Corollary 7.6. If $f(z)$ has a pole of order k at the point $z = \alpha$, then $g(z) = \frac{1}{f(z)}$ has a removable singularity at $z = \alpha$. If we define $g(\alpha) = 0$, then $g(z)$ has a zero of order k at $z = \alpha$.

Proof.

Corollary 7.7. If $f(z)$ and $g(z)$ have poles of orders m and n, respectively at the point $z = \alpha$, then their product $h(z) = f(z)\, g(z)$ has a pole of order $m + n$ at $z = \alpha$.

Proof.

Corollary 7.8. Let $f(z)$ and $g(z)$ be analytic with zeros of orders m and n, respectively at $z = \alpha$. Then their quotient $h(z) = \frac{f(z)}{g(z)}$ has the following behavior:

(i) If $m > n$, then $h(z)$ has a removable singularity at $z = \alpha$. If we define $h(\alpha) = 0$, then $h(z)$ has a zero of order $m - n$ at $z = \alpha$.

(ii) If $m < n$, then $h(z)$ has a pole of order $n - m$ at $z = \alpha$.

(iii) If $m = n$, then $h(z)$ has a removable singularity at $z = \alpha$, and can be defined so that $h(z)$ is analytic at $z = \alpha$, by $h(\alpha) = \lim_{z \to \alpha} h(z)$.

Proof.

Example 7.12. Locate the zeros and poles of $h(z) = \frac{\tan z}{z}$, and determine their order.

Solution. In Section 5.4 we saw that the zeros of $f(z) = \sin z$ occur at the points $z = n\pi$, where n is an integer. Because $f'(n\pi) = \cos n\pi \neq 0$, the zeros of $f(z)$ are simple. Similarly, the function $g(z) = z \cos z$ has simple zeros at the points $z = 0$ and $z = \left(n + \frac{1}{2}\right)\pi$, where n is an integer. From the information given, we find that $h(z) = \frac{f(z)}{g(z)}$ behaves as follows:

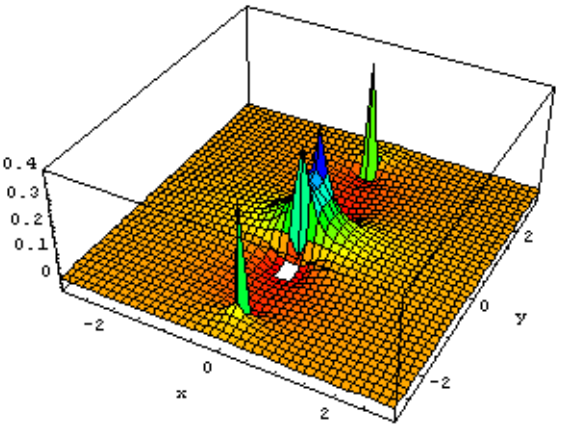
i. $h(z)$ has simple zeros at $z = n\pi$, where $n = \pm 1, \pm 2, \ldots$;

ii. $h(z)$ has simple poles at $z = \left(n + \frac{1}{2}\right)\pi$, where n is an integer; and

iii. $h(z)$ is analytic at $z = 0$ if we define $h(0) = \lim_{z \to 0} h(z) = 1$.

Explore Solution 7.12.

Example 7.13. Locate the poles of $g(z) = \frac{1}{5z^4 + 26z^2 + 5}$, and specify their order.



Solution. The roots of the quadratic equation $5z^2 + 26z + 5 = 0$ occur at the points $z = -5$ and $z = -\frac{1}{5}$. If we replace z with z^2 in this equation, the function $f(z) = 5z^4 + 26z^2 + 5$ has simple zeros at the points $z = \pm i\sqrt{5}$ and $z = \pm \frac{i}{\sqrt{5}}$. Corollary 7.5 implies that $g(z)$ has simple poles at $z = \pm i\sqrt{5}$ and $z = \pm \frac{i}{\sqrt{5}}$.

Explore Solution 7.13.

Example 7.14. Locate the zeros and poles of $g(z) = \frac{\pi \cot(\pi z)}{z^2}$, and determine their order.

Solution. The function $f(z) = z^2 \sin \pi z$ has a zero of order $k = 3$ at $z = 0$ and simple zeros at the points $z = \pm 1, \pm 2, \ldots$. Corollary 7.5 implies that $g(z)$ has a pole of order 3 at the point $z = 0$ and simple poles at the points $z = \pm 1, \pm 2, \ldots$.

Explore Solution 7.14.

Complex Analysis for Mathematics and Engineering

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7.5 Applications of Taylor and Laurent Series

In this section we show how you can use Taylor and Laurent series to derive important properties of analytic functions. We begin by showing that the zeros of an analytic function must be "isolated" unless the function is identically zero. A point α of a set T is called isolated if there exists a disk $D_r(\alpha)$ about α that does not contain any other points of T.

Theorem 7.13. Suppose $f(z)$ is analytic in a domain D containing the point $z = \alpha$ and that $f(\alpha) = 0$. If $f(z)$ is not identically zero, then there exists a punctured disk $D_r^*(\alpha)$ in which $f(z)$ has no zeros.

Proof.

The proofs of the following corollaries are given as exercises.

Corollary 7.9. Suppose that $f(z)$ is analytic in the domain D, and that $\alpha \in D$. If there exists a sequence of points $\{z_n\}$ in D such that $z_n \to \alpha$, and $f(z_n) = 0$, then $f(z) \equiv 0$ for all $\alpha \in D$.

Proof.

Corollary 7.10. Suppose that $f(z)$ and $g(z)$ are analytic in the domain D, where $\alpha \in D$. If there exists a sequence $\{z_n\}$ in D such that $z_n \to \alpha$, and $f(z_n) = g(z_n)$ for all n, then $f(z) \equiv g(z)$ for all $\alpha \in D$.

Proof.

Theorem 7.13 allows us to give a simple proof for one version of L'Hôpital's rule.

Corollary 7.11 (L'Hospital's Rule). Suppose $f(z)$ and $g(z)$ are analytic at $z = \alpha$. If $f(\alpha) = 0$ and $g(\alpha) = 0$ but $g'(\alpha) \neq 0$, then

$$\lim_{z \to \alpha} \frac{f(z)}{g(z)} = \lim_{z \to \alpha} \frac{f'(z)}{g'(z)} = \frac{f'(\alpha)}{g'(\alpha)}.$$

Proof.

We can use Theorem 7.14 to get Taylor series for quotients of analytic functions. Its proof involves ideas from Section 7.2, and we leave it as an exercise.

Theorem 7.14 (Division of Power Series). Suppose $f(z)$ and $g(z)$ are analytic at $z = \alpha$ with power series representations

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z-\alpha)^n \quad \text{for all } z \in D_r(\alpha).$$

If $g(\alpha) \neq 0$, then the quotient $\frac{f(z)}{g(z)}$ has the power series representation

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} c_n (z-\alpha)^n \quad \text{for all} \quad z \in D_r(\alpha),$$

where the coefficients satisfy the equations $a_n = b_0 c_n + b_1 c_{n-1} + \cdots + b_{n-1} c_1 + b_n c_0$.

In other words, the series for the quotient $\frac{f(z)}{g(z)}$ can be obtained by the familiar process of dividing the series for $f(z)$ by the series for $g(z)$ using the standard long division algorithm.

Proof.

Example 7.15. Find the first few terms of the Maclaurin series for $h(z) = \sec z$, if $|z| < \frac{\pi}{2}$, and then compute $h^{(4)}(0)$.

Solution. Using long division, we see that

$$\sec z = \frac{1}{\cos z}$$

$$= \frac{1}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots}$$

$$= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots$$

Using $a_0 = 1, b_0 = 1$ in $a_0 = b_0 c_0$
we get $1 = 1 \times c_0$ which yields $c_0 = 1$.

Using $a_1 = 0, b_0 = 1, b_1 = 0, c_0 = 1$ in $a_1 = b_0 c_1 + b_1 c_0$
we get $0 = 1 \times c_1 + 0 \times 1$ which yields $c_1 = 0$.

Using $a_2 = 0, b_0 = 1, b_1 = 0, b_2 = -\frac{1}{2}, c_0 = 1, c_1 = 0$ in $a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0$
we get $0 = 1 \times c_2 + 0 \times 0 - \frac{1}{2} \times 1$ which yields $c_2 = \frac{1}{2}$.

Using $a_3 = 0, b_0 = 1, b_1 = 0, b_2 = -\frac{1}{2}, b_3 = 0, c_0 = 1, c_1 = 0, c_2 = \frac{1}{2}$
in $a_3 = b_0 c_3 + b_1 c_2 + b_2 c_1 + b_3 c_0$
we get $0 = 1 \times c_3 + 0 \times \frac{1}{2} - \frac{1}{2} \times 0 + 0 \times 1$ which yields $c_3 = 0$.

Using $a_4 = 0, b_0 = 1, b_1 = 0, b_2 = -\frac{1}{2}, b_3 = 0, b_4 = \frac{1}{24}, c_0 = 1, c_1 = 0, c_2 = \frac{1}{2}, c_3 = 0$
in $a_4 = b_0 c_4 + b_1 c_3 + b_2 c_2 + b_3 c_1 + b_4 c_0$
we get $0 = 1 \times c_4 + 0 \times 0 - \frac{1}{2} \times \frac{1}{2} + 0 \times 0 + \frac{1}{24} \times 1$ which yields $c_4 = \frac{5}{24}$.

Thus, we obtain

$$\sec z = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \ldots$$

$$= 1 + 0\, z + \frac{1}{2} z^2 + 0\, z^3 + \frac{5}{24} z^4 + \ldots$$

Moreover, using Taylor's theorem, we see that if $h(z) = \sec z$, then $\frac{h^{(4)}(0)}{4!} = \frac{5}{24}$ and we obtain $h^{(4)}(0) = 5$.

Explore Solution 7.15.

We close this section with some results concerning the behavior of complex functions at points near the different types of isolated singularities. Theorem 7.15 is due to the German mathematicia G. F. Bernhard Riemann (1826--1866).

Theorem 7.15 (Riemann's Theorem). Suppose that $f(z)$ is analytic in $D_r^*(\alpha)$. If $f(z)$ is bounded in $D_r^*(\alpha)$, then either $f(z)$ is analytic at $z = \alpha$ or $f(z)$ has a removable singularity at $z = \alpha$.

Proof.

The proof of Corollary 7.12 is given as an exercise.

Corollary 7.12. Suppose that $f(z)$ is analytic in $D_r^*(\alpha)$, then $f(z)$ can be defined to be analytic at $z = \alpha$ iff $\lim_{z \to \alpha} f(z)$ exists and is finite.

Proof.

Theorem 7.16. Suppose that $f(z)$ is analytic in $D_r^*(\alpha)$. The function $f(z)$ has a pole of order k at $z = \alpha$ iff $\lim_{z \to \alpha} |f(z)| = \infty$.

Proof.

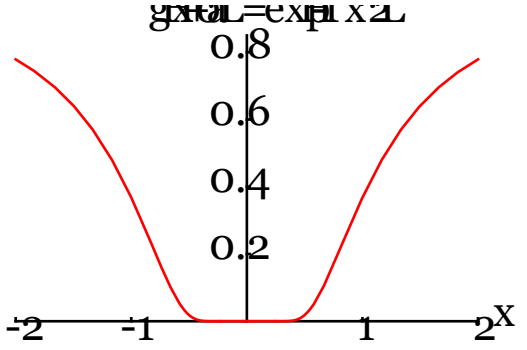
Theorem 7.17. The function $f(z)$ has an essential singularity at $z = \alpha$ iff $\lim_{z \to \alpha} |f(z)|$ does not exist.

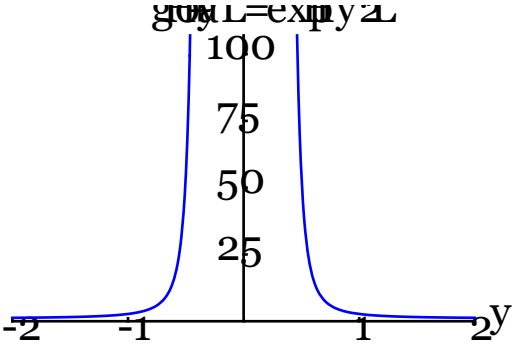
Proof.

Example 7.16. Show that the function $g(z)$ defined by

$$g(z) = \begin{cases} \exp\left(\frac{-1}{z^2}\right), & \text{when } z \neq 0, \text{ and} \\ 0, & \text{when } z = 0, \end{cases}$$

is not continuous at $z = 0$.





Solution. In Exercise 20, Section 7.2, we asked you to show this relation by computing limits along the real and imaginary axes. Note, however, that the Laurent series for $g(z)$ in the annulus $D_r^*(0)$ is

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-1}{z^2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z^{2n}},$$

so that $z = 0$ is an essential singularity for $g(z)$. According to Theorem 7.17, $\lim_{z \to 0} |g(z)|$ doesn't exist, so $g(z)$ is not continuous at $z = 0$.

Explore Solution 7.16.

Complex Analysis for Mathematics and Engineering

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Chapter 8 Residue Theory

8.1 The Residue Theorem

Overview

We now have the necessary machinery to see some amazing applications of the tools we developed in the last few chapters. You will learn how Laurent expansions can give useful information concerning seemingly unrelated properties of complex functions. You will also learn how the ideas of complex analysis make the solution of very complicated integrals of real-valued functions as easy - literally - as the computation of residues. We begin with a theorem relating residues to the evaluation of complex integrals.

The Cauchy integral formulae in Section 6.5 are useful in evaluating contour integrals over a simple closed contour C where the integrand has the form $\frac{f(z)}{(z-z_0)^k}$ and f is an analytic function. In this case, the singularity of the integrand is at worst a pole of order k at z_0. We begin this section by extending this result to integrals that have a finite number of isolated singularities inside the contour C. This new method can be used in cases where the integrand has an essential singularity at z_0 and is an important extension of the previous method.

Definition 8.1 (Residue). Let $f(z)$ have a nonremovable isolated singularity at the point z_0. Then $f(z)$ has the Laurent series representation for all z in some disk $D_R^*(z_0)$ given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.$$

The coefficient a_{-1} of $\frac{1}{z-z_0}$ is called the residue of $f(z)$ at z_0 and we use the notation

$$\text{Res}[f, z_0] = a_{-1}.$$

Example 8.1. If $f(z) = e^{\frac{2}{z}}$, then the Laurent series of f about the point $z_0 = 0$ has the form

$$f(z) = 1 + 2\frac{1}{z} + \frac{2^2}{2!\,z^2} + \frac{2^3}{3!\,z^3} + \frac{2^4}{4!\,z^4} + \frac{2^5}{5!\,z^5} + \ldots, \text{ and}$$

$$\text{Res}[f, 0] = a_{-1} = 2.$$

Explore Solution 8.1.

Example 8.2. Find $\text{Res}[g, z_0]$ if $g(z) = \frac{3}{2z + z^2 - z^3}$.

Solution. Using Example 7.7, we find that $g(z)$ has three Laurent series representations involving powers of z. The Laurent series valid in the punctured disk $D_1^*(0) = \{z : 0 < |z| < 1\}$ is

$$g(z) = \sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{2^{n+1}}\right) z^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} + 1}{2^{n+1}} z^{n-1}.$$

Computing the first few coefficients, we obtain

$$\begin{aligned} g(z) &= \frac{(-1)^0 2^1+1}{2}\frac{1}{z} + \frac{(-1)^1 2^2+1}{2^2} + \frac{(-1)^2 2^3+1}{2^3} z + \frac{(-1)^3 2^4+1}{2^4} z^2 \\ &\quad + \frac{(-1)^4 2^5+1}{2^5} z^3 + \frac{(-1)^5 2^6+1}{2^6} z^4 + \frac{(-1)^n 2^{n+1}+1}{2^{n+1}} z^{n-1} + \ldots \\ &= \frac{3}{2}\frac{1}{z} - \frac{3}{2^2} + \frac{9}{2^3} z - \frac{15}{2^4} z^2 + \frac{33}{2^5} z^3 - \frac{63}{2^6} z^4 + \ldots \end{aligned}$$

Therefore, $\text{Res}[f, 0] = a_{-1} = \frac{3}{2}$.

Explore Solution 8.2.

Recall that, for a function $f(z)$ analytic in $D_R^*(z_0)$ and for any r with $0 < r < R$, the Laurent series coefficients of $f(z)$ are given by

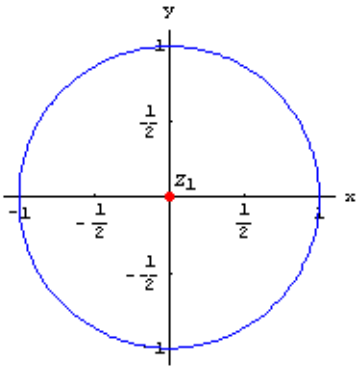
(8-1) $$a_n = \frac{1}{2\pi i}\int_{C_r^+(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots,$$

where $C_r^+(z_0)$ denotes the circle $C_r^+(z_0) = \{z : |z - z_0| = r\}$ with positive orientation. This gives us an important fact concerning $\text{Res}[f, z_0]$. If we set $n = -1$ in Equation (8-1) and replace $C_r^+(z_0)$ with any positively oriented simple closed contour C containing z_0, provided z_0 is the still only singularity of $f(z)$ that lies inside C, then we obtain

(8-2) $$\int_C f(z)\, dz = 2\pi i\, a_{-1} = 2\pi i\, \text{Res}[f, z_0].$$

If we are able to find the Laurent series expansion for $f(z)$, then above equation gives us an important tool for evaluating contour integrals.

Example 8.3. Evaluate $\int_{C_1^+(0)} \exp\left(\frac{2}{z}\right) dz$ where $C_1^+(0)$ denotes the circle $C_1^+(0) = \{z : |z| = 1\}$ with positive orientation.



Solution. In Example 8.1 we showed that the residue of $f(z) = e^{\frac{2}{z}}$ at $z_0 = 0$ is Res[f, 0] = 2. Using Equation (8-2), we get

$$\int_{C_1^+(0)} \exp\left(\frac{2}{z}\right) dz = 2\pi i \, \text{Res}[f, 0] = 4\pi i.$$

Explore Solution 8.3.

Theorem 8.1 (Cauchy's Residue Theorem). Let D be a simply connected domain, and let C be a simple closed positively oriented contour that lies in D. If f(z) is analytic inside C and on C, except at the points $z_1, z_2, \ldots, z_n$ that lie inside C, then

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}[f, z_k].$$

The situation is illustrated in Figure 8.1.

Proof.

Proof of Theorem 8.1 is in the book.

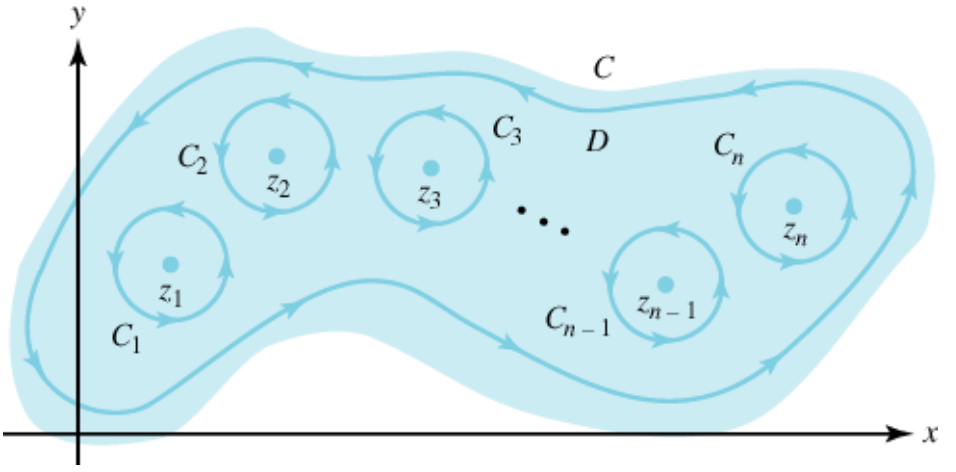


Figure 8.1 The domain D and contour C and the singular points $z_1, z_2, \ldots, z_n$ in the statement of Cauchy's residue theorem.

The calculation of a Laurent series expansion is tedious in most circumstances. Since the residue at z_0 involves only the coefficient a_{-1} in the Laurent expansion, we seek a method to calculate the residue from special information about the nature of the singularity at z_0.

If f(z) has a removable singularity at z_0, then $a_{-1} = 0$ for $n = 1, 2, \ldots$. Therefore,

$\text{Res}[f, z_0] = 0$. Theorem 8.2 gives methods for evaluating residues at poles.

Theorem 8.2 (Residues at Poles).

(i) If $f(z)$ has a simple pole at z_0, then $\text{Res}[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z)$.

(ii) If $f(z)$ has a pole of order 2 at z_0, then $\text{Res}[f, z_0] = \lim_{z \to z_0} \frac{d}{dz} \left((z - z_0)^2 f(z) \right)$.

(iii) If $f(z)$ has a pole of order 3 at z_0, then
$\text{Res}[f, z_0] = \frac{1}{2!} \lim_{z \to z_0} \frac{d^2}{dz^2} \left((z - z_0)^3 f(z) \right)$.

(v) If $f(z)$ has a pole of order k at z_0, then
$\text{Res}[f, z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} \left((z - z_0)^k f(z) \right)$.

Proof.

Example 8.4. Find the residue of $f(z) = \frac{\pi \cot(\pi z)}{z^2}$ at $z_0 = 0$.

Solution. We write $f(z) = \frac{\pi \cot(\pi z)}{z^2} = \frac{\pi \cos(\pi z)}{z^2 \sin(\pi z)}$. Because $z^2 \sin(\pi z)$ has a zero of order 3 at $z_0 = 0$ and $\pi \cot(\pi z) \neq 0$. Thus $f(z)$ has a pole of order 3 at $z_0 = 0$. By part (iii) of Theorem 8.2, we have

$$\text{Res}[f, 0] = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} (z^3 f(z)) = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left(z^3 \frac{\pi \cot(\pi z)}{z^2} \right)$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} (\pi z \cot(\pi z))$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d}{dz} (\pi \cot(\pi z) - \pi^2 z \csc^2(\pi z))$$

$$= \frac{1}{2!} \lim_{z \to 0} (-\pi^2 \csc^2(\pi z) - \pi^2 (\csc^2(\pi z) - 2 \pi z \csc^2(\pi z) \cot(\pi z)))$$

$$= \frac{1}{2!} \lim_{z \to 0} (2 \pi^3 z \cot(\pi z) \csc^2(\pi z) - 2 \pi^2 \csc^2(\pi z))$$

$$= \pi^2 \lim_{z \to 0} (\pi z \cot(\pi z) \csc^2(\pi z) - \csc^2(\pi z))$$

$$= \pi^2 \lim_{z \to 0} (\pi z \cot(\pi z) - 1) \csc^2(\pi z)$$

$$= \pi^2 \lim_{z \to 0} \frac{(\pi z \cot(\pi z) - 1) \sin(\pi z)}{\sin^3(\pi z)}$$

$$= \pi^2 \lim_{z \to 0} \frac{\pi z \cos(\pi z) - \sin(\pi z)}{\sin^3(\pi z)}$$

This last limit involves an indeterminate form, which we evaluate by using L'Hôpital's rule:

$$\text{Res}[f, 0] = \pi^2 \lim_{z \to 0} \frac{\pi z \cos(\pi z) - \sin(\pi z)}{\sin^3(\pi z)}$$

$$= \pi^2 \lim_{z \to 0} \frac{-\pi^2 z \sin(\pi z) + \pi \cos(\pi z) - \pi \cos(\pi z)}{3\pi \cos(\pi z) \sin^2(\pi z)}$$

$$= \pi^2 \lim_{z \to 0} \frac{-\pi^2 z \sin(\pi z)}{3\pi \cos(\pi z) \sin^2(\pi z)}$$

$$= \pi^2 \lim_{z \to 0} \frac{-\pi z}{3 \cos(\pi z) \sin(\pi z)}$$

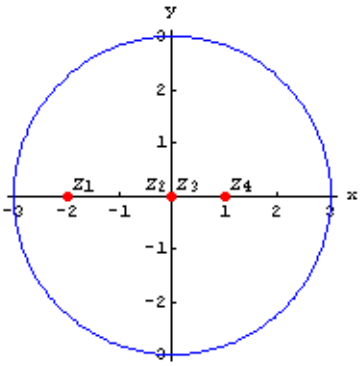
$$= \frac{-\pi^2}{3} \lim_{z \to 0} \frac{\pi z}{\cos(\pi z) \sin(\pi z)}$$

$$= \frac{-\pi^2}{3} \lim_{z \to 0} \frac{\pi z}{\sin(\pi z)} \lim_{z \to 0} \frac{1}{\cos(\pi z)}$$

$$= \frac{-\pi^2}{3} (1) (1) = -\frac{\pi^2}{3}$$

Explore Solution 8.4.

Example 8.5. Find $\int_{C_3^+(0)} \frac{1}{z^4 + z^3 - 2z^2} dz$ where $C_3^+(0)$ denotes the circle $C_3^+(0) = \{z : |z| = 3\}$ with positive orientation.



Solution. We write the integrand as $f(z) = \frac{1}{z^4 + z^3 - 2z^2} = \frac{1}{z^2 (z+2) (z-1)}$.

The singularities of $f(z)$ that lie inside $C_3^+(0)$ are simple poles at the points $z = 1$ and $z = -2$, and a pole of order 2 at the origin. We compute the residues as follows:

$$\text{Res}[f, 0] = \lim_{z \to 0} \frac{d}{dz} (z^2 f(z)) = \lim_{z \to 0} \frac{d}{dz} \left(z^2 \frac{1}{z^2 (z+2) (z-1)} \right)$$

$$= \lim_{z \to 0} \frac{d}{dz} \left(\frac{1}{(z+2) (z-1)} \right) = \lim_{z \to 0} \frac{d}{dz} \left(\frac{1}{z^2+z-2} \right)$$

$$= \lim_{z \to 0} \frac{-2z-1}{(z^2+z-2)^2} = -\frac{1}{4}$$

$$\text{Res}[f, 1] = \lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} (z-1) \frac{1}{z^2 (z+2) (z-1)}$$

$$= \lim_{z \to 1} \frac{1}{z^2 (z+2)} = \frac{1}{3}$$

$$\text{Res}[f, -2] = \lim_{z \to -2} (z - (-2))\, f(z) = \lim_{z \to -2} (z+2) \frac{1}{z^2 (z+2)(z-1)}$$

$$= \lim_{z \to -2} \frac{1}{z^2 (z-1)} = -\frac{1}{12}$$

Finally, the residue theorem yields

$$\int_{C_3^+(0)} \frac{1}{z^4 + z^3 - 2z^2} dz = 2\pi i \left(\text{Res}[f, 0] + \text{Res}[f, 1] + \text{Res}[f, -2]\right)$$

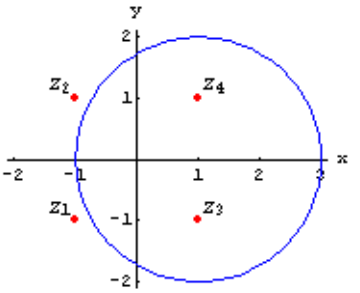
$$= 2\pi i \left(-\frac{1}{4} + \frac{1}{3} - \frac{1}{12}\right)$$

$$= 0$$

The answer, $\int_{C_3^+(0)} \frac{1}{z^4 + z^3 - 2z^2} dz = 0$, is not at all obvious, and all the preceding calculations are required to get it.

Explore Solution 8.5.

Example 8.6. Find $\int_{C_2^+(1)} \frac{1}{z^4 + 4} dz$ where $C_2^+(1)$ denotes the circle $C_2^+(1) = \{z : |z - 1| = 2\}$ with positive orientation.



Solution. The singularities of the integrand $f(z) = \frac{1}{z^4 + 4}$ that lie inside $C_2^+(1)$ are simple poles occurring at the points $z = 1 \pm i$, as the points $z = -1 \pm i$, lie outside $C_2^+(1)$. Factoring the denominator is tedious, so we use a different approach. If z_0 is any one of the singularities of $f(z)$, then we can use L'Hôpital's rule to compute $\text{Res}[f, z_0]$:

$$\text{Res}[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} (z - z_0) \frac{1}{z^4 + 4}$$

$$= \lim_{z \to z_0} \frac{(z - z_0)}{z^4 + 4} = \text{'}\frac{0}{0}\text{'}$$

$$= \lim_{z \to z_0} \frac{1}{4z^3} = \frac{1}{4 z_0^3}$$

Since $z_0^4 = -4$, we can simplify this expression further to yield

$$\text{Res}[f, z_0] = \frac{1}{4 z_0^3} = \frac{z_0}{4 z_0^4} = \frac{z_0}{4(-4)} = -\frac{1}{16} z_0$$

We now use the residue theorem to get

$$\int_{C_2^+(1)} \frac{1}{z^4+4}\,dz = 2\pi i\,(\text{Res}[f, 1+i] + \text{Res}[f, 1-i])$$

$$= 2\pi i\left(-\frac{1+i}{16} - \frac{1-i}{16}\right) = 2\pi i\left(\frac{-1-i}{16} + \frac{-1+i}{16}\right)$$

$$= 2\pi i\left(-\frac{1}{8}\right)$$

$$= -\frac{i\pi}{4}$$

Explore Solution 8.6.

The theory of residues can be used to expand the quotient of two polynomials into its partial fraction representation.

Example 8.7. Let $P(z)$ be a polynomial of degree at most 2. If a, b and c are distinct complex numbers, then

$$f(z) = \frac{P(z)}{(z-a)(z-b)(z-c)} = \frac{\mathbb{A}}{z-a} + \frac{\mathbb{B}}{z-b} + \frac{\mathbb{C}}{z-c},$$

where

$$\mathbb{A} = \text{Res}[f, a] = \frac{P(a)}{(a-b)(a-c)},$$

$$\mathbb{B} = \text{Res}[f, b] = \frac{P(b)}{(b-a)(b-c)}, \text{ and}$$

$$\mathbb{C} = \text{Res}[f, c] = \frac{P(c)}{(c-a)(c-b)}.$$

Solution. It will suffice to prove that $A = \text{Res}[f, a]$. We expand $f(z)$ in its Laurent series about the point $z = a$ by writing the three terms $\frac{A}{z-a}$, $\frac{\mathbb{B}}{z-b}$, and $\frac{\mathbb{C}}{z-c}$ in their Laurent series about the point $z = a$ and adding them. The term $\frac{\mathbb{A}}{z-a}$ is itself a one-term Laurent series about the point $z = a$. The term $\frac{\mathbb{B}}{z-b}$ is analytic at the point $z = a$, and its Laurent series is actually a Taylor series given by

$$\frac{\mathbb{B}}{z-b} = \left(\frac{-\mathbb{B}}{b-a}\right)\left(\frac{1}{1-\frac{z-a}{b-a}}\right)$$

$$= \left(\frac{-\mathbb{B}}{b-a}\right)\sum_{n=0}^{\infty}\left(\frac{z-a}{b-a}\right)^n$$

$$= \left(\frac{-\mathbb{B}}{b-a}\right)\sum_{n=0}^{\infty}\frac{(z-a)^n}{(b-a)^n}$$

$$= -\sum_{n=0}^{\infty}\frac{\mathbb{B}(z-a)^n}{(b-a)^{n+1}}$$

which is valid for $|z-a| < |b-a|$.

Likewise, the Laurent expansion of the term $\frac{\mathbb{C}}{z-c}$ is

$$\frac{\mathbb{C}}{z-c} = -\sum_{n=0}^{\infty}\frac{\mathbb{C}(z-a)^n}{(c-a)^{n+1}},$$

which is valid for $|z-a| < |c-a|$. Thus the Laurent series of $f(z)$ about the point $z = a$ is

$$f(z) = \frac{\mathbb{A}}{z-a} - \sum_{n=0}^{\infty} \frac{\mathbb{B}\,(z-a)^n}{(b-a)^{n+1}} - \sum_{n=0}^{\infty} \frac{\mathbb{C}\,(z-a)^n}{(c-a)^{n+1}},$$

which is valid for $|z-a| < R$, where $R = \min\{|b-a|, |c-a|\}$. Therefore $\text{Res}[f, a] = \mathbb{A}$, and calculation reveals that

$$\text{Res}[f, a] = \lim_{z\to a} (z-a) f(z) = \lim_{z\to a} (z-a) \frac{P(z)}{(z-a)(z-b)(z-c)}$$

$$= \lim_{z\to a} \frac{P(z)}{(z-b)(z-c)} = \frac{P(a)}{(a-b)(a-c)}$$

Explore Solution 8.7.

Example 8.8. Express $f(z) = \frac{3z+2}{z(z-1)(z-2)}$ in partial fractions.

Solution. In Example 8.7 use $a = 0, b = 1, c = 2$ and $P(z) = 3z + 2$. Computing the residues, we obtain

$$\mathbb{A} = \text{Res}[f, 0] = \frac{P(0)}{(0-1)(0-2)} = \frac{2}{(-1)(-2)} = \frac{2}{1} = 1,$$

$$\mathbb{B} = \text{Res}[f, 1] = \frac{P(1)}{(1-0)(1-2)} = \frac{5}{(1)(-1)} = \frac{5}{-1} = -5, \text{ and}$$

$$\mathbb{C} = \text{Res}[f, 2] = \frac{P(2)}{(2-0)(2-1)} = \frac{8}{(2)(1)} = \frac{8}{2} = 4.$$

The formula for $f(z)$ in Example 8.7 gives us

$$f(z) = \frac{\mathbb{A}}{z-0} + \frac{\mathbb{B}}{z-1} + \frac{\mathbb{C}}{z-2}$$

$$= \frac{1}{z-0} + \frac{-5}{z-1} + \frac{4}{z-2}$$

$$= \frac{1}{z} - \frac{5}{z-1} + \frac{4}{z-2}$$

Explore Solution 8.8.

Remark 8.1. If a repeated root occurs, then the process is similar, and it is easy to show that if $P(z)$ has degree of at most 2, then

$$f(z) = \frac{P(z)}{(z-a)^2(z-b)} = \frac{\mathbb{A}}{(z-a)^2} + \frac{\mathbb{B}}{z-a} + \frac{\mathbb{C}}{z-b},$$

where

$$\mathbb{A} = \text{Res}[(z-a) f(z), a],$$

$$\mathbb{B} = \text{Res}[f, a], \text{ and}$$

$$\mathbb{C} = \text{Res}[f, b].$$

Example 8.9. Express $f(z) = \frac{z^2+3z+2}{z^2(z-1)}$ in partial fractions.

Solution. Using the Remark 8.1 and $a = 0$, $b = 1$ and $P(z) = z^2 + 3z + 2$, we have

$$f(z) = \frac{P(z)}{(z-0)^2 (z-1)} = \frac{\mathbb{A}}{(z-0)^2} + \frac{\mathbb{B}}{z-0} + \frac{\mathbb{C}}{z-1}$$

where

$$\begin{aligned}
\mathbb{A} &= \text{Res}[(z-0) f(z), 0] = \text{Res}\left[z \frac{z^2+3z+2}{z^2(z-1)}, 0\right] \\
&= \text{Res}\left[\frac{z^2+3z+2}{z(z-1)}, 0\right] \\
&= \lim_{z \to 0} (z-0) \frac{z^2+3z+2}{z(z-1)} \\
&= \frac{2}{-1} = -2
\end{aligned}$$

$$\begin{aligned}
\mathbb{B} &= \text{Res}[f, 0] = \text{Res}\left[\frac{z^2+3z+2}{z^2(z-1)}, 0\right] \\
&= \lim_{z \to 0} \frac{d}{dz} \left((z-0)^2 \frac{z^2+3z+2}{z^2(z-1)}\right) \\
&= \lim_{z \to 0} \frac{d}{dz} \left(\frac{z^2+3z+2}{z-1}\right) \\
&= \lim_{z \to 0} \frac{(2z+3)(z-1) - (1)(z^2+3z+2)}{(z-1)^2} \\
&= \lim_{z \to 0} \frac{z^2-2z-5}{(z-1)^2} \\
&= \frac{-5}{1} = -5
\end{aligned}$$

$$\begin{aligned}
\mathbb{C} &= \text{Res}\left[\frac{z^2+3z+2}{z^2(z-1)}, 1\right] \\
&= \lim_{z \to 1} (z-1) \frac{z^2+3z+2}{z^2(z-1)} \\
&= \lim_{z \to 1} \frac{z^2+3z+2}{z^2} \\
&= \frac{1+3+2}{1} = 6
\end{aligned}$$

Thus,

$$\begin{aligned}
f(z) &= \frac{z^2+3z+2}{z^2(z-1)} = \frac{\mathbb{A}}{(z-0)^2} + \frac{\mathbb{B}}{z-0} + \frac{\mathbb{C}}{z-1} \\
&= \frac{-2}{z^2} + \frac{-5}{z} + \frac{6}{z-1} \\
&= -\frac{2}{z^2} - \frac{5}{z} + \frac{6}{z-1}
\end{aligned}$$

Explore Solution 8.9.

Extra Example 1. Express of $f[z] = \frac{1}{z^4-1}$ in partial fractions.

Explore Solution for Extra Example 1.

Complex Analysis for Mathematics and Engineering
by
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8.2 Trigonometric Integrals

As indicated at the beginning of this chapter, we can evaluate certain definite real integrals with the aid of the residue theorem. One way to do this is by interpreting the definite integral as the parametric form of an integral of an analytic function along a simple closed contour.

Suppose we wish to evaluate an integral of the form

(8-3) $$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta,$$

where $F(u,v)$ is a function of the two real variables u and v. Consider the unit circle $C_1(0) = \{z : |z| = 1\}$ with parametrization

$$C_1^+(0): \; z = \cos\theta + i\sin\theta = e^{i\theta}, \quad \text{for} \quad 0 \le \theta \le 2\pi;$$

which gives us the following symbolic differentials.

(8-4) $$dz = (-\sin\theta + i\cos\theta)\, d\theta = i e^{i\theta}\, d\theta, \quad \text{and}$$
$$d\theta = \frac{dz}{i e^{i\theta}} = \frac{dz}{iz}.$$

Combining $z = \cos\theta + i\sin\theta$ with $\frac{1}{z} = \cos\theta - i\sin\theta$, we can obtain

(8-5) $$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Using the substitutions for $\cos\theta$, $\sin\theta$, and $d\theta$ in Expression (8-3) transforms the definite integral into a contour integral

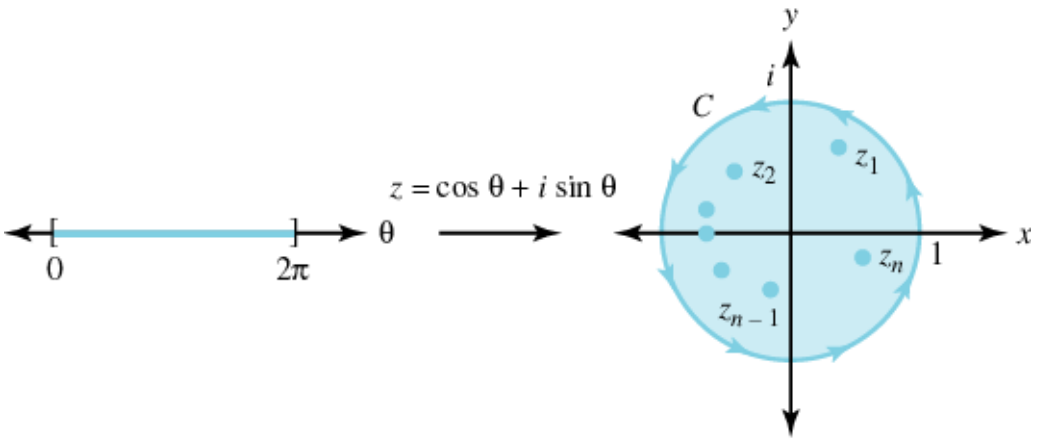
$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_{C_1(0)} f(z)\, dz,$$

where the new integrand is $f(z) = \frac{1}{iz} F\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)$.

Suppose that $f(z)$ is analytic inside and on the unit circle $C_1(0)$, except at the points $z_1, z_2, \ldots, z_n$ that lie interior to $C_1(0)$. Then the residue theorem gives

(8-6) $$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = 2\pi i \sum_{k=1}^{n} \text{Res}[f, z_k].$$

The situation is illustrated in Figure 8.2.



(a) The interval $[0, 2\pi]$ of integration for $F(\cos\theta, \sin\theta)$.

(b) The contour C of integration for $f(z)$.

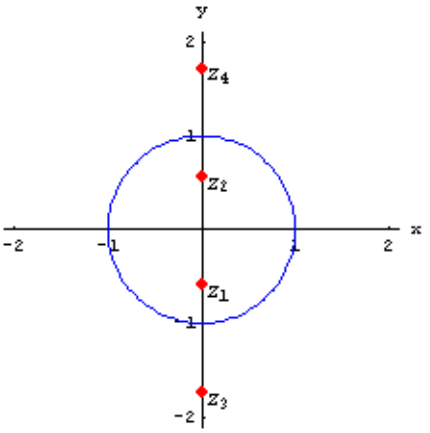
Figure 8.2 The change of variables from a definite integral on $[0, 2\pi]$ to a contour integral around C.

Some of these problems can be solved using *Mathematica*'s table of integrals.

We need to use the following substitution procedure.

```
Clear[f0, F, sub, θ, z];
sub[F_, θ_] :=
  Module[{},
   Clear[f0];
   f0 = F;
   f0 = ReplaceAll[f0, {Cos[θ] → (z + 1/z)/2, Sin[θ] → (z - 1/z)/(2 i)}];
   f0 = ReplaceAll[f0, {Cos[2 θ] → (z^2 + 1/z^2)/2, Sin[2 θ] → (z^2 - 1/z^2)/(2 i)}];
   f0 = ReplaceAll[f0, {Cos[3 θ] → (z^3 + 1/z^3)/2, Sin[3 θ] → (z^3 - 1/z^3)/(2 i)}];
   f0 = f0/(i z);
   f0 = Together[f0];
   f0 = Cancel[f0];
   Return[f0]; ];
```

Example 8.10. Evaluate $\int_0^{2\pi} \frac{1}{1 + 3\cos^2\theta}\, d\theta$ by using complex analysis.



Solution. Using Substitutions (8-4) and (8-5), we transform the integral to

$$\oint_{C_1^+(0)} \frac{1}{1+3\left(\frac{z+z^{-1}}{2}\right)^2}\,\frac{1}{i\,z}\,dz \;=\; \oint_{C_1^+(0)} \frac{-4\,i\,z}{3\,z^4+10\,z^2+3}\,dz \;=\; \oint_{C_1^+(0)} f(z)\,dz$$

where $f(z) = \frac{-4\,i\,z}{3\,z^4+10\,z^2+3}$. The singularities of $f(z)$ are poles located at the points where $3\,z^4 + 10\,z^2 + 3 = 0$ or equivalently, where $3\,(z^2)^2 + 10\,z^2 + 3 = 0$. Using the quadratic formula, we see that the singular points satisfy the relation $z^2 = \frac{-10\pm\sqrt{100-36}}{6} = \frac{-5\pm4}{3}$ which implies that $z^2 = 3$ or $z^2 = -\frac{1}{3}$. Hence the only singularities that lie inside the unit circle are simple poles corresponding to the solutions of $z^2 = -\frac{1}{3}$, which are the two points $z_1 = -\frac{i}{\sqrt{3}}$ and $z_2 = \frac{i}{\sqrt{3}}$. We use Theorem 8.2 and L'Hôpital's rule to get the residues at z_1 and z_2:

$$\begin{aligned}
\text{Res}[f, z_k] &= \lim_{z\to z_k}\,(z - z_k)\,f(z)\\
&= \lim_{z\to z_k}\,(z - z_k)\,\frac{-4\,i\,z}{3\,z^4+10\,z^2+3}\\
&= \lim_{z\to z_k}\,\frac{-4\,i\,z\,(z-z_k)}{3\,z^4+10\,z^2+3}\\
&= \lim_{z\to z_k}\,\frac{-4\,i\,\left(z^2 - z\,z_k\right)}{3\,z^4+10\,z^2+3} \;=\; \text{'}\,\frac{0}{0}\,\text{'}\\
&= \lim_{z\to z_k}\,\frac{-4\,i\,(2\,z - z_k)}{12\,z^3+20\,z}\\
&= \lim_{z\to z_k}\,\frac{-4\,i\,(2\,z_k - z_k)}{12\,z_k^3+20\,z_k}\\
&= \frac{-\,i\,4\,z_k}{\left(12\,z_k^2+20\right)\,z_k}\\
&= \frac{-\,i\,4}{\left(12\,z_k^2+20\right)}\\
&= \frac{-\,i}{3\,z_k^2+5}
\end{aligned}$$

As $z_k = \pm\frac{i}{\sqrt{3}}$ and $z_k^2 = \frac{-1}{3}$, the residues are given by

$\text{Res}[f, z_k] = \text{Res}\left[f, \frac{\pm i}{\sqrt{3}}\right] = \frac{-i}{3\left(\frac{-1}{3}\right)+5} = \frac{-i}{-1+5} = -\frac{i}{4}$. We now use Equation (8-6) to compute the value of the integral:

$$\int_0^{2\pi} \frac{1}{1+3\cos^2\theta}\, d\theta = 2\pi i\,(\text{Res}[f, z_1] + \text{Res}[f, z_2])$$

$$= 2\pi i\left(\text{Res}\left[f, \frac{i}{\sqrt{3}}\right] + \text{Res}\left[f, \frac{-i}{\sqrt{3}}\right]\right)$$

$$= 2\pi i\left(-\frac{i}{4} - \frac{i}{4}\right) = 2\pi i\left(-\frac{i}{2}\right)$$

$$= \pi$$

Explore Solution 8.10.

Example 8.11. Evaluate $\int_0^{2\pi} \frac{1}{1+3\cos^2 t}\, dt$ by using a computer algebra system.

Solution. We can obtain the antiderivative of $\frac{1}{1+3\cos^2 t}$ by using software such as Mathematica or MAPLE. It is $\int \frac{1}{1+3\cos^2 t}\, dt = -\frac{1}{2}\,\text{Arctan}(\cot t) = g(t)$.

Since $\cot 0$ and $\cot 2\pi$ are not defined, the computations for both $g(0)$ and $g(2\pi)$ are indeterminate. The graph $s = g(t)$ shown in Figure 8.3 reveals another problem:

The integrand $\frac{1}{1+3\cos^2 t}$ is a continuous function for all t, but the function $g(t)$ has a discontinuity at $t = \pi$. This condition appears to be a violation of the fundamental theorem of calculus, which asserts that the integral of a continuous function must be differentiable and hence continuous. The problem is that $g(t)$ is not an antiderivative of $\frac{1}{1+3\cos^2 t}$ for all t in the interval $[0, 2\pi]$. Oddly, it is the antiderivative at all points except 0, π, and 2π, which you can verify by computing $g'(t)$ and showing that it equals $\frac{1}{1+3\cos^2 t}$ whenever $g(t)$ is defined.

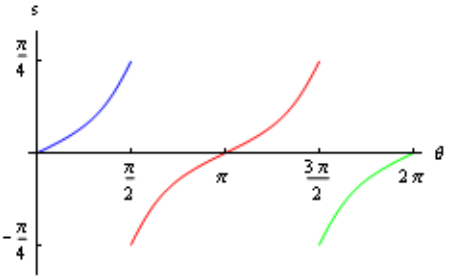


Figure 8.3 Graph of $g(t) = \int \frac{1}{1+3\cos^2 t}\, dt = \frac{1}{2}\,\text{ArcTan}\left[\frac{\text{Tan}[\theta]}{2}\right]$.

The integration algorithm used by computer algebra systems here (the Risch-Norman algorithm) gives the antiderivative $g(t) = -\frac{1}{2}\,\text{Arctan}(\cot t)$, and we must take great care in using this information. We get the proper value of the integral by using $g(t)$ on the open subintervals $(0, \pi)$ and $(0, 2\pi)$ where it is continuous, and taking appropriate limits:

$$\int_0^{2\pi} \frac{1}{1+3\cos^2 t}\,dt = \int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos^2 t}\,dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{1+3\cos^2 t}\,dt + \int_{\frac{3\pi}{2}}^{2\pi} \frac{1}{1+3\cos^2 t}\,dt$$

$$= \lim_{\substack{t\to\frac{\pi}{2}^- \\ s\to 0^+}} \int_s^t \frac{1}{1+3\cos^2 t}\,dt + \lim_{\substack{t\to\frac{3\pi}{2}^- \\ s\to\frac{\pi}{2}^+}} \int_s^t \frac{1}{1+3\cos^2 t}\,dt + \lim_{\substack{t\to 2\pi^- \\ s\to\frac{3\pi}{2}^+}} \int_s^t \frac{1}{1+3\cos^2 t}\,dt$$

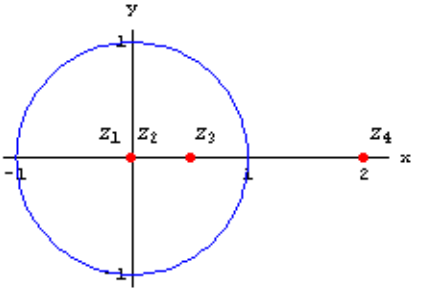
$$= \left(\lim_{t\to\frac{\pi}{2}^-} g(t) - \lim_{s\to 0^+} g(s)\right) + \left(\lim_{t\to\frac{3\pi}{2}^-} g(t) - \lim_{s\to\frac{\pi}{2}^+} g(s)\right) + \left(\lim_{t\to 2\pi^-} g(t) - \lim_{s\to\frac{3\pi}{2}^+} g(s)\right)$$

$$= \left(\frac{\pi}{4} - 0\right) + \left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) + \left(0 - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} = \pi$$

Explore Solution 8.11.

Example 8.12. Evaluate $\int_0^{2\pi} \frac{\cos(2\theta)}{5-4\cos(\theta)}\,d\theta$.



Solution. For values of z that lie on the unit circle $C_1(0)$, we have

$$z^2 = \cos 2\theta + i \sin 2\theta \quad \text{and} \quad z^{-2} = \cos 2\theta - i \sin 2\theta$$

We solve for $\cos 2\theta$ and $\sin 2\theta$ to obtain the substitutions

$$\cos 2\theta = \frac{1}{2}(z^2 + z^{-2}) \quad \text{and} \quad \sin 2\theta = \frac{1}{2i}(z^2 - z^{-2}).$$

Using the identity for $\cos 2\theta$ along with Substitutions (8-4) and (8-5), we rewrite the integral as

$$\oint_{C_1^+(0)} \frac{\frac{1}{2}(z^2+z^{-2})}{5-4\left(\frac{z+z^{-1}}{2}\right)} \frac{1}{iz}\,dz = \oint_{C_1^+(0)} \frac{i(z^4+1)}{2z^2(2z^2-5z+2)}\,dz$$

$$= \oint_{C_1^+(0)} \frac{i(z^4+1)}{2z^2(z-2)(2z-1)}\,dz = \oint_{C_1^+(0)} f(z)\,dz$$

where $f(z) = \frac{i(z^4+1)}{2z^2(z-2)(2z-1)}$. The singularities of $f(t)$ lying inside $C_1(0)$ are poles located at the points $z_1 = z_2 = 0$ and $z_3 = \frac{1}{2}$. We use Theorem 8.2 to get the residues:

$$\text{Res}[f, 0] = \lim_{z \to 0} \frac{d}{dz} (z^2 f(z)) = \lim_{z \to 0} \frac{d}{dz} \left(z^2 \frac{i (z^4+1)}{2 z^2 (z-2) (2 z-1)} \right)$$

$$= \lim_{z \to 0} \frac{d}{dz} \left(i \frac{z^4+1}{2 (z-2) (2 z-1)} \right) = \lim_{z \to 0} \frac{d}{dz} \left(i \frac{z^4+1}{2 (2 z^2-5 z+2)} \right)$$

$$= \lim_{z \to 0} \left(i \frac{4 z^3 (2 z^2-5 z+2) - (4 z-5) (z^4+1)}{2 (2 z^2-5 z+2)^2} \right)$$

$$= i \frac{(4 \times 0^3) (2 \times 0^2-5 \times 0+2) - (4 \times 0-5) (0^4+1)}{2 (2 \times 0^2-5 \times 0+2)^2} = i \frac{(0) \times (2) - (-5) (1)}{2 (2)^2}$$

$$= \frac{5 i}{8}$$

$$\text{Res}\left[f, \frac{1}{2}\right] = \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) f(z) = \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{i (z^4+1)}{2 z^2 (z-2) (2 z-1)}$$

$$= \lim_{z \to \frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{i (z^4+1)}{4 z^2 (z-2) \left(z-\frac{1}{2}\right)} = \lim_{z \to \frac{1}{2}} i \frac{z^4+1}{4 z^2 (z-2)}$$

$$= i \frac{\left(\frac{1}{2}\right)^4+1}{4 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}-2\right)} = i \frac{\frac{1}{16}+1}{4 \left(\frac{1}{4}\right) \left(\frac{1}{2}-2\right)} = i \frac{\frac{17}{16}}{4 \left(\frac{1}{4}\right) \left(\frac{-3}{2}\right)}$$

$$= -\frac{17 i}{24}$$

Therefore we conclude that

$$\int_0^{2\pi} \frac{\cos(2\theta)}{5 - 4\cos(\theta)} d\theta = 2\pi i (\text{Res}[f, z_1] + \text{Res}[f, z_2])$$

$$= 2\pi i \left(\text{Res}[f, 0] + \text{Res}\left[f, \frac{1}{2}\right]\right)$$

$$= 2\pi i \left(\frac{5 i}{8} - \frac{17 i}{24}\right) = 2\pi i \left(-\frac{i}{12}\right)$$

$$= \frac{\pi}{6}$$

Explore Solution 8.12.

Complex Analysis for Mathematics and Engineering
by
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Jones and Bartlett Learning

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8.3 Improper Integrals of Rational Functions

An important application of the theory of residues is the evaluation of certain types of improper integrals. Let $f(x)$ be a continuous function of the real variable x on the interval $0 \le x < \infty$. Recall from calculus that the improper integral of $f(x)$ over $[0, \infty)$ is defined by

$$\int_0^\infty f(x)\,dx = \lim_{b\to\infty} \int_0^b f(x)\,dx,$$

provided that the limit exists. If $f(x)$ is defined for all real x, then the integral of $f(x)$ over $(-\infty, \infty)$ is defined by

(8-7) $$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty} \int_a^0 f(x)\,dx + \lim_{b\to\infty} \int_0^b f(x)\,dx,$$

provided both limits exist. If the integral in Equation (8-7) exists, we can obtain its value by taking a single limit:

(8-8) $$\int_{-\infty}^\infty f(x)\,dx = \lim_{R\to\infty} \int_{-R}^R f(x)\,dx.$$

For some functions the limit on the right side of Equation (8-8) exists, but the limit on the right side of Equation (8-7) doesn't exist.

Example 8.13. $\lim_{R\to\infty} \int_{-R}^R x\,dx = \lim_{R\to\infty} \left(\int_R^0 x\,dx + \int_0^R x\,dx \right)$

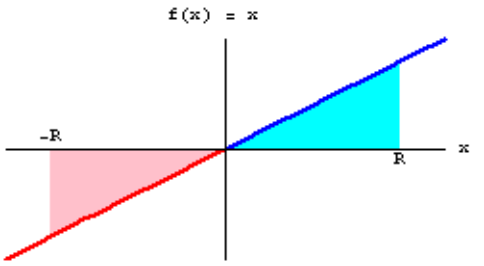
$$= \lim_{R\to\infty} \left(-\frac{R^2}{2} + \frac{R^2}{2} \right) = \lim_{R\to\infty} (0)$$

$$= 0$$

but Equation (8-7) tells us that the improper integral of $f(x) = x$ over $(-\infty, \infty)$ doesn't exist, because the calculation

$\int_{-\infty}^{\infty} x\,dx = \lim_{a\to-\infty}\int_{a}^{0} x\,dx + \lim_{b\to\infty}\int_{0}^{b} x\,dx =$ '$\infty-\infty$' is undefined .

Therefore we can use Equation (8-8) to extend the notion of the value of an improper integral, as Definition 8.2 indicates.



Explore Solution 8.13.

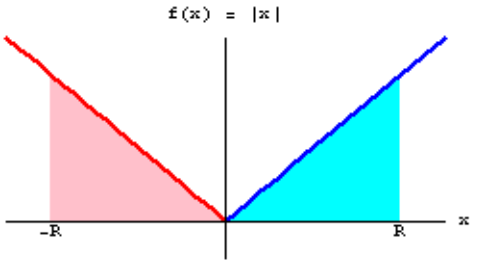
Extra Example 1. $\lim_{R\to\infty}\int_{-R}^{R} |x|\,dx = \lim_{R\to\infty}\left(\int_{R}^{0} |x|\,dx + \int_{0}^{R} |x|\,dx\right)$

$= \lim_{R\to\infty}\left(\frac{R^2}{2} + \frac{R^2}{2}\right) = \lim_{R\to\infty}(R^2)$

$= \infty$

and Equation (8-7) tells us that the improper integral of $f(x) = |x|$ over $(-\infty, \infty)$ also diverges to '∞', because

$\int_{-\infty}^{\infty} |x|\,dx = \lim_{a\to-\infty}\int_{a}^{0} |x|\,dx + \lim_{b\to\infty}\int_{0}^{b} |x|\,dx = \infty+\infty = \infty.$



Explore Extra Solution 1.

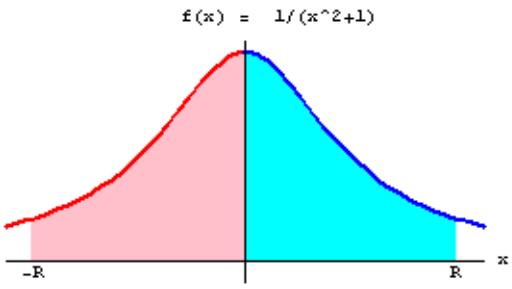
Definition 8.2 (Cauchy Principal Value - P.V.). Let $f(x)$ be a continuous real valued function for all x. The Cauchy principal value (P.V.) of the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined by

$$P.V.\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx,$$

provided the limit exists.

Example 8.13 shows that $P.V.\int_{-\infty}^{\infty} x\,dx = 0$.

Example 8.14. Find the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$.



Solution.

$$P.V.\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+1}\,dx$$

$$= \lim_{R\to\infty} [\text{Arctan}(R) - \text{Arctan}(-R)]$$

$$= \lim_{R\to\infty} \text{Arctan}(R) - \lim_{R\to\infty} \text{Arctan}(-R)$$

$$= \frac{\pi}{2} - \frac{-\pi}{2}$$

$$= \pi$$

Explore Solution 8.14.

If $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, then $f(x)$ is called a rational function. Techniques in calculus were developed to integrate rational functions. We now show how the Residue Theorem can be used to obtain the Cauchy principal value of the integral of $f(x)$ over $(-\infty, \infty)$.

Theorem 8.3 (Contour Integration for Rational Functions). Let $f(z) = \frac{P(z)}{Q(z)}$ where $P(x)$ and $Q(x)$ are polynomials, of degree m and n , respectively. If $Q(x) \neq 0$ for all real x and $n \geq m+2$, then the Cauchy Principal Value (P.V.) of the integral is

$$P.V.\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{j=1}^{k} \text{Res}\left[\frac{P}{Q}, z_j\right].$$

where $z_1, z_2, \ldots, z_k$ are the poles of $\frac{P(z)}{Q(z)}$ that lie in the upper half plane. The situation is illustrated in Figure 8.4.

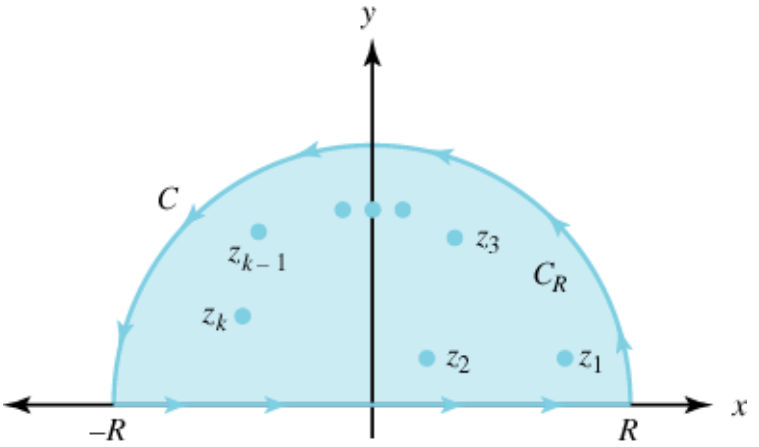
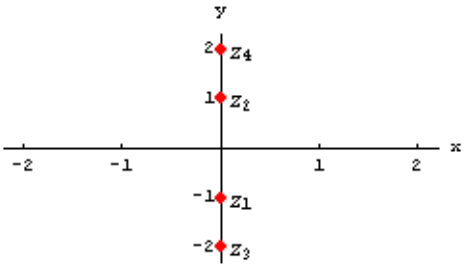
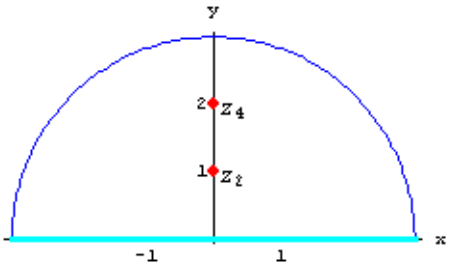


Figure 8.4 The poles $z_1, z_2, \ldots, z_k$ of $\frac{P(x)}{Q(x)}$ that lie in the upper half-plane.

Proof.

Example 8.15. Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx$.





Solution. We write the integrand as $f(z) = \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$. We see that $f(z)$ has simple poles at the points $z = \pm i$ and $z = \pm 2i$, and that the points $z_2 = i$ and $z_4 = 2i$, are the only singularities of $f(z)$ in the upper half-plane. Computing the residues, we obtain

$$\text{Res}[f, i] = \lim_{z \to i} (z - i) f(z) = \lim_{z \to i} (z - i) \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$$

$$= \lim_{z \to i} \frac{1}{(z+i)(z+2i)(z-2i)} = \frac{1}{(i+i)(i+2i)(i-2i)} = \frac{1}{(2i)(3i)(-i)}$$

$$= -\frac{i}{6}$$

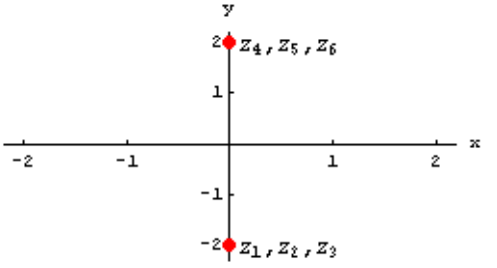
$$\begin{aligned}
\text{Res}[f, 2i] &= \lim_{z\to 2i}\ (z-2i)\, f(z) \\
&= \lim_{z\to 2i}\ (z-2i)\,\frac{1}{(z+i)(z-i)(z+2i)(z-2i)} \\
&= \lim_{z\to 2i}\ \frac{1}{(z+i)(z-i)(z+2i)} = \frac{1}{(2i+i)(2i-i)(2i+2i)} \\
&= \frac{1}{(3i)(i)(4i)} \\
&= \frac{i}{12}
\end{aligned}$$

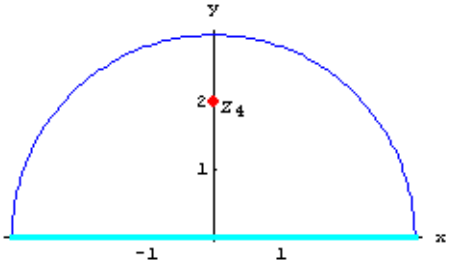
Using Theorem 8.3, we conclude that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx &= 2\pi i\,(\text{Res}[f, i] + \text{Res}[f, 2i]) \\
&= 2\pi i\left(-\frac{i}{6} + \frac{i}{12}\right) \\
&= 2\pi i\left(-\frac{i}{12}\right) \\
&= \frac{\pi}{6}
\end{aligned}$$

Explore Solution 8.15.

Example 8.16. Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^3}\,dx$.





Solution. The integrand $f(z) = \frac{1}{(z^2+4)^3} = \frac{1}{(z+2i)^3(z-2i)^3}$ has a poles of order 3 at the points $z = \pm 2i$, and $z = 2i$ is the only singularity of $f(z)$ in the upper half-plane. Computing the residue at $z_4 = 2i$, we get

$$\text{Res}[f, 2i] = \frac{1}{2!}\lim_{z\to 2i}\frac{d^2}{dz^2}\left((z-2i)^3 f(z)\right)$$

$$= \frac{1}{2}\lim_{z\to 2i}\frac{d^2}{dz^2}\left((z-2i)^3\frac{1}{(z+2i)^3(z-2i)^3}\right)$$

$$= \frac{1}{2}\lim_{z\to 2i}\frac{d^2}{dz^2}\left(\frac{1}{(z+2i)^3}\right)$$

$$= \frac{1}{2}\lim_{z\to 2i}\frac{d}{dz}\left(\frac{-3}{(z+2i)^4}\right)$$

$$= \frac{1}{2}\lim_{z\to 2i}\left(\frac{12}{(z+2i)^5}\right)$$

$$= \frac{1}{2}\left(\frac{12}{(2i+2i)^5}\right) = \frac{1}{2}\left(\frac{12}{(4i)^5}\right) = \frac{1}{2}\left(\frac{12}{1024i}\right)$$

$$= -\frac{3i}{512}$$

Therefore

$$\int_{-\infty}^{\infty}\frac{1}{\left(x^2+4\right)^3}dx = 2\pi i\,\text{Res}[f, 2i] = 2\pi i\left(\frac{-3i}{512}\right) = \frac{3\pi}{256}.$$

Explore Solution 8.16.

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8.4 Improper Integrals Involving Trigonometric Functions

Let $P(x)$ and $Q(x)$ be polynomials of degree m and n, respectively, where $n \geq m+1$. We can show (but omit the proof) that if $Q(x) \neq 0$ for all real x, then

$$P.V.\int_{-\infty}^{\infty} \frac{P(x)\cos(\alpha x)}{Q(x)}\,dx \quad \text{and} \quad P.V.\int_{-\infty}^{\infty} \frac{P(x)\sin(\alpha x)}{Q(x)}\,dx$$

are convergent improper integrals. You may encounter integrals of this type in the study of Fourier transforms and Fourier integrals. We now show how to evaluate them.

Particularly important is our use of the identities

$$\cos(\alpha x) = \mathrm{Re}[e^{i\alpha x}] \quad \text{and} \quad \sin(\alpha x) = \mathrm{Im}[e^{i\alpha x}]$$

where α is a positive real number. The crucial step in the proof of Theorem 8.4 wouldn't hold if we were to use $\cos(\alpha z)$ and $\sin(\alpha z)$ are used instead of $e^{i\alpha z}$, as you will see when you get to Lemma 8.1.

Theorem 8.4 (Contour Integration for Improper Trig. Integrals). Let $P(x)$ and $Q(x)$ be polynomials with real coefficients, of degree m and n, respectively, where $n \geq m+1$ and $Q(x) \neq 0$ for all real x. If $\alpha > 0$ and

(8-12) $$f(z) = \frac{e^{i\alpha z}P(z)}{Q(z)}, \quad \text{then}$$

(8-13) $$P.V.\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\cos(\alpha x)\,dx = -2\pi \sum_{j=1}^{k} \mathrm{Im}(\mathrm{Res}[f, z_j]), \quad \text{and}$$

(8-14) $$P.V.\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\sin(\alpha x)\,dx = 2\pi \sum_{j=1}^{k} \mathrm{Re}(\mathrm{Res}[f, z_j]),$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half plane and $\mathrm{Re}(\mathrm{Res}[f, z_j])$

and Im[Res[f, z_j]] are the real and imaginary parts of Res[f, z_j], respectively.

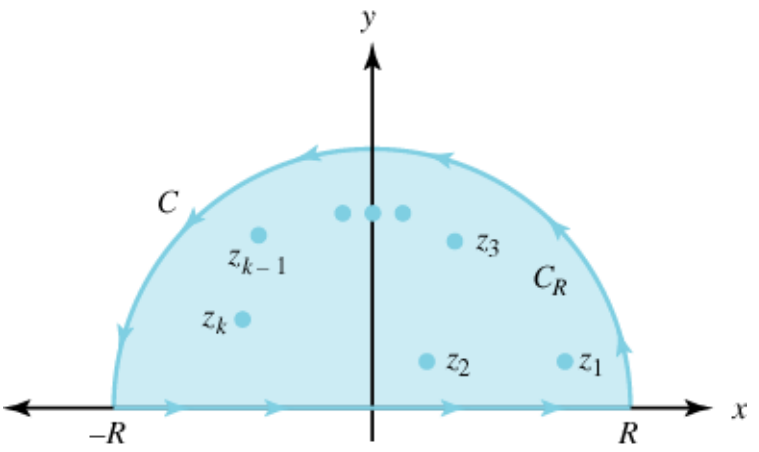
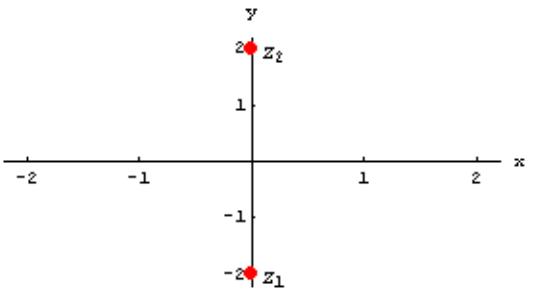


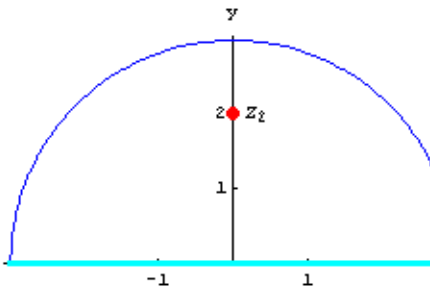
Figure 8.4 The poles $z_1, z_2, \ldots, z_k$ of $\frac{P(x)}{Q(x)}\cos(\alpha x)$ or $\frac{P(x)}{Q(x)}\sin(\alpha x)$ that lie in the upper half-plane.

Proof.

The proof of Theorem 8.4 is similar to the proof of Theorem 8.3. Before turning to the proof, we illustrate how to use Theorem 8.4.

Example 8.17. Evaluate P.V. $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4}\,dx$.





Solution. The function f(z) in Equation (8-12) is $f(z) = \frac{z e^{iz}}{z^2+4} = \frac{z e^{iz}}{(z+2i)(z-2i)}$, which has a simple pole at the point $z_2 = 2i$ in the upper half-plane. Calculating the residue yields

$$\text{Res}[f, 2i] = \lim_{z \to 2i} (z - 2i) f(z)$$

$$= \lim_{z \to 2i} (z - 2i) \frac{z e^{iz}}{(z+2i)(z-2i)}$$

$$= \lim_{z \to 2i} \frac{z e^{iz}}{z+2i} = \frac{(2i) e^{i(2i)}}{2i+2i}$$

$$= \frac{(2i) e^{-2}}{4i}$$

$$= \frac{1}{2e^2}$$

Using Equation (8-14) gives

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+4}\,dx = 2\pi\,\text{Re}\,(\text{Res}[f, z_1])$$

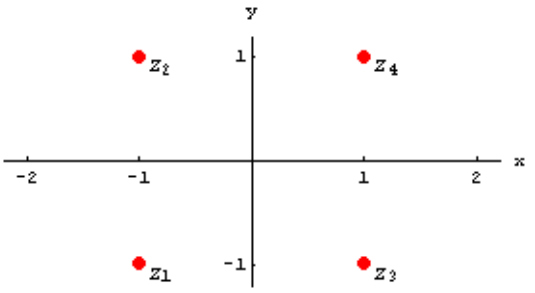
$$= 2\pi\,\text{Re}\,(\text{Res}[f, 2i])$$

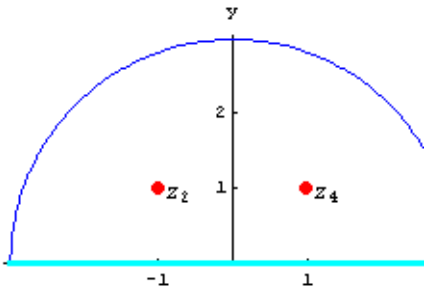
$$= 2\pi\,\text{Re}\left(\frac{1}{2e^2}\right) = 2\pi\left(\frac{1}{2e^2}\right)$$

$$= \frac{\pi}{e^2}$$

Explore Solution 8.17.

Example 8.18. Evaluate $\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos x}{x^4+4}\,dx$.





Solution. The function $f(z)$ in Equation (8-12) is $f(z) = \frac{e^{iz}}{z^4+4} = \frac{e^{iz}}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}$, which has simple poles at the points $z_2 = -1+i$ and $z_4 = 1+i$ in the upper half-plane. We get the residues with the aid of L'Hôpital's rule:

$$\text{Res}[f, 1+i] = \lim_{z\to 1+i}(z-1-i)\,f(z)$$

$$= \lim_{z\to 1+i}\frac{(z-1-i)\,e^{iz}}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}$$

$$= \lim_{z\to 1+i}\frac{(z-1-i)\,e^{iz}}{z^4+4} = \text{`}\frac{0}{0}\text{'}$$

$$= \lim_{z\to 1+i}\frac{(1)\,e^{iz}+(z-1-i)\,i\,e^{iz}}{4z^3}$$

$$= \lim_{z\to 1+i}\frac{(1+i\,(z-1-i))\,e^{iz}}{4z^3}$$

$$= \frac{(1+i\,(1+i-1-i))\,e^{i(1+i)}}{4(1+i)^3} = \frac{(1)\,e^{-1+i}}{4(1+i)^3}$$

$$= \frac{e^{-1+i}}{-8+8i} = \frac{(-1-i)\,e^{-1+i}}{16}$$

$$= \frac{(-1-i)\,e^{-1+i}}{16} = \frac{(-1-i)\,e^{-1}(\cos 1+i\sin 1)}{16}$$

$$= \frac{\sin 1-\cos 1-i\,(\cos 1+\sin 1)}{16e}$$

Similarly,

$$\text{Res}[f, -1+i] = \lim_{z \to -1+i} (z+1-i) f(z)$$

$$= \lim_{z \to -1+i} \frac{(z-1-i)\, e^{iz}}{(z+1+i)(z+1-i)(z-1+i)(z-1-i)}$$

$$= \frac{\cos 1 - \sin 1 - i(\cos 1 + \sin 1)}{16e}$$

Using Equation (8-13), we get

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{x^4+4}\, dx = -2\pi (\text{Im}(\text{Res}[f, z_1]) + \text{Im}(\text{Res}[f, z_2]))$$

$$= -2\pi (\text{Im}(\text{Res}[f, 1+i]) + \text{Im}(\text{Res}[f, -1+i]))$$

$$= -2\pi \left(\text{Im}\left(\frac{\sin 1 - \cos 1 - i(\cos 1 + \sin 1)}{16e}\right) + \text{Im}\left(\frac{\cos 1 - \sin 1 - i(\cos 1 + \sin 1)}{16e}\right)\right)$$

$$= -2\pi \left(\frac{-(\cos 1 + \sin 1)}{16e} + \frac{-(\cos 1 + \sin 1)}{16e}\right) = -2\pi \left(\frac{-(\cos 1 + \sin 1)}{8e}\right)$$

$$= \frac{\pi(\cos 1 + \sin 1)}{4e}$$

Explore Solution 8.18.

We are almost ready to give the proof of Theorem 8.4, but first we need one preliminary result known as Jordan's lemma.

Lemma 8.1, (Jordan's Lemma). Suppose that $P(z)$ and $Q(z)$ are polynomials of degree m and n, respectively, where $n \geq m+1$. If C_R is the upper semicircle $z = R e^{i\theta}$ for $0 \leq \theta \leq 2\pi$, then

$$\lim_{R \to \infty} \int_{C_R} \frac{e^{iz} P(z)}{Q(z)}\, dz = 0.$$

Proof.

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8.5 Indented Contour Integrals

If $f(x)$ is continuous on the interval $b < x \le c$, but discontinuous at b, then the improper integral of $f(x)$ over $[b, c]$ is defined by

$$\int_b^c f(x)\,dx = \lim_{r\to b^+} \int_r^c f(x)\,dx$$

provided that the limit exists. Similarly, if $f(x)$ is continuous on the interval $a \le x < b$, but discontinuous at b, then the improper integral of $f(x)$ over $[a, b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{r\to b^-} \int_a^r f(x)\,dx$$

provided that the limit exists. For example,

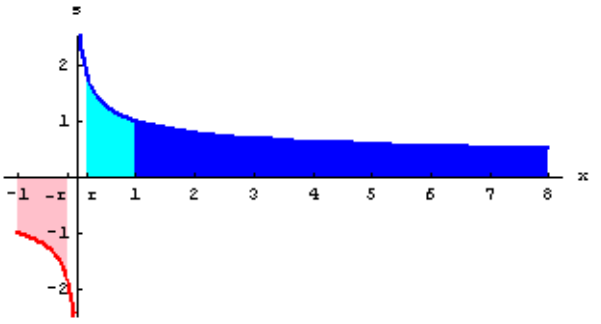
$$\int_0^9 \frac{1}{2\sqrt{x}}\,dx = \lim_{r\to 0^+} \int_r^9 \frac{1}{2\sqrt{x}}\,dx = \lim_{r\to 0^+} \sqrt{x}\Big|_{x=r}^{x=9} = 3 - \lim_{r\to 0^+}\sqrt{x} = 3.$$

If we let $f(x)$ be continuous for all values of x in the interval $[a, c]$, except at the value $x = b$, where $a < b < c$. The Cauchy principal value of $f(x)$ over $[a, c]$ is defined by

$$P.V.\int_a^c f(x)\,dx = \lim_{r\to 0^+}\left(\int_a^{b-r} f(x)\,dx + \int_{b+r}^c f(x)\,dx\right),$$

provided that the limit exists.

Example 8.19. Evaluate $P.V.\int_{-1}^{8} \frac{1}{x^{1/3}}\,dx$.



Solution. Evaluating the integrals and computing limits gives

$$\text{P.V.}\int_{-1}^{8}\frac{1}{x^{1/3}}\,dx = \lim_{r\to 0^+}\left(\int_{-1}^{-r}\frac{1}{x^{1/3}}\,dx + \int_{r}^{8}\frac{1}{x^{1/3}}\,dx\right)$$

$$= \lim_{r\to 0^+}\left(\frac{3}{2}x^{2/3}\Big|_{x=-1}^{-r} + \frac{3}{2}x^{2/3}\Big|_{x=r}^{8}\right)$$

$$= \lim_{r\to 0^+}\left(\frac{3}{2}r^{2/3} - \frac{3}{2} + 6 - \frac{3}{2}r^{2/3}\right)$$

$$= \left(\frac{3}{2}0^{2/3} - \frac{3}{2} + 6 - \frac{3}{2}0^{2/3}\right)$$

$$= \left(-\frac{3}{2} + 6\right)$$

$$= \frac{9}{2}$$

Explore Solution 8.19.

In this section we show how to use residues to evaluate the Cauchy principal value of the integral of $f(x)$ over $(-\infty, \infty)$ when the integrand $f(x)$ has simple poles on the x axis. We state our main results and then look at some examples before giving proofs.

Theorem 8.5. Let $f(z) = \frac{P(z)}{Q(z)}$ where $P(x)$ and $Q(x)$ are polynomials with real coefficients of degree m and n, respectively, and $n \geq m + 2$. If $Q(x)$ has simple zeros at the points $t_1, t_2, \ldots, t_L$ on the x-axis, then

(8-20)
$$\text{P.V.}\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx = 2\pi i\sum_{j=1}^{k}\text{Res}\left[\frac{P}{Q}, z_j\right] + \pi i\sum_{j=1}^{L}\text{Res}\left[\frac{P}{Q}, t_j\right],$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half-plane.

Proof.

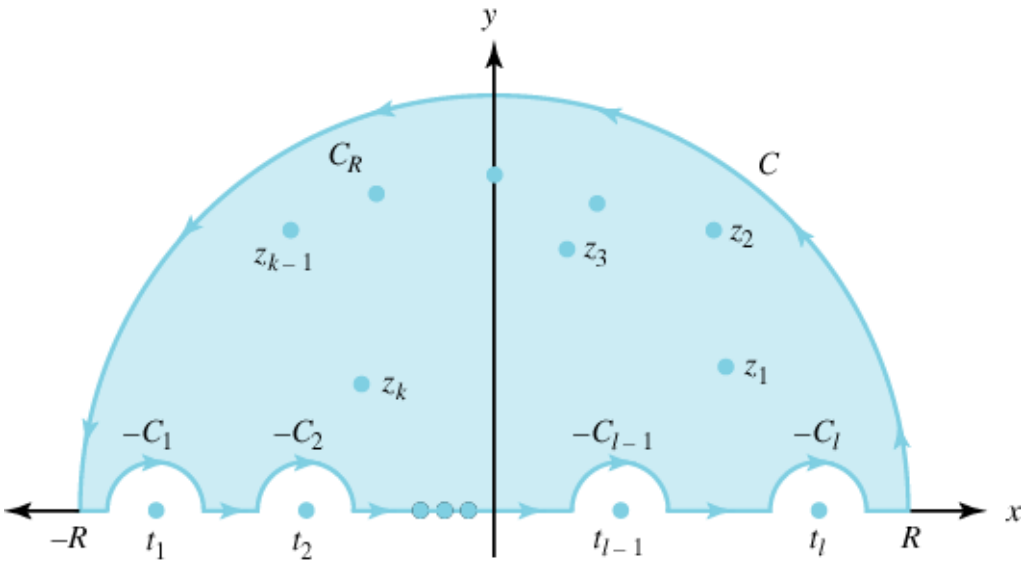


Figure 8.6 The poles $t_1, t_2, \ldots, t_L$ of $f(z)$ that lie on the x-axis and the poles $z_1, z_2, \ldots, z_k$ that lie above the semicircles $C_1, C_2, \ldots, C_L$.

Theorem 8.6. Let $P(x)$ and $Q(x)$ be polynomials, of degree m and n, respectively, where $n \geq m+1$ and let $Q(x)$ have simple zeros at the points $t_1, t_2, \ldots, t_L$ on the x-axis. If α is a positive real number and if $f(z) = \frac{e^{i\alpha z} P(z)}{Q(z)}$, then we can compute the Cauchy Principal Value (P.V.) of the following integrals

(8-21)

$$P.V.\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos\alpha x\, dx = -2\pi \sum_{j=1}^{k} \text{Im}(\text{Res}[f, z_j]) - \pi \sum_{j=1}^{L} \text{Im}(\text{Res}[f, t_j]),$$

and

(8-22)

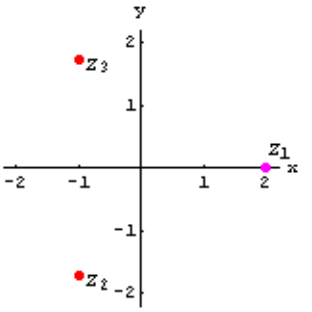
$$P.V.\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin\alpha x\, dx = 2\pi \sum_{j=1}^{k} \text{Re}(\text{Res}[f, z_j]) + \pi \sum_{j=1}^{L} \text{Re}(\text{Res}[f, t_j]),$$

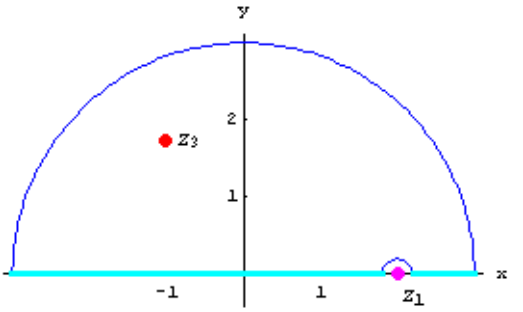
where $z_1, z_2, \ldots, z_k$ are the poles of $f(z)$ that lie in the upper half plane.

Proof.

Remark. The formulas in these theorems give the Cauchy principal value of the integral, which pays special attention to the manner in which any limits are taken. They are similar to those in Sections 8.3 and 8.4, except here we add one-half of the value of each residue at the points $t_1, t_2, \ldots, t_r$ on the x-axis.

Example 8.20. Evaluate $P.V.\int_{-\infty}^{\infty} \frac{x}{x^3-8}\, dx$ by using complex analysis.





Solution. The integrand

$$f(z) = \frac{P(z)}{Q(z)} = \frac{z}{z^3-8} = \frac{z}{(z-2)\left(z^2+2z+4\right)} = \frac{z}{(z-2)\left(z+1+i\sqrt{3}\right)\left(z+1-i\sqrt{3}\right)}$$

has simple poles at the points $t_1 = 2$ on the x axis and $z_1 = -1 + i\sqrt{3}$ in the upper half-plane. Computing the residues we get

$$\text{Res}[f, 2] = \lim_{z\to 2}(z-2)f(z) = \lim_{z\to 2}(z-2)\frac{z}{(z-2)\left(z^2+2z+4\right)} = \lim_{z\to 2}\frac{z}{\left(z^2+2z+4\right)} = \frac{2}{\left(2^2+2\times 2+4\right)} = \frac{2}{(4+4+4)} = \frac{1}{6}$$

$$\text{Res}\left[f, -1+i\sqrt{3}\right] = \lim_{z\to -1+i\sqrt{3}}\left(z+1-i\sqrt{3}\right)f(z) = \lim_{z\to -1+i\sqrt{3}}\left(z+1-i\sqrt{3}\right)\frac{z}{(z-2)\left(z+1+i\sqrt{3}\right)\left(z+1-i\sqrt{3}\right)} = \lim_{z\to -1+i\sqrt{3}}\frac{z}{(z-2)\left(z+1+i\sqrt{3}\right)} = \frac{-1+i\sqrt{3}}{\left(-1+i\sqrt{3}-2\right)\left(-1+i\sqrt{3}+1+i\sqrt{3}\right)} = \frac{-1+i\sqrt{3}}{\left(-3+i\sqrt{3}\right)\left(2i\sqrt{3}\right)} = \frac{-1+i\sqrt{3}}{-6-6i\sqrt{3}} = \frac{\left(-1+i\sqrt{3}\right)\left(-6+6i\sqrt{3}\right)}{\left(-6-6i\sqrt{3}\right)\left(-6+6i\sqrt{3}\right)} = \frac{-12-12i\sqrt{3}}{144} = \frac{-1-i\sqrt{3}}{12}$$

By Theorem 8.5,

$$\int_{-\infty}^{\infty} \frac{x}{x^3-8}\,dx = 2\pi i\,\text{Res}\left[\frac{P}{Q}, z_1\right] + \pi i\,\text{Res}\left[\frac{P}{Q}, t_1\right]$$

$$= 2\pi i\left(\frac{-1-i\sqrt{3}}{12}\right) + \pi i\left(\frac{1}{6}\right)$$

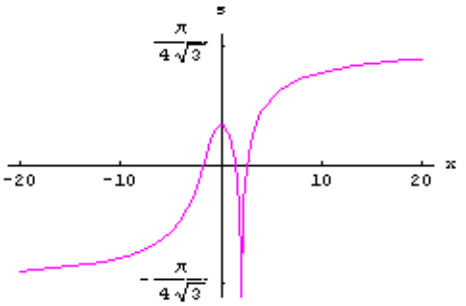
$$= \pi i\left(\frac{-1-i\sqrt{3}}{6}\right) + \pi i\left(\frac{1}{6}\right)$$

$$= \pi i\left(\frac{-i\sqrt{3}}{6}\right)$$

$$= \frac{\pi\sqrt{3}}{6}$$

Explore Solution 8.20.

Example 8.21. Evaluate $P.V.\int \frac{t}{t^3-8}\,dt$ by using a computer algebra system.



Solution. Computer algebra systems such as Mathematica or MAPLE give the indefinite integral

$$\int \frac{t}{t^3-8}\,dt = \frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{\text{Log}\,(t-2)}{6} - \frac{\text{Log}\left(t^2+2t+4\right)}{12} = g\,(t).$$

However, for real numbers, we should write the second term as $\frac{\text{Log}\left[(t-2)^2\right]}{12}$ and use the equivalent formula:

$$g\,(t) = \frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{\text{Log}\left[(t-2)^2\right]}{12} - \frac{\text{Log}\left(t^2+2t+4\right)}{12}$$

$$= \frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{\text{Log}\left[(t-2)^2\right]}{12} - \frac{\text{Log}\left[(t+2)^2\right]}{12}$$

$$= \frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{12}\,\text{Log}\left[\frac{(t-2)^2}{(t+2)^2}\right]$$

$$= \frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{12}\,\text{Log}\left[\left(\frac{t-2}{t+2}\right)^2\right]$$

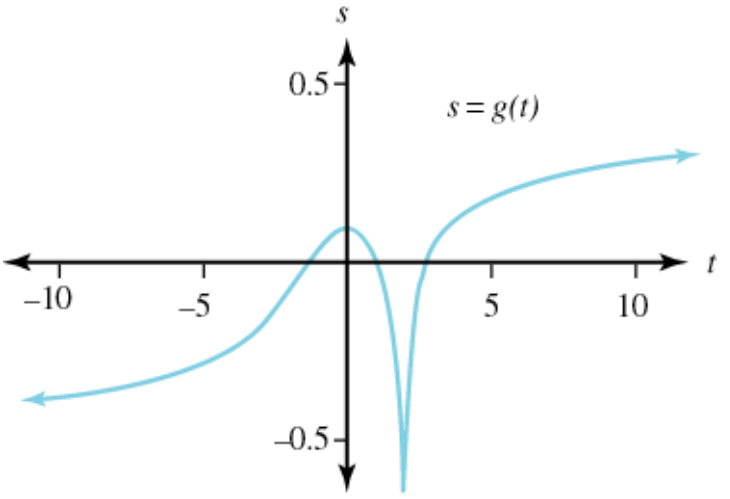


Figure 8.5 Graph of $s = g(t) = \int \frac{t}{t^3-8}\, dt$.

This antiderivative has the property that $\lim_{t\to 2} g(t) = -\infty$, as shown in Figure 8.5. We also compute

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \left(\frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{12}\,\text{Log}\left[\left(\frac{t-2}{t+2}\right)^2\right] \right)$$

$$= \lim_{t\to\infty} \frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \lim_{t\to\infty} \frac{1}{12}\,\text{Log}\left[\left(1-\frac{4}{t+2}\right)^2\right]$$

$$= \frac{\frac{\pi}{2}}{2\sqrt{3}} + \frac{1}{12}\,\text{Log}\left[(1-0)^2\right]$$

$$= \frac{\pi\sqrt{3}}{12}$$

$$\lim_{t\to-\infty} g(t) = \lim_{t\to-\infty} \left(\frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \frac{1}{12}\,\text{Log}\left[\left(\frac{t-2}{t+2}\right)^2\right] \right)$$

$$= \lim_{t\to-\infty} \frac{\text{Arctan}\left(\frac{1+t}{\sqrt{3}}\right)}{2\sqrt{3}} + \lim_{t\to-\infty} \frac{1}{12}\,\text{Log}\left[\left(1-\frac{4}{t+2}\right)^2\right]$$

$$= \frac{-\frac{\pi}{2}}{2\sqrt{3}} + \frac{1}{12}\,\text{Log}\left[(1-0)^2\right]$$

$$= -\frac{\pi\sqrt{3}}{12}$$

The Cauchy principal limit at $t = 2$ as $r \to 0$ is

$$\lim_{r\to 0^+}\ (g(2+r)-g(2-r)) =$$

$$\lim_{r\to 0^+}\left(\frac{\text{ArcTan}\,\frac{3+r}{\sqrt{3}}}{2\sqrt{3}}+\frac{\text{Log}\,\frac{r^2}{(4+r)^2}}{12}-\frac{\text{ArcTan}\,\frac{3-r}{\sqrt{3}}}{2\sqrt{3}}-\frac{\text{Log}\,\frac{r^2}{(4-r)^2}}{12}\right)$$

$$= \lim_{r\to 0^+}\left(\frac{\text{ArcTan}\,\frac{3+r}{\sqrt{3}}}{2\sqrt{3}}-\frac{\text{ArcTan}\,\frac{3-r}{\sqrt{3}}}{2\sqrt{3}}+\frac{1}{12}\,\text{Log}\,\frac{(4-r)^2}{(4+r)^2}\right)$$

$$= \frac{\text{ArcTan}\,\frac{3}{\sqrt{3}}}{2\sqrt{3}}-\frac{\text{ArcTan}\,\frac{3}{\sqrt{3}}}{2\sqrt{3}}+\frac{1}{12}\,\text{Log}\,1$$

$$= 0$$

Therefore the Cauchy principal value of the improper integral is

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{t}{t^3-8}\,dt = \lim_{r\to 0^+}\left(\int_{-\infty}^{2-r}\frac{t}{t^3-8}\,dt+\int_{2+r}^{\infty}\frac{t}{t^3-8}\,dt\right)$$

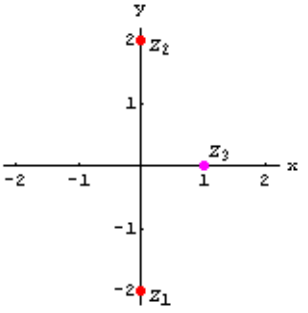
$$= \lim_{t\to\infty} g(t) - \lim_{r\to 0^+}(g(2+r)-g(2-r)) - \lim_{t\to-\infty} g(t)$$

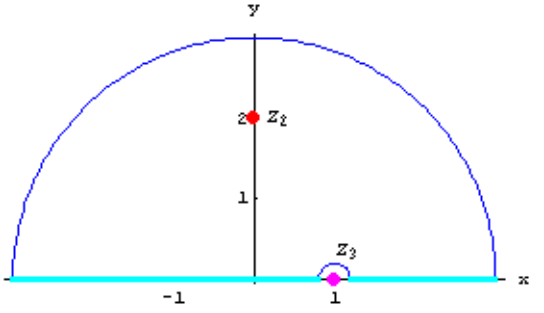
$$= \frac{\pi\sqrt{3}}{12} - 0 - \frac{-\pi\sqrt{3}}{12}$$

$$= \frac{\pi\sqrt{3}}{6}$$

Explore Solution 8.21.

Example 8.22. Show that $\text{P.V.}\int_{-\infty}^{\infty}\frac{\sin x}{(x-1)(x^2+4)}\,dx = \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right)$.





Solution. The integrand $f(z) = \frac{e^{iz}}{(z-1)(z^2+4)} = \frac{e^{iz}}{(z-1)(z+2i)(z-2i)}$ has simple poles at the points $t_1 = 1$ on the x axis and $z_1 = 2i$ in the upper half-plane. Computing the residues we get

$$\text{Res}[f, 1] = \lim_{z \to 1} (z - 1) f(z)$$

$$= \lim_{z \to 1} (z - 1) \frac{e^{i z}}{(z-1)(z^2 + 4)}$$

$$= \lim_{z \to 1} \frac{e^{i z}}{(z^2 + 4)} = \frac{e^{i}}{(1^2 + 4)}$$

$$= \frac{\cos 1 + i \sin 1}{5}$$

$$\text{Res}[f, 2i] = \lim_{z \to 2i} (z - 2i) f(z)$$

$$= \lim_{z \to 2i} (z - 2i) \frac{e^{i z}}{(z-1)(z+2i)(z-2i)}$$

$$= \lim_{z \to 2i} \frac{e^{i z}}{(z-1)(z+2i)} = \frac{e^{i(2i)}}{(2i-1)(2i+2i)}$$

$$= \frac{e^{-2}}{(-8-4i)} = \frac{e^{-2}(-8+4i)}{(-8-4i)(-8+4i)} = \frac{-8+4i}{80 e^2}$$

$$= \frac{-2+i}{20 e^2}$$

By Theorem 8.6,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+4)} \sin x \, dx = 2\pi \, \text{Re}(\text{Res}[f, z_1]) + \pi \, \text{Re}(\text{Res}[f, t_1])$$

$$= 2\pi \, \text{Re}(\text{Res}[f, 2i]) + \pi \, \text{Re}(\text{Res}[f, 1])$$

$$= 2\pi \, \text{Re}\left(\frac{-2+i}{20 e^2}\right) + \pi \, \text{Re}\left(\frac{\cos 1 + i \sin 1}{5}\right)$$

$$= 2\pi \frac{-2}{20 e^2} + \pi \frac{\cos 1}{5}$$

$$= \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right)$$

Explore Solution 8.22.

The proofs of Theorems 8.5 and 8.6 depend on the following result.

Lemma 8.2. Let $f(z)$ have a simple pole at the point t_0 on the x-axis. If the contour is $C_r : z = t_0 + r e^{i\theta}$ for $0 \le \theta \le \pi$, then

$$\lim_{r \to 0} \int_{C_r} f(z) \, dz = i \pi \, \text{Res}[f, t_0].$$

Proof.

Complex Analysis for Mathematics and Engineering

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8.6 Integrands with Branch Points

We now show how to evaluate certain improper real integrals involving the integrand $\frac{x^\alpha P(x)}{Q(x)}$. Since the complex function z^α is multivalued, so we must first specify the branch to be used.

Let α be a real number with $0 < \alpha < 1$. In this section we use the branch of z^α corresponding to the branch of the logarithm $\log_0(z)$ (see Equation (5-20)) as follows:

$$z^\alpha = e^{\alpha(\log_0(z))} = e^{\alpha(\ln|z| + i \arg_0 z)} = e^{\alpha(\ln r + i\theta)} = r^\alpha(\cos\alpha\theta + i\sin\alpha\theta),$$

where $z = r e^{i\theta}$ and $0 < \theta \le 2\pi$. Note that this is not the traditional principal branch of z^α and that, as defined, the function z^α is analytic in the domain $\{z = r e^{i\theta} : r > 0,\ 0 < \theta < 2\pi\}$.

Theorem 8.7. Let $P(x)$ and $Q(x)$ be polynomials of degree m and n, respectively, where $n \ge m + 2$. If $Q(x) \ne 0$ for $x > 0$, and $Q(x)$ has a zero of order at most 1 at the origin, and $f(z) = \frac{z^\alpha P(z)}{Q(z)}$, where $0 < \alpha < 1$, then

$$P.V.\int_0^\infty \frac{x^\alpha P(x)}{Q(x)}\,dx = \frac{2\pi i}{1 - e^{i\alpha 2\pi}} \sum_{j=1}^{k} \mathrm{Res}[f, z_j],$$

where $z_1, z_2, \ldots, z_k$ are the non-zero poles of $\frac{P(z)}{Q(z)}$.

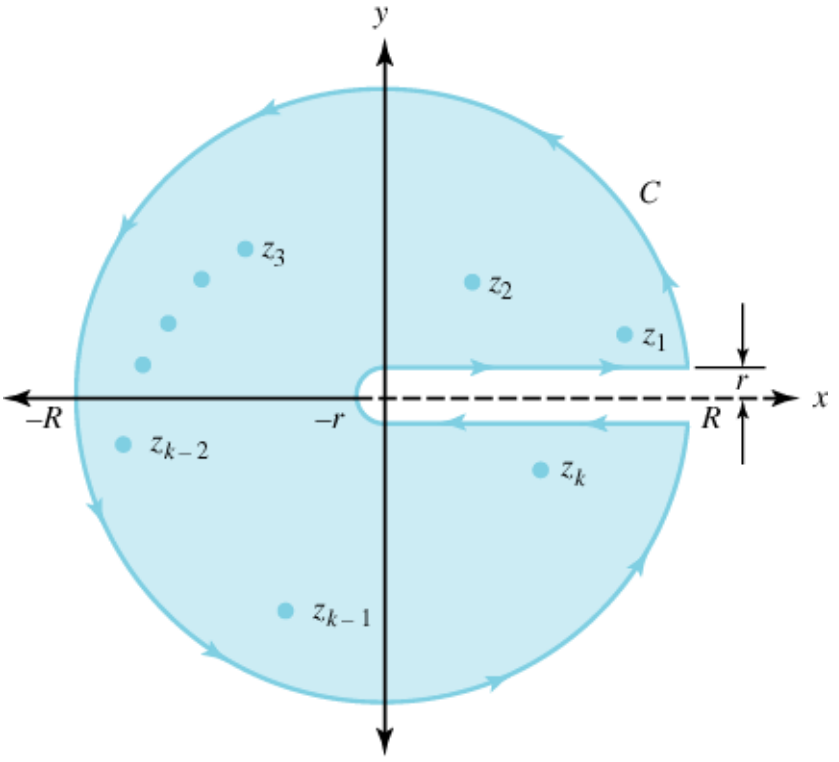
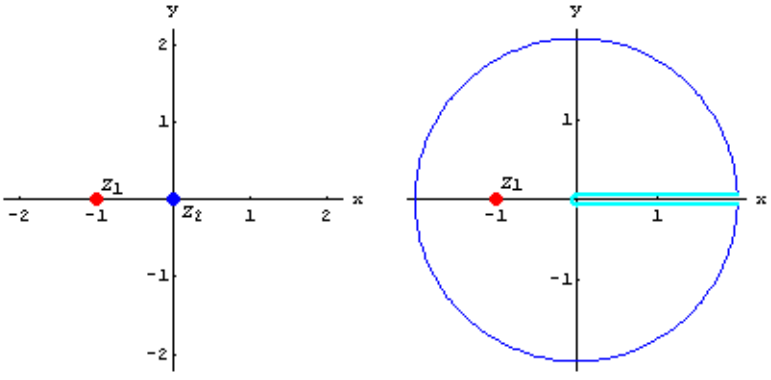


Figure 8.7 The contour C that encloses the nonzero poles $z_1, z_2, \ldots, z_k$ of $\frac{P(z)}{Q(z)}$.

Proof.

Example 8.23. Evaluate $P.V. \int_0^\infty \frac{x^\alpha}{x(x+1)}\,dx$, where $0 < \alpha < 1$.



Solution. The function $f(z) = \frac{x^\alpha}{x(x+1)}$ has a nonzero pole at the point $z_1 = -1$, and the denominator has a zero of order at most 1 (in fact, exactly 1) at the origin. Calculating the residue we get

$$\text{Res}[f, -1] = \lim_{z \to -1} (z+1) f(z)$$

$$= \lim_{z \to -1} (z+1) \frac{x^\alpha}{x(x+1)}$$

$$= \lim_{z \to -1} \frac{x^\alpha}{x} = \frac{(-1)^\alpha}{-1}$$

$$= \frac{(-1)^\alpha}{-1} = \frac{\left(e^{i\pi}\right)^\alpha}{-1}$$

$$= \frac{e^{i\alpha\pi}}{-1}$$

Using Theorem 8.7, we have

$$\text{P.V.} \int_0^\infty \frac{x^\alpha}{x(x+1)} dx = \frac{2\pi i}{1 - e^{i\alpha 2\pi}} \text{Res}[f, z_1]$$

$$= \frac{2\pi i}{1 - e^{i\alpha 2\pi}} \text{Res}[f, -1]$$

$$= \frac{2\pi i}{1 - e^{i\alpha 2\pi}} \left(\frac{e^{i\alpha\pi}}{-1}\right) = \frac{2\pi i}{e^{i\alpha\pi} - e^{-i\alpha\pi}}$$

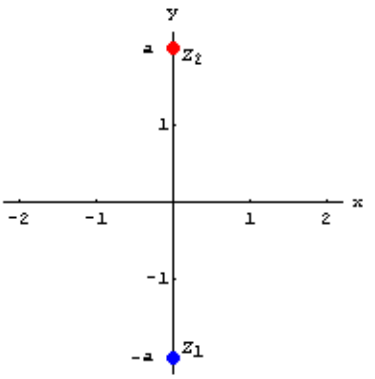
$$= \frac{\pi}{\frac{e^{i\alpha\pi} - e^{-i\alpha\pi}}{2i}}$$

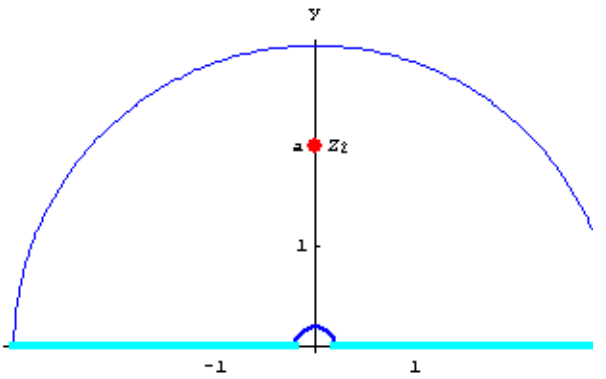
$$= \frac{\pi}{\sin \alpha\pi}$$

Explore Solution 8.23.

We can apply the preceding ideas to other multivalued functions.

Example 8.24. Evaluate $\text{P.V.} \int_0^\infty \frac{\ln x}{x^2 + a^2} dx = \frac{\pi \text{Log}[a]}{2a}$, where $a > 0$.





Solution. We use the function $f(z) = \frac{\log_{-\frac{\pi}{2}}(z)}{z^2 + a^2}$. Recall that

$$\log_{-\frac{\pi}{2}}(z) = \ln|z| + i \arg_{-\frac{\pi}{2}}(z) = \ln r + i\theta,$$

where $z = r e^{i\theta}$ and $-\frac{\pi}{2} < \theta \le \frac{3\pi}{2}$. The path C of integration will consist of the segments

$[-R, -r]$ and $[r, R]$ of the x axis together with the upper semicircles $C_r : z = r e^{i\theta}$ and $C_R : z = R e^{i\theta}$, for $0 \le \theta \le \pi$, as shown in Figure 8.8.

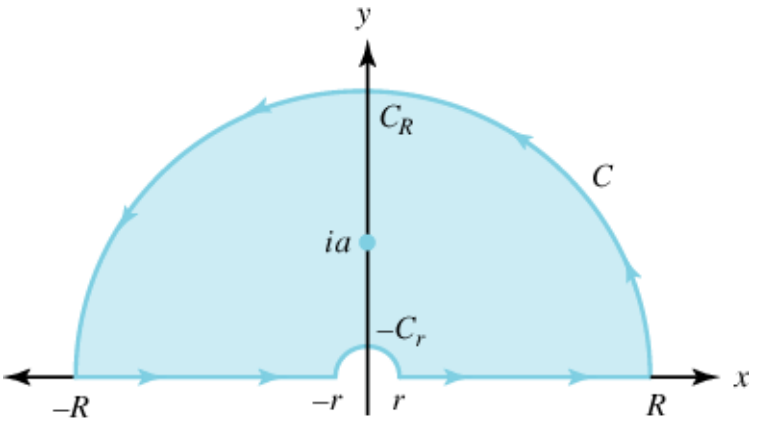


Figure 8.8 The contour C for the integrand $f(z) = \frac{\log_{-\frac{\pi}{2}}(z)}{z^2 + a^2}$.

We chose the branch $\log_{-\frac{\pi}{2}}(z)$ because it is analytic on C and its interior, hence so is the function $f(z)$ which has a simple pole in the upper half-plane at the point $z_2 = a\,i$.

$$\begin{aligned}
\text{Res}[f, z_1] &= \text{Res}[f, a\,i] \\
&= \lim_{z \to a\,i} (z - a\,i)\, f(z) \\
&= \lim_{z \to a\,i} (z - a\,i)\, \frac{\log_{-\frac{\pi}{2}}(z)}{(z + a\,i)(z - a\,i)} \\
&= \lim_{z \to a\,i} \frac{\log_{-\frac{\pi}{2}}(z)}{(z + a\,i)} \\
&= \frac{\log_{-\frac{\pi}{2}}(a\,i)}{(a\,i + a\,i)} = \frac{\log_{-\frac{\pi}{2}}(a\,i)}{2\,a\,i} \\
&= \frac{\ln a + i \arg_{-\frac{\pi}{2}}(a\,i)}{2\,a\,i} = \frac{\ln a + i\,\frac{\pi}{2}}{2\,a\,i} \\
&= \frac{\ln a}{2\,a\,i} + \frac{\pi}{4\,a}
\end{aligned}$$

This choice enables us to apply the residue theorem properly (see the hypotheses of Theorem 8.1), and we get

$$\begin{aligned}
\oint_C f(z)\, dz &= 2\pi i\, \text{Res}[f, z_1] = 2\pi i\, \text{Res}[f, a\,i] \\
&= 2\pi i \left(\frac{\ln a}{2\,a\,i} + \frac{\pi}{4\,a} \right) \\
&= \frac{\pi \ln a}{a} + i\,\frac{\pi^2}{2\,a}
\end{aligned}$$

Keeping in mind the branch of logarithm that we're using, we then have

(8-33)

$$\oint_C f(z)\,dz = \int_{-R}^{-r} f(x)\,dx + \oint_{-C_r} f(z)\,dz + \int_r^R f(x)\,dx + \oint_{C_R} f(z)\,dz$$

$$= \int_{-R}^{-r} \frac{\ln|x| + i\pi}{x^2 + a^2}\,dx + \oint_{-C_r} f(z)\,dz + \int_r^R \frac{\ln x}{x^2 + a^2}\,dx + \oint_{C_R} f(z)\,dz$$

$$= \frac{\pi \ln a}{a} + i\,\frac{\pi^2}{2a}$$

If $R^2 > r^2$, then by the ML inequality (Theorem 6.3)

$$\left| \oint_{C_R} f(z)\,dz \right| = \left| \int_0^\pi \frac{\ln R + i\theta}{R^2 e^{i2\theta} + a^2}\, i R e^{i\theta}\,d\theta \right|$$

$$\leq \frac{R(\ln R + \pi)\pi}{R^2 - a^2}$$

and L'Hôpital's rule yields $\lim_{R\to\infty} \oint_{C_R} f(z)\,dz = 0$. A similar computation shows that $\lim_{r\to 0} \oint_{C_r} f(z)\,dz = 0$.

We use these results when we take limits Equations (8-33) to get

$$\text{P.V.}\left(\int_{-\infty}^{0} \frac{\ln|x| + i\pi}{x^2 + a^2}\,dx + \int_0^\infty \frac{\ln x}{x^2 + a^2}\,dx\right) = \lim_{\substack{R\to\infty \\ r\to 0}} \left(\int_{-R}^{-r} \frac{\ln|x| + i\pi}{x^2 + a^2}\,dx + \int_r^R \frac{\ln x}{x^2 + a^2}\,dx\right)$$

$$= \oint_C f(z)\,dz$$

$$= \frac{\pi \ln a}{a} + i\,\frac{\pi^2}{2a}$$

Equating the real parts in this equation gives

$$\text{P.V.}\left(\int_{-\infty}^{0} \frac{\ln|x| + i\pi}{x^2 + a^2}\,dx + \int_0^\infty \frac{\ln x}{x^2 + a^2}\,dx\right) = \frac{\pi \ln a}{a}$$

$$\text{P.V.}\left(\int_0^\infty \frac{\ln x}{x^2 + a^2}\,dx + \int_0^\infty \frac{\ln x}{x^2 + a^2}\,dx\right) = \frac{\pi \ln a}{a}$$

$$\text{P.V.}\int_0^\infty \frac{\ln x}{x^2 + a^2}\,dx = \frac{\pi \ln a}{2a}$$

Explore Solution 8.24.

Complex Analysis for Mathematics and Engineering

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8.7 The Argument Principle and Rouché's Theorem

We now derive two results based on Cauchy's residue theorem. They have important practical applications and pertain only to functions all of whose isolated singularities are poles.

Definition 8.3 (Meromorphic Function). A function $f(z)$ is said to be meromorphic in a domain D provided the only singularities of $f(z)$ are isolated poles and removable singularities.

We make three important observations relating to this definition.

(i). Analytic functions are a special case of meromorphic functions.

(ii). Rational functions $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials, are meromorphic in the entire complex plane.

(iii). By definition, meromorphic functions have no essential singularities.

Suppose that $f(z)$ is analytic at each point on a simple closed contour C and $f(z)$ is meromorphic in the domain that is the interior of C. We assert without proof that Theorem 7.13 can be extended to meromorphic functions so that $f(z)$ has at most a finite number of zeros that lie inside C. Since the function $g(z) = \frac{1}{f(x)}$ is also meromorphic, it can have only a finite number of zeros inside C, and so $f(z)$ can have at most a finite number of poles that lie inside C.

Theorem 8.8, known as the argument principle, is useful in determining the number of zeros and poles that a function has.

Theorem 8.8 (Argument Principle). Suppose that $f(z)$ be meromorphic in the simply connected domain D. Suppose that $f(z)$ is meromorphic in the simply connected domain D and that C is a simple closed positively oriented contour in D such that $f(z)$ has no zeros or poles ($f(z) \neq 0$ and $f(z) \neq \infty$) for $z \in C$. Then

(8-34) $$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = Z_f - P_f,$$

where Z_f is the number of zeros of $f(z)$ that lie inside C and P_f is the number of poles that lie inside C.

Proof.

Corollary 8.1. Suppose that $f(z)$ is analytic in the simply connected domain D. Let C be a simple closed positively oriented contour in D such that for $z \in C$, $f(z) \neq 0$. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = Z_f,$$

where Z_f is the number of zeros of $f(z)$ that lie inside C.

Proof.

Remark 8.3. Certain feedback control systems in engineering must be stable. A test for stability involves the function $G(z) = 1 + F(z)$, where $F(z)$ is a rational function. If $G(z)$ does not have any zeros in the region $\{z : \mathrm{Re}(z) \geq 0\}$, then the system is stable. We determine the number of zeros of $G(z)$ by writing $F(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials with no common zero. Then $G(z) = \frac{Q(z)+P(z)}{Q(z)}$, and we can check for the zeros of $Q(z) + P(z)$ by using Theorem 8.8. We select a value R so that $G(z) \neq 0$ for $\{z : |z| > R\}$ and then integrate along the contour consisting of the right half of the circle $C_R(0)$ and the line segment between $i\,R$ and $-i\,R$. This method is known as the Nyquist stability criterion.

Why do we label Theorem 8.8 as the argument principle? The answer lies with a fascinating application known as the winding number.
Recall that a branch of the logarithm function, $\log_\alpha(z)$, is defined by

$$\log_\alpha(z) = \ln|z| + i\arg_\alpha(z) = \ln r + i\phi,$$

where $z = r e^{i\phi}$ and $0 < \phi \leq 2\pi$. Loosely speaking, suppose that for some branch of the logarithm, the composite function $\log_\alpha(f(z))$ were analytic in a simply connected domain D containing the contour C. This would imply that $\log_\alpha(f(z))$ is an antiderivative of the function $\frac{f'(z)}{f(z)}$ for all $z \in D$. Theorems 6.9 and 8.8 would then tell us that, as z winds around the curve C, the quantity $\log_\alpha(f(z)) = \ln|f(z)| + i\arg_\alpha f(z)$ would change by $2\pi i(Z_f - P_f)$. Since $2\pi i(Z_f - P_f)$ is purely imaginary, this result tells us that $\arg_\alpha f(z)$ would change by $2\pi i(Z_f - P_f)$ radians. In other words, as z winds around C, the integral $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$ would count how many times the curve $f(C)$ winds around the origin.

Unfortunately, we can't always claim that $\log_\alpha(f(z))$ is an antiderivative of the function $\frac{f'(z)}{f(z)}$ for all $z \in D$. If it were, the Cauchy-Goursat theorem would imply that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = 0$. Nevertheless, the heuristics that we gave - indicating that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz$ counts how many times the curve $f(C)$ winds around the origin - still hold true, as shown in the book.

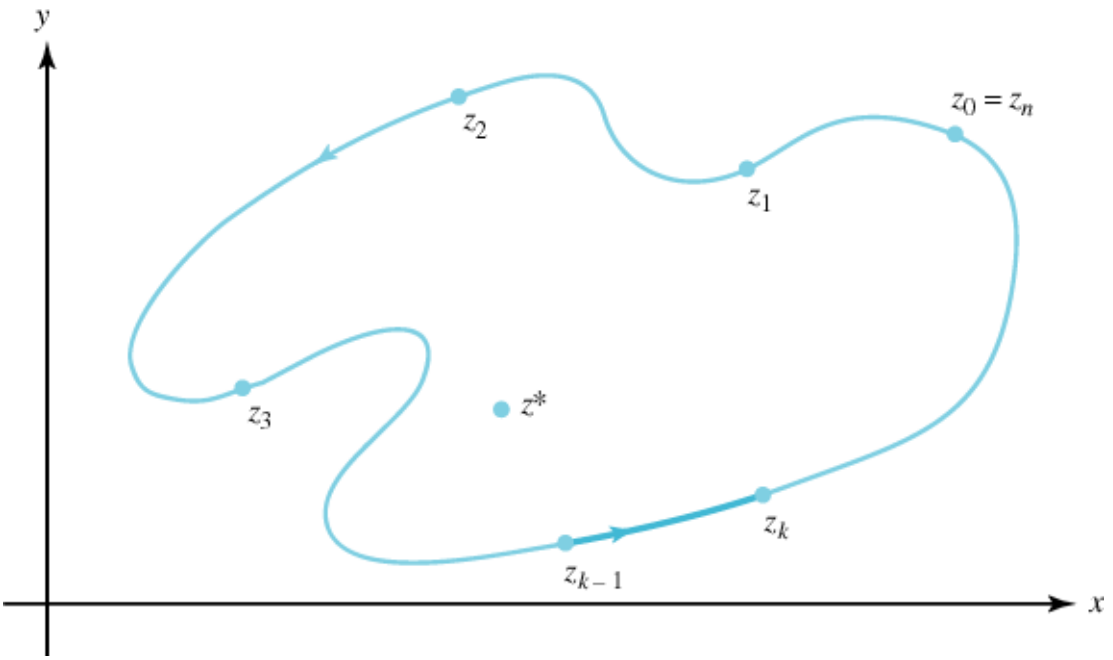


Figure 8.10 The points z_k on the contour C that winds around z^*.

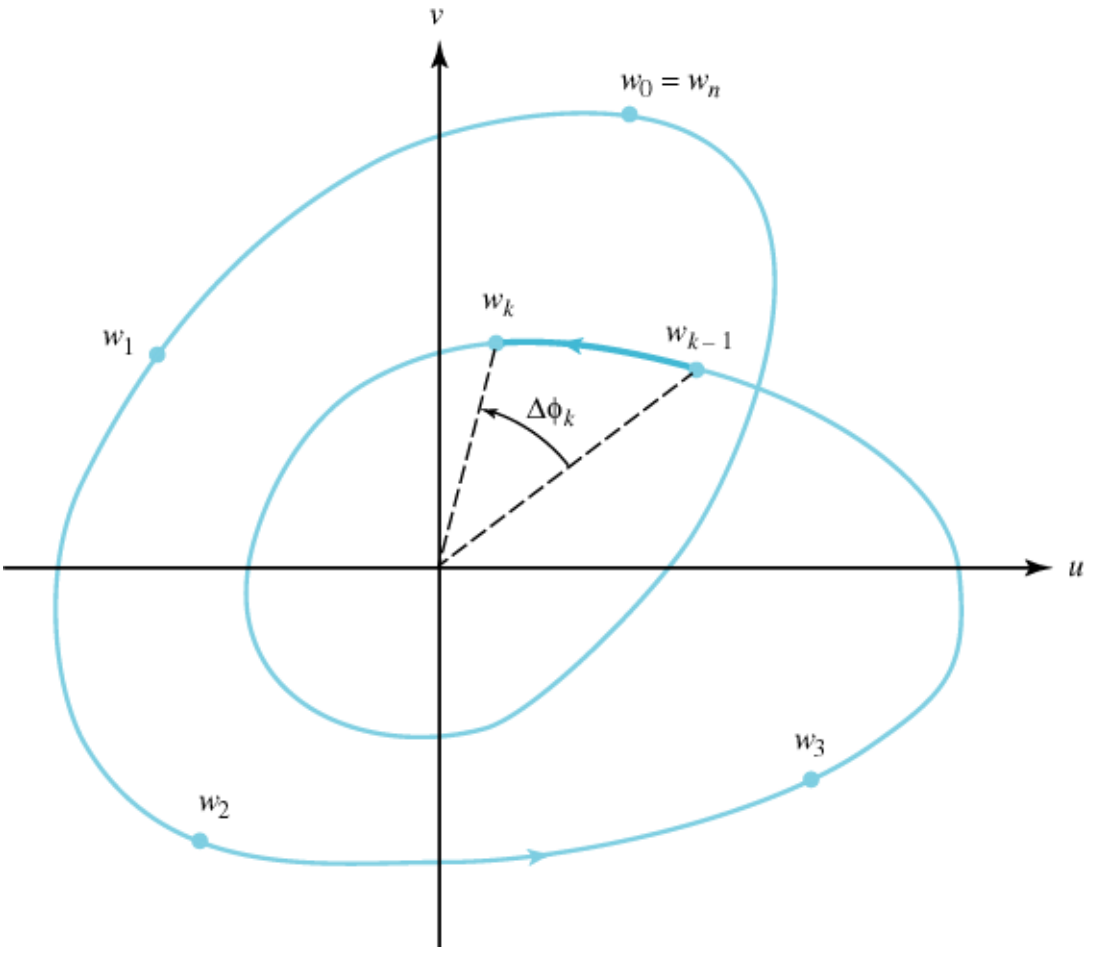


Figure 8.11 The points w_k on the contour f(C) that winds around 0.

Example 8.25. The image of the circle $C_2(0)$ under $w = f(z) = z^2 + z$ is the curve $\{(u, v) = (4\cos 2t + 2\cos t, 4\sin 2t + 2\sin t) : 0 \le t \le 2\pi\}$ shown in Figure 8.12. Note that the image curve $f(C_2(0))$ winds twice around the origin. We check this by computing

$$\frac{1}{2\pi i}\int_{C_2^+(0)} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_{C_2^+(0)} \frac{2z+1}{z^2+z}\,dz.$$

The residues of the integrand are at 0 and −1. Thus

$$\frac{1}{2\pi i}\int_{C_2^+(0)} \frac{2z+1}{z^2+z}\,dz = \text{Res}\left[\frac{2z+1}{z^2+z}, 0\right] + \text{Res}\left[\frac{2z+1}{z^2+z}, 1\right]$$

$$= 1 + 1 = 2$$

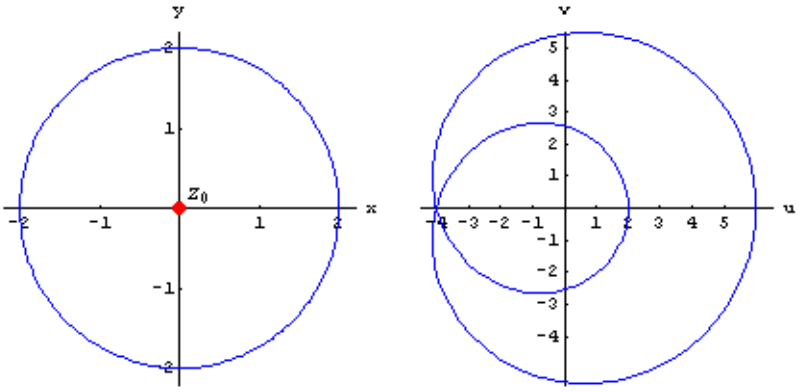


Figure 8.12 The image curve $f(C_2(0))$ under $w = f(z) = z^2 + z$.

Explore Solution 8.25.

Finally, we note that if $g(z) = f(z) - a$, then $g'(z) = f(z)$, and thus we can generalize what we've just said to compute how many times the curve $f(C)$ winds around the point a. Theorem 8.9 summarizes our discussion.

Theorem 8.9 (Winding Number). Suppose $f(z)$ is meromorphic in the simply connected domain D. If C is a simple closed, positively oriented contour in D such that for $z \in C$, $f(z) \neq 0$ and $f(z) \neq \infty$, then

$$W(f(C), a) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-a}\,dz,$$

known as the winding number of $f(C)$ about a, counts the number of times the curve $f(C)$ winds around the point a. If $a = 0$, the integral counts the number of times the curve $f(C)$ winds around the origin.

Proof.

Remark 8.4. Letting $f(z) = z$ in Theorem 8.9 gives

$$W(C, a) = \frac{1}{2\pi i}\int_C \frac{1}{z-a}\,dz = \begin{cases} 1, & \text{if } a \text{ lies inside } C, \text{ or} \\ 0, & \text{if } a \text{ lies outside } C, \end{cases}$$

which counts the number of times the curve C winds around the point a. If C is not a simple closed curve, but crosses itself perhaps several times, we can show (but omit the proof) that $W(C, a)$ still gives the number of times the curve C winds around the point a. Thus winding number is indeed an appropriate term.

We close this section with a result that will help us gain information about the location of the zeros and poles of meromorphic functions.

Theorem 8.10 (Rouche's Theorem). Suppose that $f(z)$ and $g(z)$ are meromorphic functions defined in the simply connected domain D, that C is a simply closed contour in D, and that $f(z)$ and $g(z)$ have no zeros or poles for $z \in C$. If the strict inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ holds for all $z \in C$, then $Z_f - P_f = Z_g - P_g$.

Proof.

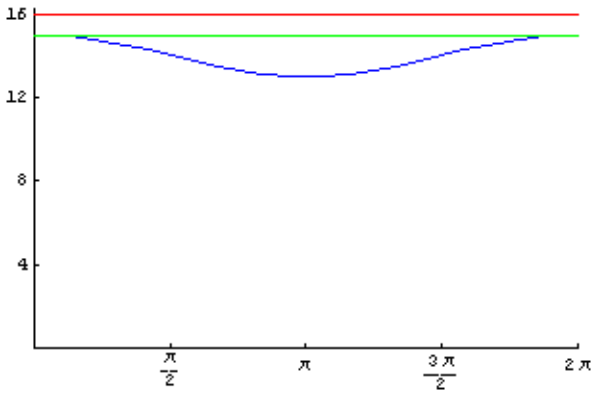
Corollary 8.2. Suppose that $f(z)$ and $g(z)$ are analytic functions defined in the simply connected

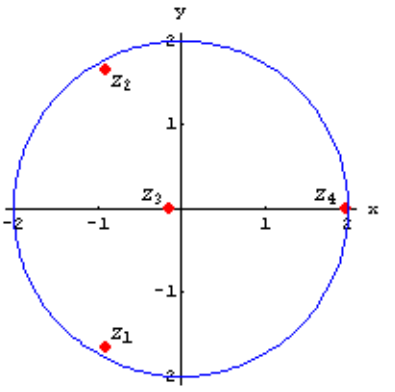
domain D, that C is a simple closed contour in D, and that $f(z)$ and $g(z)$ have no zeros for $z \in C$. If the strict inequality $|f(z)+g(z)| < |f(z)|+|g(z)|$ holds for all $z \in C$, then $Z_f = Z_g$.

Proof.

Remark 8.5. Theorem 8.10 is usually stated with the requirement that $f(z)$ and $g(z)$ satisfy the condition $|f(z)+g(z)| < |g(z)|$, for $z \in C$. The improved theorem that we gave was discovered by Irving Glicksberg (see the American Mathematical Monthly, 83 (1976), pp. 186-187). The weaker version is adequate for most purposes, however, as the following examples illustrate.

Example 8.26. Show that al four zeros of the polynomial $g(z) = z^4 - 7z - 1$ lie in the disk $D_2(0) = \{z : |z| < 2\}$.





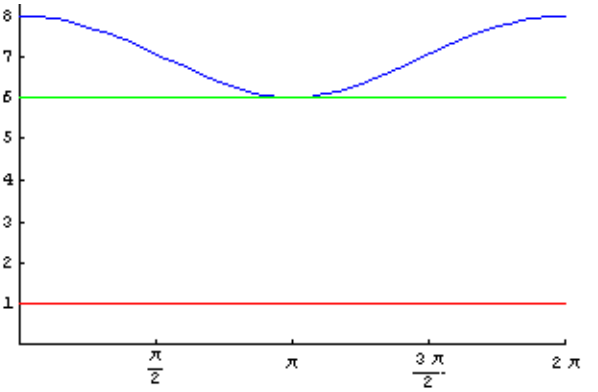
Solution. Let $f(z) = -z^4$. Then $f(z)+g(z) = -7z-1$, and at points on the circle $C_2(0) = \{z : |z| = 2\}$ we have the relation

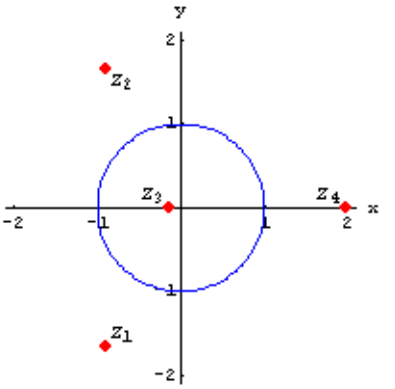
$$\begin{aligned} |f(z)+g(z)| &= |-7z-1| \\ &\le |-7z| + |-1| \\ &= 7|z| + 1 = 7\times 2 + 1 \\ &= 15 \\ &< 16 = |z^4| = |-z^4| \\ &= |f(z)| \end{aligned}$$

Of course, if $|f(z)+g(z)| < |f(z)|$, then as we indicated in Remark 8.5 we certainly have $|f(z)+g(z)| < |f(z)|+|g(z)|$, so that the conditions for applying Corollary 8.2 are satisfied on the circle $C_2(0)$. The function $f(z)$ has a zero of order 4 at the origin, so $g(z)$ must have four zeros inside $D_2(0)$.

Explore Solution 8.26.

Example 8.27. Show that the polynomial $g(z) = z^4 - 7z - 1$ has one zero in the disk $D_1(0) = \{z : |z| < 1\}$.





Solution. Let $f(z) = 7z + 1$, then $f(z) + g(z) = z^4$. At points on the circle $C_1(0) = \{z : |z| = 1\}$ we have the relation

$$|f(z) + g(z)| = |z^4|$$

$$= 1$$

$$< 6$$

$$= 7 - 1 = |7z| - |1|$$

$$\leq |7z - 1|$$

$$= |f(z)|$$

The function $f(z)$ has one zero at the point $-\frac{1}{7}$ in the disk $D_1(0)$, and the hypotheses of Corollary 8.2 hold on the circle $C_1(0)$. Therefore $g(z)$ has one zero inside $D_1(0)$.

Explore Solution 8.27.

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Chapter 9 z-transforms and applications

Overview

The z-transform is useful for the manipulation of discrete data sequences and has acquired a new significance in the formulation and analysis of discrete-time systems. It is used extensively today in the areas of applied mathematics, digital signal processing, control theory, population science, economics. These discrete models are solved with difference equations in a manner that is analogous to solving continuous models with differential equations. The role played by the z-transform in the solution of difference equations corresponds to that played by the Laplace transforms in the solution of differential equations.

9.1 The z-transform

The function notation for sequences is used in the study and application of z-transforms. Consider a function $x[t]$ defined for $t \geq 0$ that is sampled at times $t = T,\ T,\ 2\,T, 3\,T,\ \ldots$, where T is the sampling period (or rate). We can write the sample as a sequence using the notation $\{x_n = x[n\,T]\}_{n=0}^{\infty}$. Without loss of generality we will set $T = 1$ and consider real sequences such as, $\{x_n = x[n]\}_{n=0}^{\infty}$. The definition of the z-transform involves an infinite series of the reciprocals z^{-n}.

Definition 9.1 (z-transform) Given the sequence $\{x_n = x[n]\}_{n=0}^{\infty}$ the z-transform is defined as follows

$$X(z) = \mathcal{Z}[\{x_n\}_{n=0}^{\infty}] = \sum_{n=0}^{\infty} x_n\, z^{-n} = \sum_{n=0}^{\infty} x[n]\, z^{-n}, \tag{9-1}$$

which is a series involving powers of $\frac{1}{z}$.

Remark 9.1. The z-transform is defined at points $z \in \mathbb{C}$ where the Laurent series (9-1) converges. The z-transform region of convergence (ROC) for the Laurent series is chosen to be

$$|z| \geq R, \quad \text{where} \quad R = \limsup \sqrt[n]{|x_n|}\,.$$

Remark 9.2. The sequence notation $\{x_n\}_{n=0}^{\infty}$ is used in mathematics to study difference equations and the function notation $\{x[n]\}_{n=0}^{\infty}$ is used by engineers for signal processing. It's a good idea to

know both notations.

Remark 9.3. In the applications, the sequence $\{x_n = x[n]\}_{n=0}^{\infty}$ will be used for inputs and the sequence $\{y_n = y[n]\}_{n=0}^{\infty}$ will be used for outputs. We will also use the notations

$$\mathcal{Z}[x_n] = \mathcal{Z}[x[n]] = X(z), \text{ and}$$

$$\mathcal{Z}[y_n] = \mathcal{Z}[y[n]] = Y(z).$$

Theorem 9.1 (Inverse z-transform) Let $X(z)$ be the z-transform of the sequence $\{x_n = x[n]\}_{n=0}^{\infty}$ defined in the region $R < |z|$. Then x_n is given by the formula

(9-2) $$x_n = x[n] = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi i}\oint_C X(z)\, z^{n-1}\, dz,$$

where C is any positively oriented simple closed curve that lies in the region $R < |z|$ and winds around the origin.

Proof.

9.1.1 Admissible form of a z-transform

Formulas for $X(z)$ do not arise in a vacuum. In an introductory course they are expressed as linear combinations of z-transforms corresponding to elementary functions such as

$$S = \{\delta[n],\ u[n],\ b^n,\ e^{an},\ n,\ n^2,\ n^m,\ n\,b^n,\ n\,e^{an},\ b^n \operatorname{Sin}[a\,n],\ b^n \operatorname{Cos}[a\,n],$$
$$\operatorname{Sinh}[a\,n],\ \operatorname{Cosh}[a\,n],\ \text{etc.}\}.$$

In Table 9.1, we will see that the z-transform of each function in S is a rational function of the complex variable z. It can be shown that a linear combination of rational functions is a rational function. Therefore, for the examples and applications considered in this book we can restrict the z-transforms to be rational functions. This restriction is emphasized this in the following definition.

Definition 9.2 (Admissible z-transform) Given the z-transform $X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$ we say that $X(z)$ is an admissible z-transform, provided that it is a rational function, that is

(9-3) $$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \ldots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \ldots + a_{q-1} z^{q-1} + a_q z^q},$$

where $P(z)$ and $Q(z)$, are polynomials of degree p and q, respectively.

From our knowledge of rational functions, we see that an admissible z-transform is defined everywhere in the complex plane except at a finite number of isolated singularities that are poles and occur at the points where $Q(z) = 0$. The Laurent series expansion in (9-1) can be obtained by a partial fraction manipulation and followed by geometric series expansions in powers of $\frac{1}{z}$. However, the signal feature of formula (9-3) is the calculation of the inverse z-transform via residues.

Theorem 9.2 (Cauchy's Residue Theorem) Let D be a simply connected domain, and let C be a simple closed positively oriented contour that lies in D. If f(z) is analytic inside C and on C, except at the points $z_1, z_2, \ldots, z_k$ that lie inside C, then

$$\frac{1}{2\pi i}\int_C f(z)\, dz = \sum_{j=1}^{k} \operatorname{Res}[f(z), z_j].$$

Proof.

Corollary 9.1 (Inverse z-transform) Let $X(z)$ be the z-transform of the sequence $\{x_n\}$. Then x_n is given by the formula

$$x_n = x[n] = \mathcal{Z}^{-1}[X(z)] = \sum_{j=1}^{k} \text{Res}[X(z)\, z^{n-1}, z_j].$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z) = X(z)\, z^{n-1}$.

Corollary 9.2 (Inverse z-transform) Let $X(z)$ be the z-transform of the sequence. If $X(z)$ has simple poles at the points $z_1, z_2, \ldots, z_k$ then x_n is given by the formula

$$x_n = x[n] = \mathcal{Z}^{-1}[X(z)] = \sum_{j=1}^{k} \left(\lim_{z \to z_j} (z - z_j)\, X(z)\, z^{n-1} \right).$$

Example 9.1. Find the z-transform of the unit pulse or impulse sequence $x_n = \delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases}$.

Solution 9.1. This follows trivially from Equation (9-1)

$$X(z) = \mathcal{Z}[x_n] = \sum_{n=0}^{\infty} x_n z^{-n} = 1 + \sum_{n=1}^{\infty} 0\, z^{-n} = 1.$$

Explore Solution 9.1.

Example 9.2. The z-transform of the unit-step sequence $x_n = u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$ is $X(z) = \frac{z}{z-1}$.

Solution 9.2. From Equation (9-1)

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^n$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

Explore Solution 9.2.

Example 9.3. The z-transform of the sequence $x_n = b^n$ is $X(z) = \frac{z}{z-b}$.

Solution 9.3. From Definition 9.1

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} b^n z^{-n}.$$

$$= \sum_{n=0}^{\infty} \left(\frac{b^n}{z^n}\right) = \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n$$

$$= \frac{1}{1-\frac{b}{z}} = \frac{z}{z-b}$$

Explore Solution 9.3.

Example 9.4. The z-transform of the exponential sequence $x_n = e^{an}$ is $X(z) = \frac{z}{z-e^a}$.

Solution 9.4. From Definition 9.1

$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} e^{an} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{e^{an}}{z^n}\right) = \sum_{n=0}^{\infty} \left(\frac{e^{a}}{z}\right)^n$$

$$= \frac{1}{1-\frac{e^a}{z}} = \frac{z}{z-e^a}$$

Explore Solution 9.4.

9.1.2 Properties of the z-transform

Given that $\mathcal{Z}[x_n] = \mathcal{Z}[x[n]] = X(z)$ and $\mathcal{Z}[y_n] = \mathcal{Z}[y[n]] = Y(z)$. We have the following properties:

(i) Linearity. $\mathcal{Z}[c_1 x_n + c_2 y_n] = \mathcal{Z}[c_1 x[n] + c_2 y[n]] = c_1 X(z) + c_2 Y(z)$.

(ii) Delay Shift. $\mathcal{Z}[x[n-N]\,u[n-N]] = X(z)\,z^{-N}$.

(iii) Advance Shift.

$\mathcal{Z}[x[n+N]] = z^N (X(z) - x[0] - x[1] z^{-1} - x[2] z^{-2} - \ldots - x[N-1] z^{-N+1})$, or

$$\mathcal{Z}[x_{n+N}] = z^N (X(z) - x_0 - x_1 z^{-1} - x_2 z^{-2} - \ldots - x_{N-1} z^{-N+1})$$

(iv) Multiplication by n. $\mathcal{Z}[n x_n] = \mathcal{Z}[n x[n]] = -z \frac{d}{dz} X(z)$.

Example 9.5 (a). The z-transform of the sequence $x_n = \cos(a n)$ is $X(z) = \frac{z(z-\cos(a))}{z^2-2z\cos(a)+1}$.

Example 9.5 (b). The z-transform of the sequence $x_n = \sin(a n)$ is $X(z) = \frac{\sin(a)\,z}{z^2-2\cos(a)\,z+1}$.

Solution 9.5 (a).

$$\mathcal{Z}[x_n] = \mathcal{Z}[\cos(a n)] = \mathcal{Z}\left[\frac{1}{2} e^{i a n} + \frac{1}{2} e^{-i a n}\right]$$

$$= \frac{1}{2}\mathcal{Z}[e^{i a n}] + \frac{1}{2}\mathcal{Z}[e^{-i a n}] = \frac{1}{2}\frac{z}{z-e^{i a}} + \frac{1}{2}\frac{z}{z-e^{-i a}}$$

$$= \frac{1}{2}\left(\frac{z(z-e^{-i a})}{(z-e^{i a})(z-e^{-i a})} + \frac{z(z-e^{i a})}{(z-e^{i a})(z-e^{-i a})}\right)$$

$$= \frac{z(2z-e^{i a}-e^{-i a})}{2(z-e^{i a})(z-e^{-i a})} = \frac{z\left(z-\frac{e^{i a}+e^{-i a}}{2}\right)}{(z-e^{i a})(z-e^{-i a})}$$

$$= \frac{z(z-\cos(a))}{(z-e^{i a})(z-e^{-i a})} = \frac{z(z-\cos(a))}{z^2-2z\cos(a)+1}$$

Remark 9.4. When using the residue theorem to compute inverse z-transforms, the complex form is preferred, i. e.

$$\mathcal{Z}[\cos(a n)] = \frac{z\left(z-\frac{e^{i a}+e^{-i a}}{2}\right)}{(z-e^{i a})(z-e^{-i a})},$$

and

$$\mathcal{Z}[\sin(a n)] = \frac{z\left(\frac{e^{i a}-e^{-i a}}{2 i}\right)}{(z-e^{i a})(z-e^{-i a})}.$$

Solution 9.5 (b). This is left as an exercise for the reader.

Explore Solution 9.5 (a).

9.1.3 Table of z-transforms

We list the following table of z-transforms. It can also be used to find the inverse z-transform.

	Sequence	z – transform
1	$\delta[n]$	1
2	$u[n]$	$\frac{z}{z-1}$
3	b^n	$\frac{z}{z-b}$
4	$b^{n-1}\,u[n-1]$	$\frac{1}{z-b}$
5	$e^{a\,n}$	$\frac{z}{z-e^a}$
6	n	$\frac{z}{(z-1)^2}$
7	n^2	$\frac{z\,(z+1)}{(z-1)^3}$
8	$b^n\,n$	$\frac{b\,z}{(z-b)^2}$
9	$e^{a\,n}\,n$	$\frac{z\,e^a}{(z-e^a)^2}$
10	$\sin(a\,n)$	$\frac{\sin(a)\,z}{z^2-2\cos(a)\,z+1}$
11	$b^n \sin(a\,n)$	$\frac{\sin(a)\,b\,z}{z^2-2\cos(a)\,b\,z+b^2}$
12	$\cos(a\,n)$	$\frac{z\,(z-\cos(a))}{z^2-2\cos(a)\,z+1}$
13	$b^n \cos(a\,n)$	$\frac{z\,(z-b\cos(a))}{z^2-2\cos(a)\,b\,z+b^2}$

Table 9.1. z-transforms of some common sequences.

Exploration

Theorem 9.3 (Residues at Poles)

(i) If $f(z)$ has a simple pole at z_0, then the residue is

$$\text{Res}[f(z), z_0] = \lim_{z \to z_0} (z - z_0)\, f(z).$$

(ii) If $f(z)$ has a pole of order 2 at z_0, then the residue is

$$\text{Res}[f(z), z_0] = \lim_{z \to z_0} \frac{d}{dz} \left((z - z_0)^2 f(z) \right).$$

(iii) If $f(z)$ has a pole of order 3 at z_0, then the residue is

$$\text{Res}[f(z), z_0] = \frac{1}{2!} \lim_{z \to z_0} \frac{d^2}{dz^2} \left((z - z_0)^3 f(z) \right).$$

Proof.

Proof of Theorem 9.3 is in the book (see Theorem 8.2).

Subroutines for finding the inverse z-transform

Example 9.6. Find the inverse z-transform $x_n = x[n] = \mathcal{Z}^{-1}\left[\frac{2z}{2z-1}\right]$. Use (a) series, (b) table of z-transforms, (c) residues.

Solution 9.6.

Solution 9.6 (a). Method of series.

Expand $X(z) = \frac{2z}{2z-1}$ in a series involving powers of $\frac{1}{z}$.

$$X(z) = \frac{z}{z-\frac{1}{2}} = \frac{1}{1-\frac{1}{2}z^{-1}} = \sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n = \sum_{n=0}^{\infty}\frac{1}{2^n}z^{-n}$$

The sequence of coefficients in the Laurent series is what we desire, and we see that

$$x_n = x[n] = \frac{1}{2^n}.$$

Solution 9.6 (b). Writing $X(z) = \frac{z}{z-\frac{1}{2}}$ we identify $b = \frac{1}{2}$ and use line 3 in the Table 9.1 to obtain

$$x_n = x[n] = b^n = \frac{1}{2^n}.$$

Solution 9.6 (c). Writing $X(z) = \frac{z}{z-\frac{1}{2}}$ we see that $X(z)$ has a simple pole at $z_0 = \frac{1}{2}$. Using the Corollary 9.1 for finding the inverse z-transform we obtain

$$x_n = x[n] = \text{Res}[X(z)\,z^{n-1}, z_0] = \text{Res}\left[\frac{z}{z-\frac{1}{2}}z^{n-1}, \frac{1}{2}\right] = \text{Res}\left[\frac{z^n}{z-\frac{1}{2}}, \frac{1}{2}\right]$$

Using the function $f(z) = X(z)\,z^{n-1}$ and value $z_0 = \frac{1}{2}$ in Corollary 9.2 we get

$$x_n = x[n] = \lim_{z\to z_0}(z-z_0)\,f(z) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^n}{z-\frac{1}{2}} = \lim_{z\to\frac{1}{2}} z^n = \frac{1}{2^n}.$$

Explore Solution 9.6.

The following two theorems about z-transforms are useful in finding the solution to a difference equation.

Theorem 9.4 (Shifted Sequences & Initial Conditions) Define the sequence $\{x[n] = x_n\}_{n=0}^{\infty}$ and let $X(z) = \mathcal{Z}[x[n]] = \mathcal{Z}[x_n] = \sum_{n=0}^{\infty} x_n z^{-n}$ be its z-transform. Then

(i) $$\mathcal{Z}[x[n+1]] = \mathcal{Z}[x_{n+1}] = z(X(z) - x_0)$$

(ii) $$\mathcal{Z}[x[n+2]] = \mathcal{Z}[x_{n+2}] = z^2(X(z) - x_0 - x_1 z^{-1})$$

(iii) $$\mathcal{Z}[x[n+3]] = \mathcal{Z}[x_{n+3}] = z^3(X(z) - x_0 - x_1 z^{-1} - x_2 z^{-2})$$

Theorem 9.5 (Convolution) Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be sequences with z-transforms $X(z)$ and $Y(z)$, respectively. Then

$$\mathcal{Z}[x_n * y_n] = X(z)\,Y(z)$$

Proof.

9.1.4 Properties of the z-transform

The following properties of z-transforms listed in Table 9.2 are well known in the field of digital signal analysis. The reader will be asked to prove some of these properties in the exercises.

		Sequence	z – transform
	definition	$x_n = x[n]$	$X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$
1	addition	$x_n + y_n$	$X(z) + Y(z)$
2	constant multiple	$c\, x_n$	$c\, X(z)$
3	linearity	$c\, x_n + d\, y_n$	$c\, X(z) + d\, Y(z)$
4	delayed unit step	$u[n-m]$	$\frac{z^{1-m}}{z-1}$
5	time delay 1 tap	$x_{n-1}\, u[n-1]$	$\frac{1}{z} X(z)$
6	time delayed shift	$x_{n-m}\, u[n-m]$	$z^{-m} X(z)$
7	forward 1 tap	x_{n+1}	$z\,(X(z) - x_0)$
8	forward 2 taps	x_{n+2}	$z^2\,(X(z) - x_0 - x_1 z^{-1})$
9	time forward	x_{n+m}	$z^m \left(X(z) - \sum_{i=0}^{m-1} x_i z^{-i}\right)$
10	complex translation	$e^{an} x_n$	$X(z e^{-a})$
11	frequency scale	$b^n x_n$	$X\left(\frac{z}{b}\right)$
12	differentiation	$n\, x_n$	$-z\, X'(z)$
13	integration	$\frac{1}{n} x_n$	$-\int \frac{X(z)}{z} dz$
14	integration shift	$\frac{1}{n+m} x_n$	$-z^{-m} \int \frac{X(z)}{z^{m+1}} dz$
15	discrete time convolution	$x_n * y_n = \sum_{i=0}^{n} x_i\, y_{n-i}$	$X(z)\, Y(z)$
16	convolution with $y_n = 1$	$\sum_{i=0}^{n} x_i$	$\frac{z}{z-1} X(z)$
17	initial time	x_0	$\lim_{z\to\infty} X(z)$
18	final value	$\lim_{n\to\infty} x_n$	$\lim_{z\to 1} (z-1) X(z)$

Table 9.2. Some properties of the z-transform.

Exploration

Example 9.7. Given $w_n = x[n] = n + 1$. Use convolution to show that the z-transform is $W(z) = \frac{z^2}{(z-1)^2}$.

Solution 9.7.

Let both $x_n = 1$ and $y_n = 1$ be the unit step sequence, and both $X(z) = \frac{z}{(z-1)}$ and $Y(z) = \frac{z}{(z-1)}$. Then

$$W(z) = X(z)\, Y(z) = \frac{z}{(z-1)} \frac{z}{(z-1)} = \frac{z^2}{(z-1)^2},$$

so that w_n is given by the convolution

$$w_n = x_n * y_n = \sum_{i=0}^{n} x_i\, y_{n-i} = \sum_{i=0}^{n} 1 = n + 1.$$

9.1.5 Application to signal processing

Digital signal processing often involves the design of finite impulse response (FIR) filters. A

simple 3-point FIR filter can be described as

(9-4) $$y[n] = x[n] + a\,x[n-1] + b\,x[n-2].$$

Here, we choose real coefficients a and b so that the homogeneous difference equation

(9-5) $$x[n] + a\,x[n-1] + b\,x[n-2] = 0$$

has solutions $x[n] = \cos(\omega\pi n)$ and $x[n] = \sin(\omega\pi n)$. That is, if the linear combination $x[n] = c_1 \cos(\omega\pi n) + c_2 \sin(\omega\pi n)$ is input on the right side of the FIR filter equation, the output $y[n]$ on the left side of the equation will be zero.

Applying the time delay property to the z-transforms of each term in (9-4), we obtain $X(z) + a\,X(z)\,z^{-1} + b\,X(z)\,z^{-2} = Y(z)$. Factoring, we get

(9-6) $$X(z)\,(1 + a\,z^{-1} + b\,z^{-2}) = X(z)\,H(z), \text{ where}$$

(9-7) $$H(z) = 1 + a\,z^{-1} + b\,z^{-2}$$

represents the filter transfer function. Now, in order for the filter to suppress the inputs $\cos(\omega\pi n)$ and $\sin(\omega\pi n)$, we must have

$$1 + a\,z^{-1} + b\,z^{-2} = (1 - e^{i\omega\pi}\,z^{-1})\,(1 - e^{-i\omega\pi}\,z^{-1}) = z^{-2}\,(z - e^{i\omega\pi})\,(z - e^{-i\omega\pi})$$

and an easy calculation reveals that

$$a = -(e^{i\omega\pi} + e^{-i\omega\pi}) = -2\,\text{Cos}[\omega\pi], \text{ and}$$

$$b = e^{i\omega\pi}\,e^{-i\omega\pi} = 1.$$

A complete discussion of this process is given in Section 9.3 of this chapter.

Example 9.8. (FIR filter design) Use residues to find the inverse z-transform $x[n]$ of $X(z) = \frac{1}{1-\sqrt{2}\,z^{-1}+z^{-2}} = \frac{1}{(1-e^{i\pi/4}\,z^{-1})\,(1-e^{-i\pi/4}\,z^{-1})}$.
Then, write down the FIR filter equation that suppresses $x[n]$.

Solution 9.8.

Explore Solution 9.8.

9.1.6 First Order Difference Equations

The solution of difference equations is analogous to the solution of differential equations. Consider the first order homogeneous equation

$$y[n+1] - a\,y[n] = 0$$

where a is a constant. The following method is often used.

Trial solution method.

Use the trial solution $y[n] = c_1 r^n$, and substitute it into the above equation and get $c_1 r^{n+1} - a c_1 r^n = 0$. Then divide through by r^n

and simplify to obtain $r = a$. The general solution to the difference equation is

$$y[n] = c_1 a^n.$$

Familiar models of difference equations are given in the table below.

Model	Difference Equation	Solution
Exponential growth or decay	$y[n+1] = (1+r)\, y[n]$	$y[n] = y_0 (1+r)^n$
Newton's law of cooling	$y[n+1] = a L + (1-a)\, y[n]$	$y[n] = y_0 (1-a)^n + L\,(1-(1-a)^n)$
Repeated dosage drug level	$y[n+1] = a\, y[n] + b$	$y[n] = y_0 a^n + \frac{a^n - 1}{a-1} b$
Value of an annuity due	$y[n+1] = (1+r)\,(y[n] + P)$ where $y[0] = 0$	$y[n] = \frac{(1+r)^{1+n} - 1}{r} P - P$

Table 3. Some examples of first order linear difference equations.

Exploration

9.1.7 Methods for Solving First Order Difference Equations

Consider the first order linear constant coefficient difference equation (LCCDE)

$$y[n+1] - a\, y[n] = x[n] \text{ with the initial condition } y[0] = y_0.$$

Trial solution method.

First, solve the homogeneous equation $y_h[n+1] - a\, y_h[n] = 0$ and get $y_h[n] = c_1 a^n$. Then use a trial solution that is appropriate for the sequence $x[n]$ on the right side of the equation and solve to obtain a particular solution $y_p[n]$. Then the general solution is

$$y[n] = y_h[n] + y_p[n].$$

The shortcoming of this method is that an extensive list of appropriate trial solutions must be available. Details can be found in difference equations textbooks. We will emphasize techniques that use the z-transform.

z-transform method.

(i) Use the time forward property $\mathcal{Z}[y[n+1]] = z\,(Y(z) - y_0)$ and take the z-transform of each term and get

$$z\,(Y(z) - y_0) \;-\; a\,Y(z) \;=\; X(z)$$

(ii) Solve the equation in (i) for $Y(z)$.

(iii) Use partial fractions to expand $Y(z)$ in a sum of terms, and look up the inverse z-transform(s) using Table 1, to get

$$y[n] \;=\; \mathcal{Z}^{-1}[Y(z)]$$

Residue method.

Perform steps (i) and (ii) of the above z-transform method. Then find the solution using the formula

(iii) $$y[n] \;=\; \mathcal{Z}^{-1}[Y(z)] \;=\; \sum_{i=1}^{k} \text{Res}[Y(z)\,z^{n-1}, z_i].$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z) = Y(z)\,z^{n-1}$.

Convolution method.

(i) Solve the homogeneous equation $y_h[n+1] - a\,y_h[n] = 0$ and get $y_h[n] = c_1\,a^n$.

(ii) Use the transfer function $H(z) = \frac{1}{1 - a\,z^{-1}}$

and construct the unit-sample response $h[n] = \mathcal{Z}^{-1}[H(z)] = a^n$.

(iii) Construct the particular solution $y_p[n] = \mathcal{Z}^{-1}[X(z)\,H(z)]$,

in convolution form $y_p[n] = \sum_{i=0}^{n} x[n-i]\,h[i] = \sum_{i=0}^{n} x[n-i]\,a^i$.

(iv) The general solution to the nonhomogeneous difference equation is

$$y[n] \;=\; y_h[n] + y_p[n] \;=\; c_1\,a^n \;+\; \sum_{i=0}^{n} x[n-i]\,a^i.$$

(v) The constant $c_1 = y_0 - x[0]$ will produce the proper initial condition $y[0] = y_0$. Therefore,

$$y[n] \;=\; (y_0 - x[0])\,a^n \;+\; \sum_{i=0}^{n} x[n-i]\,a^i.$$

Remark 9.6. The particular solution $y_p[n]$ obtained by using convolution has the initial condition

$$y_p[0] \;=\; \sum_{i=0}^{0} x[0-i]\,h[i] \;=\; x[0]\,h[0]$$

$$\;=\; x[0]$$

Example 9.9. Solve the difference equation $y[n+1] - 2\,y[n] = 3^n$ with initial condition $y[0] = 2$.

9 (a). Use the z-transform and Tables 9.1 - 9.2 to find the solution.

9 (b). Use residues to find the solution.

Solution 9.9

Explore Solution 9.9.

Example 9.10. Solve the difference equation $y[n+1] - 2\,y[n] = n$ with initial condition $y[0] = 1$.

9.10 (a). Use the z-transform and Tables 9.1 - 9.2 to find the solution.

9.10 (b). Use residues to find the solution.

Solution 9.10.

Explore Solution 9.10.

Example 9.11. Given the repeated dosage drug level model $y[n+1] = a\,y[n] + b$ with the initial condition $y[0] = y_0$.

9.11 (a). Use the trial solution method.

9.11 (b). Use z-transforms to find the solution.

9.11 (c). Use residues to find the solution.

9.11 (d). Use convolution to find the solution.

Solution 9.11.

Solution 9.11 (a).

The first step is to solve the homogeneous equation $y_h[n+1] - a\,y_h[n] = 0$. The trial solution is $y_h[n] = r^n$, and substitution produces $r^{n+1} - a\,r^n = 0$, from which we obtain the characteristic equation $r - a = 0$. The root is $r = a$ and the homogeneous solution is

$$y_h[n] = c_1\,a^n.$$

The second step is to find a particular solution to the nonhomogeneous equation

$$y_p[n+1] - a\,y_p[n] = b.$$

Since the right hand side is a constant, we try $y_p[n] = c$, and substitution produces $c - a\,c = b$, hence $c = \frac{b}{1-a}$. The general solution is

$$y[n] = y_h[n] + y_p[n]$$

$$y[n] = c_1\,a^n + \frac{b}{1-a}$$

Then $y[0] = c_1\,a^0 + \frac{b}{1-a} = y_0$ can be solved for $c_1 = \frac{b}{a-1} + y_0$. Substituting this in the previous expression for $y[n]$ yields

$$y[n] = \left(\frac{b}{a-1} + y_0\right) a^n - \frac{b}{a-1}$$

$$y[n] = y_0\,a^n + \frac{1}{a-1}\,a^n\,b - \frac{1}{a-1}\,b$$

$$y[n] = y_0\,a^n + \frac{1}{a-1}\,(a^n - 1)\,b$$

Solution 9.11 (b).

Start with the difference equation $y[n+1] = a\,y[n] + b$. Take the z-transform of both sides

$$z\,(Y(z) - y_0) = a\,Y(z) + b\,\frac{z}{z-1}.$$

Solve for $Y(z)$ and get $Y(z) = \frac{b\,z - z\,y_0 + z^2\,y_0}{(z-1)\,(z-a)}$. Then expand and obtain

$$Y(z) = y_0\left(\frac{z}{z-a}\right) + \frac{a\,b}{a-1}\left(\frac{1}{z-a}\right) - \frac{b}{a-1}\left(\frac{1}{z-1}\right)$$

Find the inverse z-transform of each term

$$y[n] = y_0\,\mathcal{Z}^{-1}\left[\frac{z}{z-a}\right] + \frac{a\,b}{a-1}\,\mathcal{Z}^{-1}\left[\frac{1}{z-a}\right] - \frac{b}{a-1}\,\mathcal{Z}^{-1}\left[\frac{1}{z-1}\right]$$

$$y[n] = y_0\,(a^n) + \frac{a\,b}{a-1}\,(a^{n-1}\,u[n-1]) - \frac{b}{a-1}\,(u[n-1])$$

When $n = 0$ we get $y[0] = y_0\,(a^0) + 0 - 0 = y_0$, and when $n \geq 1$ the expression for $y[n]$ simplifies to be

$$y[n] = y_0\,a^n + \frac{a^n - 1}{a-1}\,b$$

Solution 11 (c).

Start with the formula for the z-transform that we found in part (b): $Y(z) = \frac{b\,z - z\,y_0 + z^2\,y_0}{(z-1)\,(z-a)}$.

Then use the Corollary and residues to find $\mathcal{Z}^{-1}[Y(z)]$.

$$y[n] = \mathcal{Z}^{-1}[Y(z)] = \text{Res}[Y(z)\,z^{n-1}, 1] + \text{Res}[Y(z)\,z^{n-1}, a]$$

$$= \text{Res}\left[\frac{b\,z - z\,y_0 + z^2\,y_0}{(z-1)\,(z-a)}\,z^{n-1}, 1\right] + \text{Res}\left[\frac{b\,z - z\,y_0 + z^2\,y_0}{(z-1)\,(z-a)}\,z^{n-1}, a\right]$$

$$= \lim_{z\to 1}\,(z-1)\,X(z)\,z^{n-1} + \lim_{z\to a}\,(z-a)\,X(z)\,z^{n-1}$$

$$= \lim_{z\to 1}\,(z-1)\,\frac{b\,z - z\,y_0 + z^2\,y_0}{(z-1)\,(z-a)}\,z^{n-1} + \lim_{z\to a}\,\frac{b\,z - z\,y_0 + z^2\,y_0}{(z-1)\,(z-a)}\,X(z)\,z^{n-1}$$

$$= \lim_{z\to 1}\,\frac{b\,z - z\,y_0 + z^2\,y_0}{z-a}\,z^{n-1} + \lim_{z\to a}\,\frac{b\,z - z\,y_0 + z^2\,y_0}{z-1}\,z^{n-1}$$

$$= \frac{b - y_0 + y_0}{1-a}\,1^{n-1} + \frac{b\,a - a\,y_0 + a^2\,y_0}{a-1}\,a^{n-1}$$

$$= \frac{b}{1-a} + \frac{a^{-1+n}\,(a\,b - a\,y_0 + a^2\,y_0)}{-1+a}$$

$$= y_0\,a^n + \frac{a^n - 1}{a-1}\,b$$

Solution 9.11 (d).

The solution to the homogeneous equation $y_h[n+1] - a\,y_h[n] = 0$ is $y_h[n] = c_1\,a^n$.

The transfer function is $H(z) = \frac{1}{1 - a\,z^{-1}}$ and the unit-sample response is

$h[n] = \mathcal{Z}^{-1}[H(z)] = a^n$.

The input sequence is $x[n] = b$ and it's z-transform is $X(z) = \frac{b\,z}{z-1}$.

The particular solution is calculated with the formula $y_p[n] = \mathcal{Z}^{-1}[X(z)\,H(z)]$ as follows:

$$y_p[n] = \mathcal{Z}^{-1}\left[\left(\frac{b\,z}{z-1}\right)\left(\frac{1}{1-a\,z^{-1}}\right)\right]$$

$$= \mathcal{Z}^{-1}\left[b - \frac{b}{(-1+a)(-1+z)} + \frac{a^2\,b}{(-1+a)(-a+z)}\right]$$

$$= b\,\delta[n] - \frac{b}{a-1}\,u[n-1] + \frac{a^{1+n}\,b}{a-1}\,u[n-1]$$

which can be simplified to obtain

$$y_p[n] = \frac{\left(a^{1+n}-1\right)b}{a-1}$$

In convolution form $y_p[n] = x[n] * h[n] = \sum_{i=0}^{n} x[n-i]\,h[i]$ and we have

$$y_p[n] = \sum_{i=0}^{n} b\,a^i = \frac{\left(a^{1+n}-1\right)b}{a-1}$$

The particular solution $y_p[n]$ obtained by using convolution has the initial condition $y_p[0] = \sum_{i=0}^{0} x[0-i]\,h[i] = x[0]\,h[0] = x[0] = b$.

The total solution to the nonhomogeneous difference equation is

$$y[n] = y_h[n] + y_p[n] = c_1\,a^n + \sum_{i=0}^{n} b\,a^i$$

$$= c_1\,a^n + \frac{\left(a^{1+n}-1\right)b}{a-1}$$

Now we compute $y_0 = y[0] = c_1\,a^0 + \frac{(a-1)\,b}{a-1} = c_1 + b$, and solve for the constant $c_1 = y_0 - b$ which will produce the proper initial condition. Therefore,

$$y[n] = (y_0 - b)\,a^n + \frac{\left(a^{1+n}-1\right)b}{a-1},$$

which can be manipulated to yield $y[n] = y_0\,a^n + \frac{a^n-1}{a-1}\,b$.

An illustration of the dosage model using the parameters $a = \frac{4}{5}$, $b = 1$ and initial condition $y[0] = y_0 = 0$ is shown in Figure 1 below.

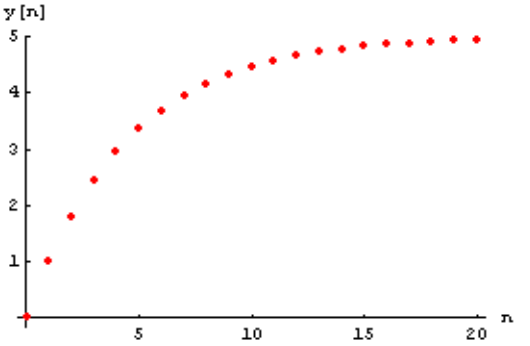

Figure 9.1. The solution to $y[n+1] = \frac{5}{4}\,y[n] + 1$ with $y[0] = y_0 = 0$.

Explore Solution 9.11 (a).

Explore Solution 9.11 (b).

Explore Solution 9.11 (c).

Explore Solution 9.11 (d).

Complex Analysis for Mathematics and Engineering
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9.2 Second Order Homogeneous Difference Equations

Before proceeding with the z-transform method, we mention a heuristic method based on substitution of a trial solution. Consider the second order homogeneous linear constant-coefficient difference equation (HLCCDE)

(9-8) $$y[n+2] - 2a\,y[n+1] + b\,y[n] = 0,$$

where a and b are constants. Using the trial solution $y[n] = r^n$, and direct substitution into (9-8) produces the equation $r^{n+2} - 2a\,r^{n+1} + b\,r^n = 0$, and dividing through by r^n produces the characteristic polynomial $r^2 - 2a\,r + b$ and characteristic equation

(9-9) $$r^2 - 2a\,r + b = 0.$$

There are three types of solutions to (9-8) which are determined by the nature of the roots in (9-9).

Case (i) If $b < a^2$ then we have real distinct roots $r_1 = a - \sqrt{a^2 - b}$ and $r_2 = a + \sqrt{a^2 - b}$, and

(9-10) $$y[n] = c_1 r_1^n + c_2 r_2^n.$$

Case (ii) If $b = a^2$ then we have a double real root $r = r_1 = r_2 = a$, and

(9-11) $$y[n] = c_1 r^n + c_2\, n\, r^n.$$

Case (iii) If $b > a^2$ then we have complex roots $r_1 = a - i\sqrt{b - a^2}$ and $r_2 = a + i\sqrt{b - a^2}$, and

(9-12) $$y[n] = k_1 r_1^n + k_2 r_2^n.$$

The solution for case (iii) can also be written as the following linear combination

(9-13) $$y[n] = c_1 r^n \cos(\phi\, n) + c_2 r^n \sin(\phi\, n),$$

where $r = \sqrt{b}$ and $\phi = \arctan\left[\frac{\sqrt{b-a^2}}{a}\right]$.

Caution. Be sure to use the value of arctan that lies in the range $0 < \phi \leq \pi$.

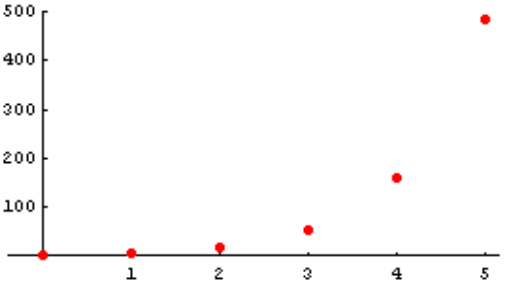
Remark About Stability

Stability depends on the location of the roots of the characteristic polynomial. Without loss if generality, if $|r_1| \leq |r_2| < 1$ then both roots lie inside the unit circle and the solutions are asymptotically stable and tends to zero as $n \to \infty$. If $r_1 \neq r_2$ and $|r_1| \leq |r_2| = 1$ then a root lies on the unit circle and the solutions are stable. If $r_1 = r_2 = \pm 1$ then there is an unstable solution. If $1 < |r_2|$ then at least one root lies outside the unit circle and there is an unstable solution.

The following subroutine uses the characteristic equation to construct solutions to a second order homogeneous difference equation.

Mathematica Subroutine (Solution of a Difference Equation).

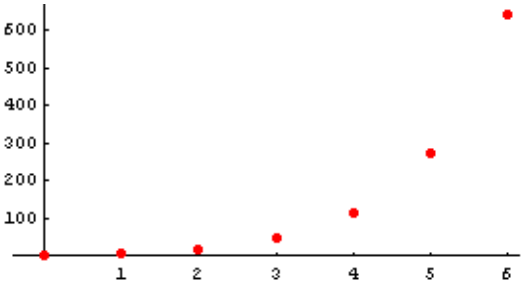
Example 9.12. Solve $y[n+2] - 4\,y[n+1] + 3\,y[n] = 0$ with $y[0] = y_0 = 1$ and $y[1] = y_1 = 5$.



Solution 9.12.

Explore Solution 9.12.

Example 9.13. Solve $y[n+2] - 4\,y[n+1] + 4\,y[n] = 0$ with $y[0] = y_0 = 1$ and $y[1] = y_1 = 5$.



Solution 9.13.

The characteristic equation $r^2 - 4\,r + 4 = (r-2)^2 = 0$ has equal roots $r = r_1 = r_2 = 2$ hence the general solution is $y[n] = c_1\,2^n + c_2\,2^n\,n$. Use the initial conditions and form the linear system

$$y[0] = c_1 + c_2\,0 = 1$$

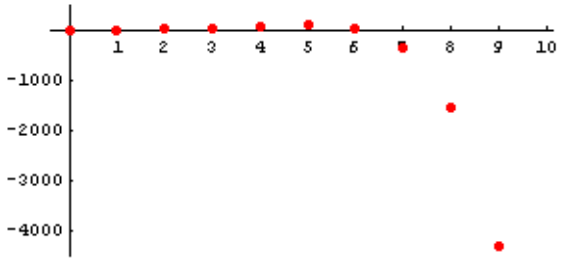
$$y[1] = c_1\,2 + c_2\,2 = 5$$

then solve for the constants and get $c_1 = 1, c_2 = \frac{3}{2}$. Hence the solution is

$$y[n] = 2^n + \frac{3}{2} n\, 2^n = 2^n + 3\, n\, 2^{n-1}.$$

Explore Solution 9.13.

Example 9.14. Solve $y[n+2] - 4\, y[n+1] + 5\, y[n] = 0$ with $y[0] = y_0 = 1$ and $y[1] = y_1 = 5$.



Solution 9.14.

Explore Solution 9.14.

Higher Order Difference Equations

The general form of a P^{th} order linear constant coefficient difference equation (LCCDE), is

(9-14)
$$y[n] + a_1\, y[n-1] + a_2\, y[n-2] + \ldots + a_{N-1}\, y[n-P+1] + a_P\, y[n-P]$$
$$= b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2] + \ldots + b_{M-1}\, x[n-Q+1] + b_Q\, x[n-Q]$$

where $\{a_p\}_{p=1}^{P}$ and $\{b_q\}_{q=0}^{Q}$. The sequence $\{x_n = x[n]\}_{n=0}^{\infty}$ is given and the sequence $\{x_n = x[n]\}_{n=0}^{\infty}$ is output. The integer P is the order of the difference equation. The compact form of writing this difference equation is

(9-15)
$$y[n] + \sum_{p=1}^{P} a_p\, y[n-p] = \sum_{q=0}^{Q} b_q\, x[n-q].$$

This formula can be expressed in a recursive form:

(9-16)
$$y[n] = \sum_{q=0}^{Q} b_q\, x[n-q] - \sum_{p=1}^{P} a_1\, y[n-p].$$

This form of the LCCDE explicitly shows that the present output $y[n]$ is a function of the past output values $y[n-p]$, for $p = 1, 2, \ldots, P$; and the present input $x[n]$, and the previous inputs $x[n-q]$ for $q = 1, 2, \ldots, Q$.

Now we would like to emphasize the method of z-transforms for solving difference equations. Applying the linearity and time delay shift property of the z-transform to equation (9-15), we obtain

(9-17)
$$Y(z) + \sum_{p=1}^{P} a_p\, Y(z)\, z^{-p} = \sum_{q=0}^{Q} b_q\, X(z)\, z^{-q}.$$

This can be rearranged as $Y(z)\left(1 + \sum_{p=1}^{P} a_p\, z^{-p}\right) = X(z) \sum_{q=0}^{Q} b_q\, z^{-q}$ and then solved for the quotient $H(z) = \frac{Y(z)}{X(z)}$. The sequence $h[n] = \mathcal{Z}^{-1}[H(z)]$ can be used to construct a particular

using the convolution sum, as follows:

(9-18) $$y_p[n] = \sum_{i=0}^{n} h[n-i]\, x[i].$$

Remark. This particular solution $y_p[n]$ does not involve initial conditions for (9-14). We will illustrate how to use convolution at the end of this section.

Difference Equations with Initial Conditions

Often times a difference equation involves only one input on the right hand side of (9-14) and we write

$$y[n] + a_1\, y[n-1] + a_2\, y[n-2] + \ldots + a_{N-1}\, y[n-P+1] + a_P\, y[n-P] = x[n]$$

then we could shift the index and use the form

$$y[n+P] + a_1\, y[n+P-1] + a_2\, y[n+P-2] + \ldots + a_{N-1}\, y[P+1] + a_P\, y[n] = x[n+P].$$

Consider the first order linear constant coefficient difference equation (LCCDE)

(9-19) $$y[n+2] - 2\,a\,y[n+1] + b\,y[n] = x[n+2]$$

with the initial conditions $y[0] = y_0$ and $y[1] = y_1$ (and implicitly we have $x[0] = x_0$ and $x[1] = x_1$).

Step (i) Using the time forward properties

$$\mathcal{Z}[y[n+1]] = z\,(Y(z) - y_0),$$
$$\mathcal{Z}[y[n+2]] = z^2\,(Y(z) - y_0 - y_1\, z^{-1}),$$
$$\mathcal{Z}[x[n+2]] = z^2\,(X(z) - x_0 - x_1\, z^{-1})$$

and take the $\mathcal{Z}$-transform of each term and get the equation

(9-20)
$$z^2\,(Y(z) - y_0 - y_1\, z^{-1}) - 2\,a\,z\,(Y(z) - y_0) + b\,Y(z) = z^2\,(X(z) - x_0 - x_1\, z^{-1}).$$

Step (ii) Solve equation (9-20) for $Y(z)$.

Step (iii) Use partial fractions to expand $Y(z)$ in a sum of terms, and look up the inverse $\mathcal{Z}$-transform(s) using Table 1, to get the solution

$$y[n] = \mathcal{Z}^{-1}[Y(z)].$$

Step (iv) Alternate calculation using residues. Perform steps (i) and (ii) listed above. Then find $y[n]$ using residues

$$y[n] = \mathcal{Z}^{-1}[Y(z)] = \sum_{i=1}^{k} \text{Res}[Y(z)\, z^{n-1}, z_i].$$

where $z_1, z_2, \ldots, z_k$ are the poles of $f(z) = Y(z)\, z^{n-1}$.

Remark The function $f(z) = Y(z)\, z^{n-1}$ has real coefficients. Hence, if , and if z_i and $\overline{z_i}$ are poles, then we can use the computational fact:

(9-21) $$\text{Res}[f(z), \overline{z_i}] = \overline{\text{Res}[f(z), z_i]}$$

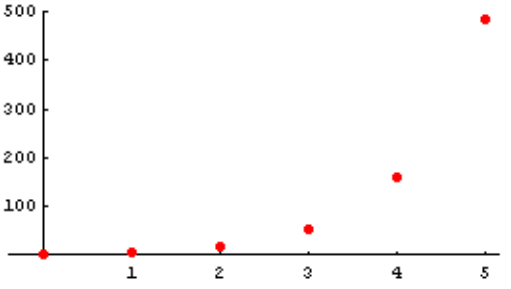
We now show how to obtain answers to Examples 9.12-9.14 using z-transform methods.

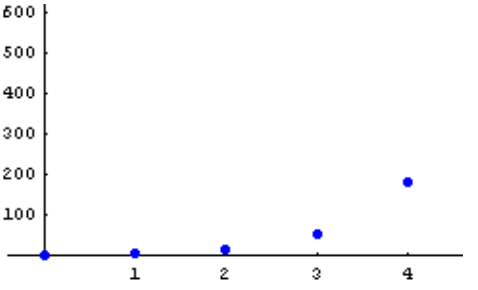
The following subroutine uses z-transforms to construct solutions to a second order homogeneous difference equation.

Mathematica Subroutines (Solution of a Difference Equation).

Example 9.15 (a). Use z-transform methods to solve $y[n+2] - 4\,y[n+1] + 3\,y[n] = 0$ with $y[0] = y_0 = 1$ and $y[1] = y_1 = 5$.

9.15 (b). Use z-transform methods to solve $y[n+2] - 4\,y[n+1] + 3\,y[n] = 2^{n+2}$ with $y[0] = y_0 = 1$ and $y[1] = y_1 = 3$.





Solution 9.15 (a).

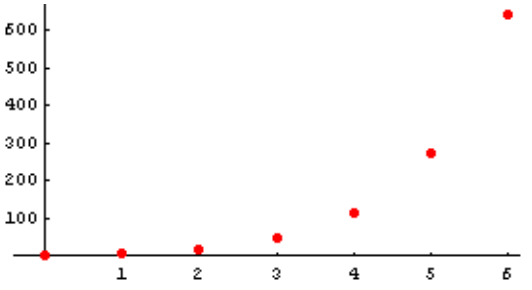
Solution 9.15 (b).

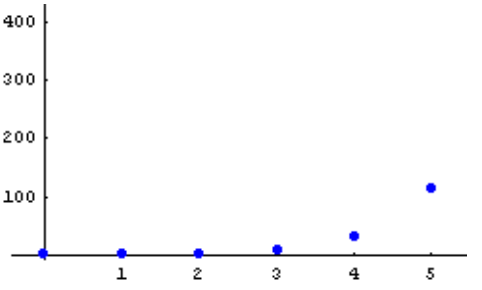
Explore Solution 9.15 (a).

Explore Solution 9.15 (b).

Example 9.16 (a). Use z-transform methods to solve $y[n+2] - 4\,y[n+1] + 4\,y[n] = 0$ with $y[0] = y_0 = 1$ and $y[1] = y_1 = 5$.

9.16 (b). Use z-transform methods to solve $y[n+2] - 4\,y[n+1] + 4\,y[n] = 3^n$ with $y[0] = y_0 = 2$ and $y[1] = y_1 = 3$.





Solution 9.16 (a).

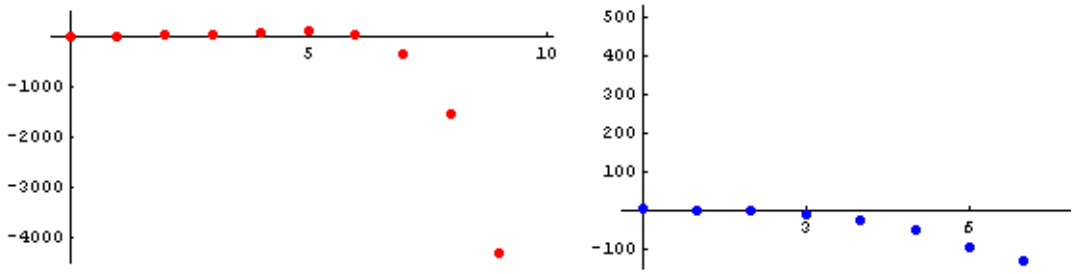
Solution 9.16 (b).

Explore Solution 9.16 (a).

Explore Solution 9.16 (b).

Example 9.17 (a). Use z-transform methods to solve $y[n+2] - 4\,y[n+1] + 5\,y[n] = 0$ with $y[0] = y_0 = 1$ and $y[1] = y_1 = 5$.

with $y[0] = y_0 = 1$ and $y[1] = y_1 = 0$.



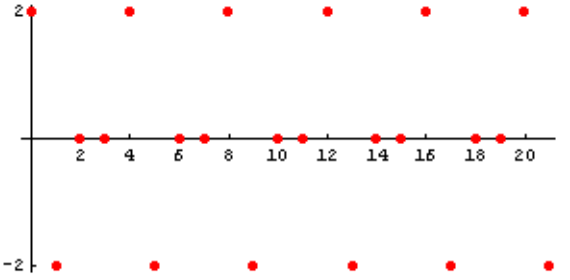
Solution 9.17 (a).

Solution 9.17 (b).

Explore Solution 9.17 (a).

Explore Solution 9.17 (b).

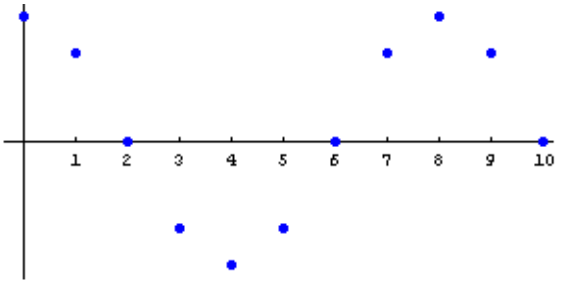
Example 9.18. Solve $y[n+3] + y[n+2] + y[n+1] + y[n] = 0$ with $y[0] = 2$, $y[1] = -2$, and $y[2] = 0$.



Solution 9.18.

Explore Solution 9.18.

Example 9.19. Solve $y[n+2] - \sqrt{2}\, y[n+1] + y[n] = 0$ with $y[0] = 2$ and $y[1] = \sqrt{2}$.



Solution 9.19.

Explore Solution 9.19.

Convolution for Solving a Non-homogeneous Equation

(i) Solve the homogeneous equation $y_h[n+2] - 2\,a\,y_h[n+1] + b\,y_h[n] = 0$ and get $y_h[n]$.

(ii) Use the transfer function $H(z) = \frac{1}{1 - 2\,a\,z^{-1} + b\,z^{-2}}$.

(iii) Construct the particular solution using convolution

$$y_p[n] = \mathcal{Z}^{-1}[H(z)\,X(z)], \text{ or}$$

$$y_p[n] = \sum_{i=0}^{n} h[i]\,x[n-i].$$

(iv) The total solution to the nonhomogeneous difference equation is

$$y[n] = y_h[n] + y_p[n].$$

Example 9.20 (a). Find the general solution to $y[n+2] - \frac{2}{3}\sqrt{2}\,y[n+1] + \frac{4}{9}y[n] = 0$.

9.20 (b). Find the general solution to $y[n+2] - \frac{2}{3}\sqrt{2}\,y[n+1] + \frac{4}{9}y[n] = \cos\left(\frac{\pi}{4}n\right)$.

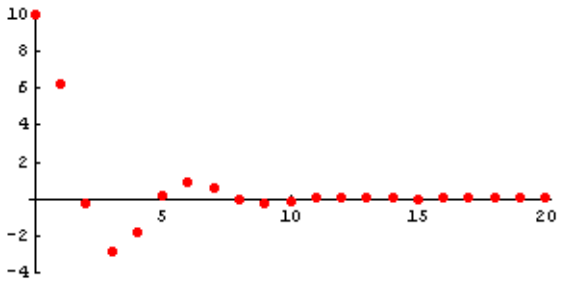


Figure 9.2. A typical solution to $y[n+2] - \frac{2}{3}\sqrt{2}\,y[n+1] + \frac{4}{9}y[n] = 0$.

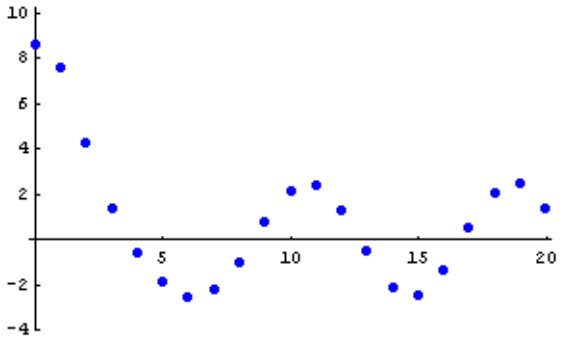


Figure 9.3. A typical solution to $y[n+2] - \frac{2}{3}\sqrt{2}\,y[n+1] + \frac{4}{9}y[n] = \cos\left(\frac{\pi}{4}n\right)$.

Solution 9.20 (a).

Solution 9.20 (b).

Explore Solution 9.20 (a).

Explore Solution 9.20 (b).

Complex Analysis for Mathematics and Engineering
by
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9.3.1 Introduction to Filtering

In the field of signal processing the design of digital signal filters involves the process of suppressing certain frequencies and boosting others. A simplified filter model is

(9-22)
$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3],$$

where the input signal is $x_n = x[n]$ is modified to obtain the output signal $y_n = y[n]$ using the recursion formula

(9-23)
$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] - a_1 y[n-1] - a_2 y[n-2].$$

The implementation of (9-23) is straightforward and only requires starting values, then $y_n = y[n]$ is obtained by simple iteration. Since the signals must have a starting point, it is common to require that $x_n = 0$ and $y_n = 0$ for $n < 0$. We emphasize this concept by making the following definition.

Definition 9.3 (Causal Sequence) Given the the input $\{x_n\}_{n=-\infty}^{\infty}$ and output $\{y_n\}_{n=-\infty}^{\infty}$ sequences. If $x_n = 0$ and $y_n = 0$ for $n < 0$, the sequence is said to be causal.

Given the causal sequence $\{x_n = x[n]\}_{n=0}^{\infty}$, it is easy to calculate the solution $\{y_n = y[n]\}_{n=0}^{\infty}$ to (9-23). Use the fact that these sequences are causal:

(9-24) $$x_{-3} = 0,\ x_{-2} = 0,\ x_{-1} = 0 \quad \text{and} \quad y_{-2} = 0,\ y_{-1} = 0.$$

Then compute

(9-25)
$$y_0 = b_0 x_0,$$
$$y_1 = b_0 x_1 + b_1 x_0 - a_1 y_0,$$
$$y_2 = b_0 x_2 + b_1 x_1 + b_2 x_0 - a_1 y_1 - a_2 y_0,$$
$$y_3 = b_0 x_3 + b_1 x_2 + b_2 x_1 + b_3 x_0 - a_1 y_2 - a_2 y_1.$$

The general iterative step is

(9-26)
$$y_n = b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2} + b_3 x_{n-3} - a_1 y_{n-1} - a_2 y_{n-2}.$$

9.3.2 The Basic Filters

The following three simplified basic filters serve as illustrations.

(i) Zeroing Out Filter $y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3]$, (note that $a_1 = 0$ and $a_2 = 0$).

(ii) Boosting Up Filter $y[n] = b_0 x[n] - a_1 y[n-1] - a_2 y[n-2]$, (note that $b_1 = 0$, $b_2 = 0$ and $b_3 = 0$).

(iii) Combination Filter

$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] - a_1 y[n-1] - a_2 y[n-2]$.

The transfer function $H(z)$ for these model filters has the following general form

(9-27)
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2}},$$

where the z-transforms of the input and output sequences are $X(z) = \mathcal{Z}[x_n]$ and $Y(z) = \mathcal{Z}[y_n]$, respectively. In the previous section we mentioned that the general solution to a homogeneous difference equation $y[n] + a_1 y[n-1] + a_2 y[n-2] = 0$ is stable only if the zeros of the characteristic equation lie inside the unit circle. Similarly, if a filter is stable then the poles of the transfer function $H(z)$ must all lie inside the unit circle.

Before developing the general theory, we would like to investigate the amplitude response $A(\theta)$ when the input signal is a linear combination of $\cos(\theta n)$ and $\sin(\theta n)$. The amplitude response for the frequency θ uses the complex unit signal $z = e^{i\theta}$, and is defined to be

(9-28)
$$A(\theta) = |H(e^{i\theta})|.$$

The formula for $A(\theta)$ will be rigorously explained after a few introductory examples.

Example 9.21. Given the filter $y[n] = x[n] - \sqrt{2}\, x[n-1] + x[n-2]$.

9.21 (a). Show that it is a zeroing out filter for the signals $\cos\left(\frac{\pi}{4} n\right)$ and $\sin\left(\frac{\pi}{4} n\right)$ and calculate the amplitude response $A\left(\frac{\pi}{4}\right) = A(0.785398)$.

9.21 (b). Calculate the amplitude responses $A(0.10)$ and $A(0.77)$ and investigate the the filtered signal for $x[n] = \cos(0.10\, n) + \sin(0.77\, n)$.

9.21 (c). Calculate the amplitude responses $A(0.10)$ and $A\left(\frac{2\pi}{3}\right)$ and investigate the the filtered signal for $x[n] = \cos(0.10\,n) + 0.20 \sin\left(\frac{2\pi}{3}\,n\right)$.

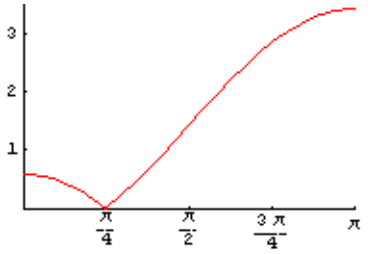


Figure 9.4. The amplitude response $A(\theta) = \left|1 - \sqrt{2}\, z^{-1} + z^{-2}\right|$ for $y[n] = x[n] - \sqrt{2}\, x[n-1] + x[n-2]$.

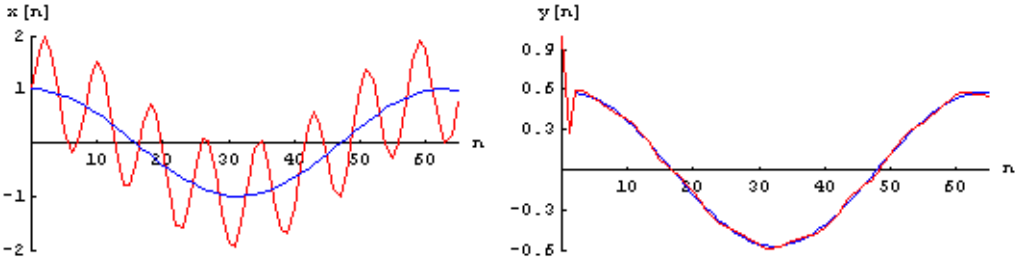


Figure 9.5. The input $x[n] = \cos(0.10\,n) + \sin(0.77\,n)$ and output $y[n]$.

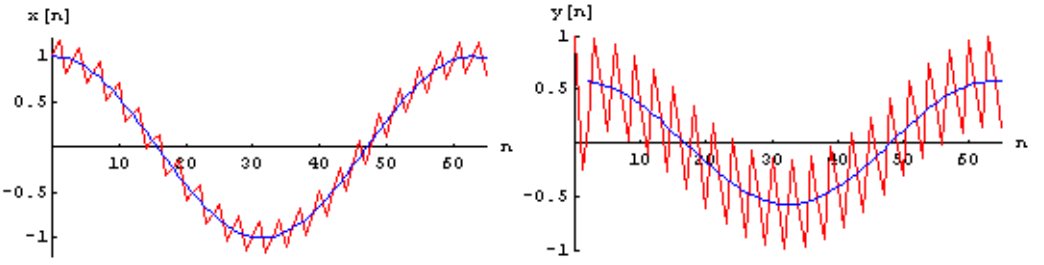


Figure 9.6. The input $x[n] = \cos(0.10\,n) + 0.20 \sin\left(\frac{2\pi}{3}\,n\right)$ and output $y[n]$.

Solution 9.21.

Explore Solution 9.21.

Example 9.22. Given the filter $y[n] = x[n] + \frac{2}{3}\sqrt{2}\, y[n-1] - \frac{4}{9}\, y[n-2]$.

9.22 (a). Show that it is a boosting up filter for the signals $\cos\left(\frac{\pi}{4}\,n\right)$ and $\sin\left(\frac{\pi}{4}\,n\right)$ and calculate the amplitude response $A\left(\frac{\pi}{4}\right)$.

9.22 (b). Calculate the amplitude responses $A\left(\frac{\pi}{8}\right)$ and $A(2.60)$ and investigate the the filtered signal for $x[n] = \cos\left(\frac{\pi}{8}\,n\right) + \sin\left(\frac{\pi}{4}\,n\right) + \sin(2.60\,n)$.

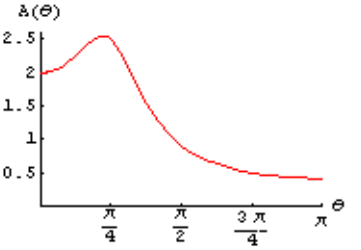


Figure 9.7. The amplitude response $A(\theta) = \left|\frac{1}{\quad}\right|$ for

$$y[n] = x[n] + \frac{2}{3}\sqrt{2}\, y[n-1] - \frac{4}{9} y[n-2].$$

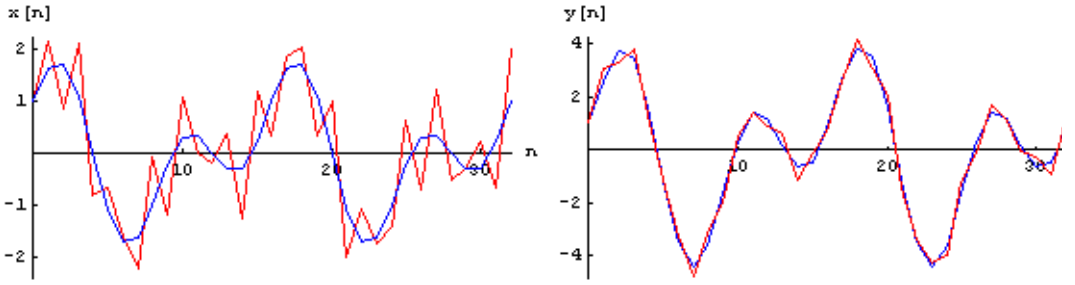


Figure 9.8. The input $x[n] = \cos\left(\frac{\pi}{8} n\right) + \sin\left(\frac{\pi}{4} n\right) + \sin(2.60\, n)$ and output $y[n]$.

Solution 9.22.

Explore Solution 9.22.

9.3.3 The General Filter Equation

The general form of a P^{th} order filter difference equation is

(9-29)

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \ldots + a_{P-1} y[n-P+1] + a_P y[n-P]$$

$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \ldots + b_{Q-1} x[n-Q+1] + b_Q x[n-Q]$$

where $\{a_p\}_{p=1}^{P}$ and $\{b_q\}_{q=0}^{Q}$ are constants. Note carefully that the terms involved are of the form $y[n-p]$ and $y[n-q]$ where $p \geq 0$ and $q \geq 0$, which makes these terms time delayed. The compact form of writing the difference equation is

(9-30)
$$y[n] + \sum_{p=0}^{P} a_p y[n-p] = \sum_{q=0}^{Q} b_q x[n-q],$$

where the input signal $x_n = x[n]$ is modified to obtain the output signal $y_n = y[n]$ using the recursion formula

(9-31)
$$y[n] = \sum_{q=0}^{Q} b_q x[n-q] - \sum_{p=1}^{P} a_p y[n-p].$$

The portion $\sum_{q=0}^{Q} b_q x[n-q]$ will "zero out" signals and $\sum_{p=1}^{P} a_p y[n-p]$ will "boost up" signals.

Remark 9.14. Formula (9-31) is called the recursion equation and the recursion coefficients are $\{a_p\}_{p=1}^{P}$ and $\{b_q\}_{q=0}^{Q}$. It explicitly shows that the present output $y[n]$ is a function of the past values $y[n-p]$, for $p = 1, 2, \ldots, P$, the present input $x[n]$, and the previous inputs $x[n-q]$ for $q = 1, 2, \ldots, Q$. The sequences can be regarded as signals and they are zero for negative indices. With this information we can now define the general formula for the transfer function $H(z)$. Using the time delayed-shift property for causal sequences and taking the z-transform of each term in (9-31), we obtain

(9-32)
$$Y[z] = -\sum_{p=1}^{P} a_p Y[z]\, z^{-p} + \sum_{q=0}^{Q} b_q X[z]\, z^{-q}.$$

We can factor $X[z]$ and $Y[z]$ out of the summations and write this in an equivalent form

(9-33)
$$Y[z]\left(1 + \sum_{p=1}^{P} a_p z^{-p}\right) = X[z] \sum_{q=0}^{Q} b_q z^{-q}.$$

$$H(z) = \frac{Y[z]}{X[z]} = \frac{\sum_{q=0}^{Q} b_q z^{-q}}{1 + \sum_{p=1}^{P} a_p z^{-p}}$$

which leads to the following important definition.

Definition 9.4 (Transfer Function) The transfer function corresponding to the P^{th} order difference equation (8) is given by

(9-34) $$H(z) = \frac{Y[z]}{X[z]} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{Q-1} z^{-Q+1} + b_Q z^{-Q}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{P-1} z^{-P+1} + a_P z^{-P}}.$$

Formula (9-34) is the transfer function for an infinite impulse response filter (IIR filter). In the special case when the denominator is unity it becomes the transfer function for a finite impulse response filter (FIR filter).

Definition 9.5 (Unit-Sample Response) The sequence $h[n] = \mathcal{Z}^{-1}[H(z)]$ corresponding to the transfer function $H(z) = \frac{Y(z)}{X(z)}$ is called the unit-sample response.

Theorem 9.6 (Output Response) The output response $y[n]$ of a the filter (10) given an input signal $x[n]$ is given by the inverse z-transformation

(9-35) $$y[n] = \mathcal{Z}^{-1}[Y(z)] = \mathcal{Z}^{-1}[H(z)\,X(z)],$$

and in convolution form it is given by

(9-36) $$y[n] = \sum_{i=0}^{n} x[n-i]\,h[i].$$

Another important use of the transfer function is to study how a filter affects various frequencies. In practice, a continuous time signal is sampled at a frequency f_S that is at least twice the highest input signal frequency to avoid frequency fold-over, or aliasing. That is because the Fourier transform of a sampled signal is periodic with period $\omega_S = \frac{2\pi}{f_S}$, though we will not prove this here. Aliasing prevents accurate recovery of the original signal from its samples.

Now it can be shown that the argument of the Fourier transform maps onto the z-plane unit circle via the formula

(9-37) $\theta = 2\pi \frac{f}{f_S}$, where θ is called the normalized frequency.

Therefore the z-transform evaluated on the unit circle is also periodic, except with period 2π.

Definition 9.6 (Amplitude Response) The amplitude response $A(\theta)$ is defined to be the magnitude of the transfer function evaluated at the complex unit signal $z = e^{i\theta}$. The formula is

(9-38) $A(\theta) = |H(e^{i\theta})|$ over the interval $[0, \pi]$.

Proof.

Sinusoidal signals are linear combinations of $e^{i\theta}$ and $e^{-i\theta}$. To determine the amplitude response we input $x[n] = e^{in\theta}$ which has z-transform $X(z) = \mathcal{Z}[x[n]] = \frac{z}{z - e^{i\theta}}$
The z-transform of the output is $Y(z) = X(z)\,H(z) = \frac{z}{z - e^{i\theta}} H(z)$ which can be written as

(9-39) $$Y(z) = X(z)\,H(z) = z \frac{1}{z - e^{i\theta}} \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{Q-1} z^{-Q+1} + b_Q z^{-Q}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{P-1} z^{-P+1} + a_P z^{-P}}$$

It is possible to use a technique like partial fraction expansions to write (9-39) in the form

(9-40) $$Y(z) = z\,\frac{F_0}{z - e^{i\theta}} + Q(z) = F_0\,\frac{z}{z - e^{i\theta}} + Q(z),$$

where $F_0 = H(e^{i\theta})$.

For stable solutions we have already mentioned (see Section 9.2.1) that the poles of the transfer function must all lie inside the unit circle. Hence the terms in the solution $y[n]$ corresponding to the poles of $Q(z)$ are all transient and decay to zero as $n \to \infty$. The steady state portion of the solution $y[n]$ is the inverse of the term $F_0 \mathcal{Z}\left[\frac{z}{z - e^{i\theta}}\right]$ and its magnitude is

(9-41) $$A(\theta) = |H(e^{i\theta})|\,|e^{in\theta}| = |H(e^{i\theta})|.$$

This proof is in the book.

The fundamental theorem of algebra implies that the numerator has Q roots (called zeros) and the denominator has P roots (called poles). The zeros $\{z_q\}_{q=1}^{Q}$ may be chosen in conjugate pairs on the unit circle and $|z_q| = 1$ for $q = 1, 2, \ldots, Q$. For stability, all the poles $\{w_p\}_{p=1}^{P}$ must inside the unit circle and $|w_p| < 1$ for $p = 1, 2, \ldots, P$. Furthermore, the poles are chosen to be real numbers and/or in conjugate pairs. This will guarantee that the recursion coefficients are all real numbers. IIR filters may be all pole or zero-pole and stability is a concern; FIR filters and all zero-filters are always stable.

9.3.4 Design of Filters

In practice recursion formula (10) is used to calculate the output signal. However, digital filter design is based on the above theory. One starts by selecting the location of zeros and poles corresponding to filter design requirements and constructing the transfer function $H(z) = \frac{Y(z)}{X(z)}$. Since the coefficients in $H(z)$ are real, all zeros and poles having an imaginary component must occur in conjugate pairs. Then the recursion coefficients are identified in (13) and used in (10) to write the recursive filter. Both the numerator and denominator of $H(z)$ can be factored into quadratic factors with real coefficients and possibly one or two linear factors with real coefficients. The following principles are used to construct $H(z)$.

(i) Zeroing Out Factors

To filter out the signals $\cos(\theta n)$ and $\sin(\theta n)$, use factors of the form

$$\left(\frac{z - e^{i\theta}}{z}\right)\left(\frac{z - e^{-i\theta}}{z}\right) = 1 - (e^{i\theta} + e^{-i\theta})\,z^{-1} + z^{-2} \quad \text{if } 0 < \theta < \pi, \quad \text{and}$$

$$\left(\frac{z - e^{i\pi}}{z}\right) = 1 + z^{-1} \quad \text{if } \theta = \pi,$$

in the numerator of $H(z)$. They will contribute to the term

(9-42) $$b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_{Q-1} z^{-Q+1} + b_Q z^{-Q} = \frac{z^Q b_0 + b_1 z^{Q-1} + b_2 z^{Q-2} + \ldots + b_{Q-1} z + b_Q}{z^Q}.$$

(ii) Boosting Up Factors

To amplify the signals $\cos(\theta n)$ and $\sin(\theta n)$, use factors of the form

$$\left(\frac{z - \rho\, e^{i\phi}}{z}\right)\left(\frac{z - \rho\, e^{-i\phi}}{z}\right) = 1 - \rho\,(e^{i\phi} + e^{-i\phi})\,z^{-1} + \rho^2 z^{-2} \quad \text{if } 0 < \rho < 1$$

$$\left(\frac{z-\rho\, e^{i\pi}}{z}\right) = 1 + \rho\, z^{-1} \quad \text{if } 0 < \rho < 1 \text{ and } \phi = \pi, \quad \text{and}$$

$$\left(\frac{z-\rho}{z}\right) = 1 - \rho\, z^{-1} \quad \text{if } 0 < \rho < 1 \text{ and } \phi = 0,$$

in the denominator of $H(z)$. They will contribute to the term

(9-43) $$1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{P-1} z^{-P+1} + a_P z^{-P} = \frac{z^P + a_1 z^{P-1} + a_2 z^{P-2} + \ldots + a_{P-1} z + a_P}{z^P}.$$

(iii) Attenuating Factors

To attenuate the signals $\cos(\theta n)$ and $\sin(\theta n)$, use factors of the form

$$\left(\frac{z-\rho\, e^{i\theta}}{z}\right)\left(\frac{z-\rho\, e^{-i\theta}}{z}\right) = 1 - \rho\,(e^{i\theta} + e^{-i\theta})\, z^{-1} + \rho^2 z^{-2} \quad \text{if } 0 < \rho < 1$$

and $0 < \theta < \pi$, and

$$\left(\frac{z-\rho\, e^{i\pi}}{z}\right) = 1 + \rho\, z^{-1} \quad \text{if } 0 < \rho < 1 \text{ and } \theta = \pi.$$

The factor

$$\left(\frac{z-\rho}{z}\right) = 1 - \rho\, z^{-1} \quad \text{if } 0 < \rho < 1 \text{ and } \phi = 0,$$

is a special case that attenuates low frequency signals. These factors will contribute to the term (9-42).

(iv) Combination of Factors

The transfer function $H(z)$ could have a zero or pole at the origin, but this has no net effect on the output signal. The other zeros and poles determine the nature of the filter. A conjugate pair of zeros $e^{\pm i\theta}$ of $H(z)$ on the unit circle will "zero-out" the signals $\cos(\theta n)$ and $\sin(\theta n)$. If $0 < \rho \approx 1$ the conjugate pair of zeros $\rho\, e^{\pm i\theta}$ of $H(z)$ will attenuate the signals $\cos(\theta n)$ and $\sin(\theta n)$, and the conjugate pair of poles $\rho\, e^{\pm i\phi}$ of $H(z)$ will amplify the signals $\cos(\theta n)$ and $\sin(\theta n)$. It is useful to plot the location of the zeros and poles and notice their magnitude and argument. As a general rule, zeros are used to attenuate signals and poles are used to amplify signals. The primary goal of filter design is to construct $H(z)$ so that the amplitude response $A(\theta)$ has a desired shape. The following examples have been chosen to illustrate these concepts. Books on digital signal filter design will explain the process in detail.

Example 9.23 (a). The filter $y[n] = x[n] + x[n-2]$ is designed to zero out $\cos\left(\frac{\pi}{2} n\right)$ and $\sin\left(\frac{\pi}{2} n\right)$.

9.23 (b). The moving average filter $y[n] = \frac{1}{4}(x[n] + x[n-1] + x[n-2] + x[n-3])$ is designed to zero out $\cos(n\pi)$, $\cos\left(\frac{\pi}{2} n\right)$ and $\sin\left(\frac{\pi}{2} n\right)$.

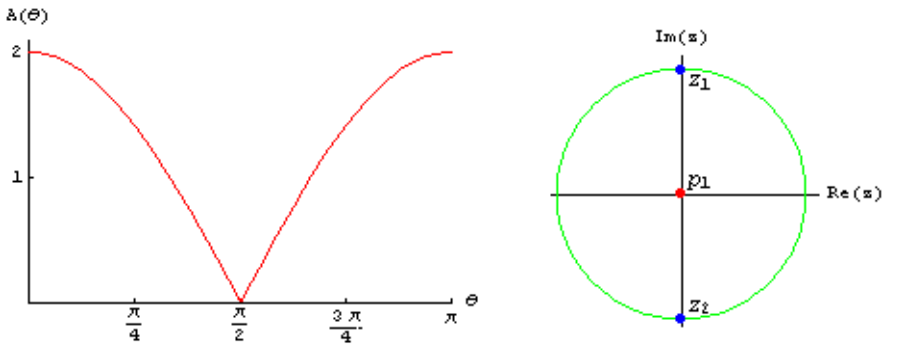


Figure 9.9. Amplitude response $A(\Theta)$, and zero-pole plot for $y[n] = \frac{1}{4}\sum_{k=0}^{3} x[n-k]$.

Solution 9.23.

Explore Solution 9.23.

Example 9.24. The moving average filter

$y[n] = \frac{1}{8}(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] + x[n-5] + x[n-6] + x[n-7])$

is designed to zero out

$\cos(n\pi)$, $\cos\left(\frac{3\pi}{4}n\right)$, $\sin\left(\frac{3\pi}{4}n\right)$, $\cos\left(\frac{\pi}{2}n\right)$, $\sin\left(\frac{\pi}{2}n\right)$, $\cos\left(\frac{\pi}{4}n\right)$ and $\sin\left(\frac{\pi}{4}n\right)$.

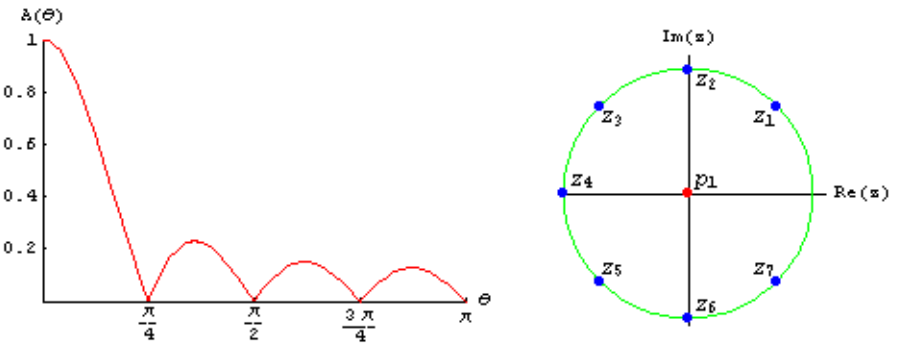


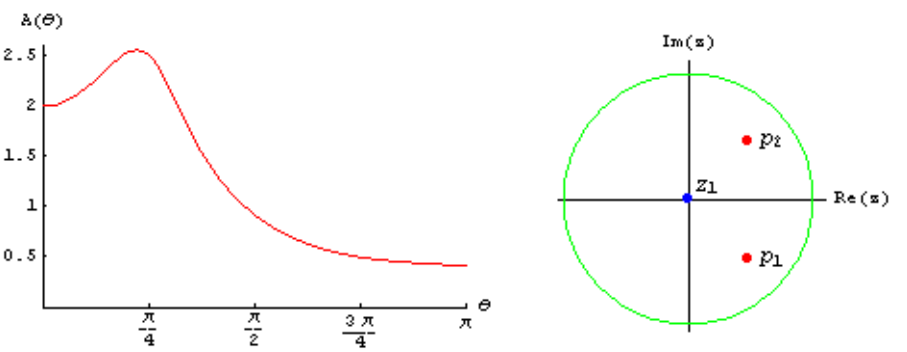
Figure 9.10. Amplitude response $A(\Theta)$, and zero-pole plot for $y[n] = \frac{1}{8}\sum_{k=0}^{7} x[n-k]$.

Solution 9.24.

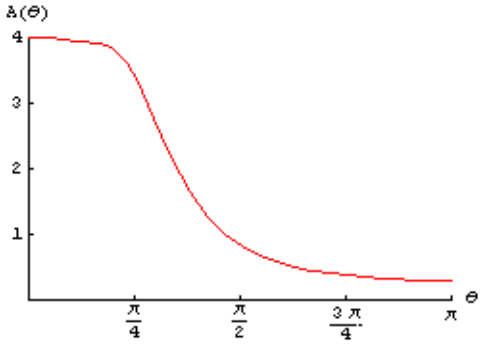
Explore Solution 9.24.

Example 9.25 (a). Design a filter with poles $\frac{2}{3}e^{i\pi/4}$ and $\frac{2}{3}e^{-i\pi/4}$ for boosting up signals near $\cos\left(\frac{\pi}{4}n\right)$ and $\sin\left(\frac{\pi}{4}n\right)$.

9.25 (b). Include the additional pole at $\frac{1}{2}e^{i0} = \frac{1}{2}$ to the filter design in (a) so that it also boosts up low frequency signals.



$y[n] = x[n] + \frac{2}{3}\sqrt{2}\, y[n-1] - \frac{4}{9} y[n-2]$.



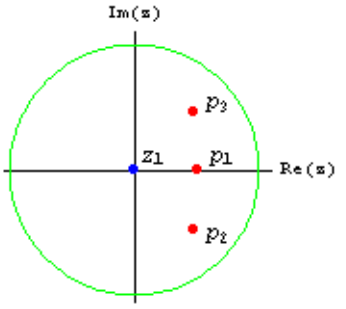
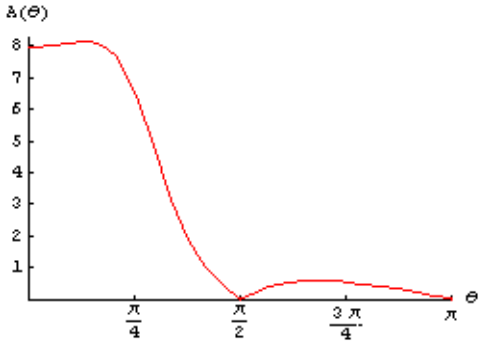


Figure 9.12. Amplitude response $A(\Theta)$, and zero-pole plot for
$y[n] = x[n] + \frac{1}{6}\left(3 + 4\sqrt{2}\right) y[n-1] - \frac{1}{9}\left(4 + 3\sqrt{2}\right) y[n-2] + \frac{2}{9} y[n-3]$.

Solution 9.25.

Explore Solution 9.25.

Example 9.26. Design a combination filter using the zeros $e^{\pm i\pi/2}$ and $e^{i\pi}$ for zeroing out $\cos(\pi n)$, $\cos\left(\frac{\pi}{2} n\right)$ and $\sin\left(\frac{\pi}{2} n\right)$ and poles $\frac{2}{3} e^{\pm i\pi/4}$ boosting up some of the low frequencies.



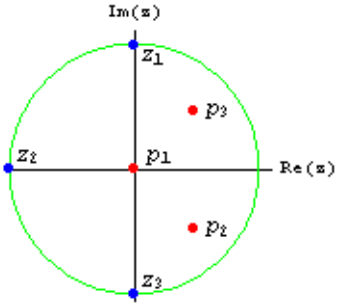


Figure 9.13. Amplitude response $A(\Theta)$, and zero-pole plot for
$y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + \frac{2}{3}\sqrt{2}\, y[n-1] - \frac{4}{9} y[n-2]$.

Solution 9.26.

Explore Solution 9.26.

Low Frequency Filter

The previous examples emphasize high frequency filters. To boost up high frequencies place the poles near the higher frequencies. The following example is similar to Example 9.25. The poles have been changed from $\frac{2}{3} e^{i\pi/4}$, $\frac{2}{3} e^{-i\pi/4}$ and $\frac{1}{2} e^{i0}$ to $\frac{2}{3} e^{i3\pi/4}$, $\frac{2}{3} e^{-i3\pi/4}$ and $\frac{1}{2} e^{i\pi}$.

Extra Example 1 (a). Design a filter with poles $\frac{2}{3} e^{i3\pi/4}$ and $\frac{2}{3} e^{-i3\pi/4}$ for boosting up signals near $\cos\left(\frac{3\pi}{4} n\right)$ and $\sin\left(\frac{3\pi}{4} n\right)$.

1 (b). Include the additional pole at $\frac{1}{2} e^{i\pi} = -\frac{1}{2}$ to the filter design in (a) so that it also boosts up high frequency signals.

Explore Extra Solution 1.

Poles can be placed near frequencies to boost them up and zeros for attenuating the other frequencies.

Extra Example 2. Design a combination filter boosting up some of the low frequencies where $\frac{\pi}{3} < \Theta < \frac{\pi}{2}$ and attenuating frequencies $\Theta < \frac{22}{100}\,\pi$ and $\frac{63}{100}\,\pi < \Theta$.

Explore Extra Solution 2.

A signal processing engineer uses complex analysis to construct filters with the desired amplitude and phase response characteristics. Finite impulse response (FIR) filters have only zeros, whereas infinite impulse response (IIR) filters have poles and may have zeros as well. The area of filter design involves many types, such as: low pass, high pass, all pass, band pass and band stop. Special forms of such filters include, but are not limited to Bessel, Butterworth, Chebyshev, Gaussian, moving average, single pole, Remez, etc. More information about filter design can be found in books on digital signal processing.

Algorithm (Analysis of Filters) The following subroutine will analyze the transfer function for $\sum_{p=0}^{P} a_p\, y[n-p] = \sum_{q=0}^{Q} b_q\, x[n-q]$. The z-transform is

$$Y[z] \sum_{p=0}^{P} a_p\, z^{-p} = X[z] \sum_{q=0}^{Q} b_q\, z^{-q}$$

The functions $A[z] = \sum_{p=0}^{P} a_p\, z^{-p}$ and $B[z] = \sum_{q=0}^{Q} b_q\, z^{-q}$ must be defined.

Mathematica Subroutine (Analysis of Filters)

Complex Analysis for Mathematics and Engineering

by

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Chapter 10 Conformal Mapping

Overview

The terminology "conformal mapping" should have a familiar sound. In 1569 the Flemish cartographer Gerardus Mercator (1512--1594) devised a cylindrical map projection that preserves angles. The Mercator projection is still used today for world maps. Another map projection known to the ancient Greeks is the stereographic projection. It is also conformal (i.e., angle preserving), and we introduced it in Chapter 2 when we defined the Riemann sphere. In complex analysis a function preserves angles if and only if it is analytic or anti-analytic (i.e., the conjugate of an analytic function). A significant result, known as Riemann mapping theorem, states that any simply connected domain (other than the entire complex plane) can be mapped conformally onto the unit disk.

10.1 Basic Properties of Conformal Mappings

Let $f(z)$ be an analytic function in the domain D, and let z_0 be a point in D. If $f'(z_0) \neq 0$, then we can express $f(z)$ in the form

(10-1) $$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0),$$

where $\eta(z) \to 0$ as $z \to z_0$. If z is near z_0, then the transformation $w = f(z)$ has the linear approximation

$$S(z) = A + B(z - z_0),$$

where $A = f(z_0)$ and $B = f'(z_0)$. Because $\eta(z) \to 0$ when $z \to z_0$, for points near z_0 the transformation $w = f(z)$ has an effect much like the linear mapping $w = S(z)$. The effect of the linear mapping S is a rotation of the plane through the angle $\alpha = \text{Arg}(f'(z_0))$, followed by a magnification by the factor $|f'(z_0)|$, followed by a rigid translation by the vector $A + Bz_0$. Consequently, the mapping $w = S(z)$ preserves angles at the point z_0. We now show that the mapping $w = f(z)$ also preserves angles at z_0.

For a smooth curve that passes through the point $z(0) = z_0$, we use the notation

$$C: z(t) = x(t) + i\, y(t),$$

for $-1 \le t \le 1$.

A vector $\vec{T}$ tangent to C at the point z_0 is given by

$$\vec{T} = z'(0),$$

where the complex number $z'(0)$ is expressed as a vector.

The angle of inclination of $\vec{T}$ with respect to the positive x axis is

$$\beta = \text{Arg}\, z'(0).$$

The image of C under the mapping $w = f(z)$ is the curve K in the w plane given by the formula

$$K: w(t) = u(x(t), y(t)) + i\, v(x(t), y(t)).$$

We can use the chain rule to show that a vector $\vec{T^*}$ tangent to K at the point $w_0 = f(z_0)$ is given by

$$\vec{T^*} = w'(0) = f'(z_0)\, z'(0).$$

The angle of inclination of $\vec{T^*}$ with respect to the positive u axis is

$$\gamma = \text{Arg}\, w'(0) = \text{Arg}\, f'(z_0) + \text{Arg}\, z'(0) = \alpha + \beta,$$

where $\alpha = \text{Arg}\, f'(z_0)$.

Therefore the effect of the transformation $w = f(z)$ is to rotate the angle of inclination of the tangent vector $\vec{T}$ at z_0 through the angle $\alpha = \text{Arg}\, f'(z_0)$ to obtain the angle of inclination of the tangent vector $\vec{T^*}$ at $w_0 = f(z_0)$. This situation is illustrated in Figure 10.1.

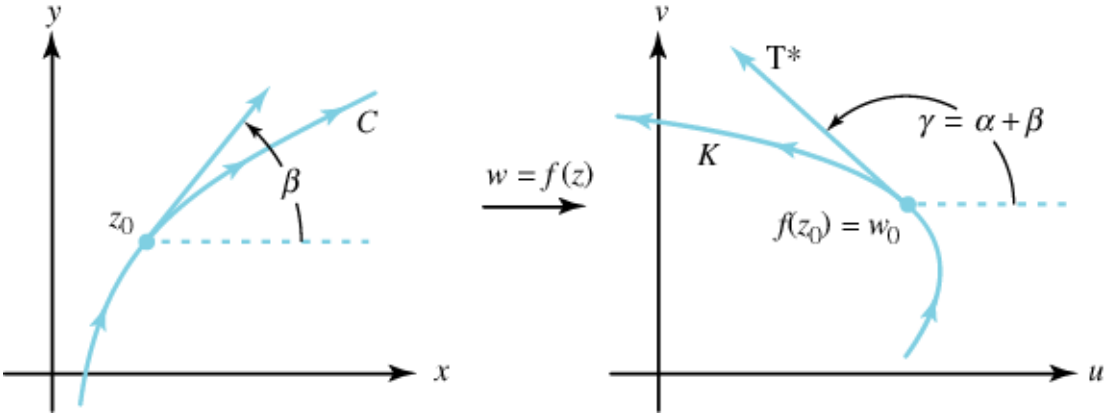


Figure 10.1 The tangents at the points z_0 and w_0, where $f(z)$ is an analytic function and $f'(z_0) \neq 0$.

A mapping $w = f(z)$ is said to be angle preserving, or conformal at z_0, if it preserves angles between oriented curves in magnitude as well as in orientation. Theorem 10.1 shows where a mapping by an analytic function is conformal.

Theorem 10.1 (Conformal Mapping). Let $f(z)$ be an analytic function in the domain D, and let z_0 be a point in D. If $f'(z_0) \neq 0$, then $f(z)$ is conformal at z_0.

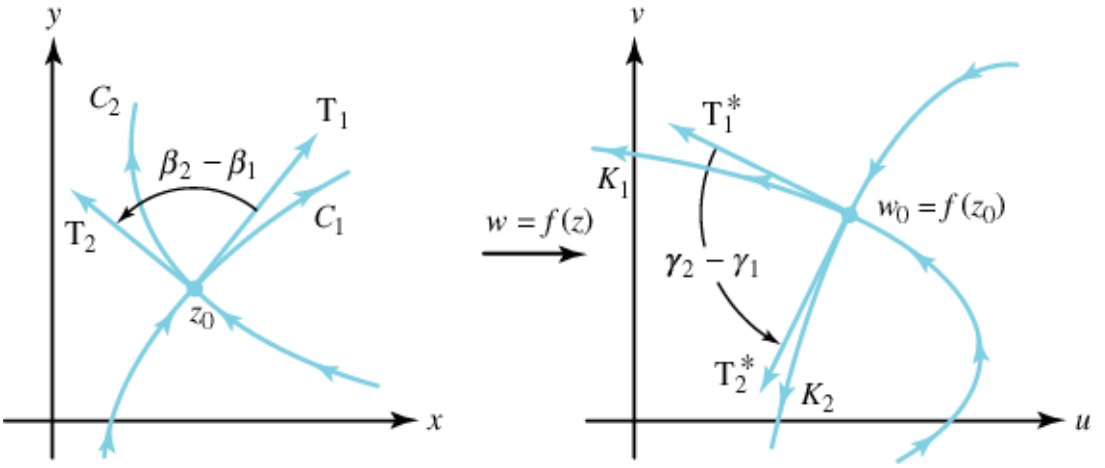
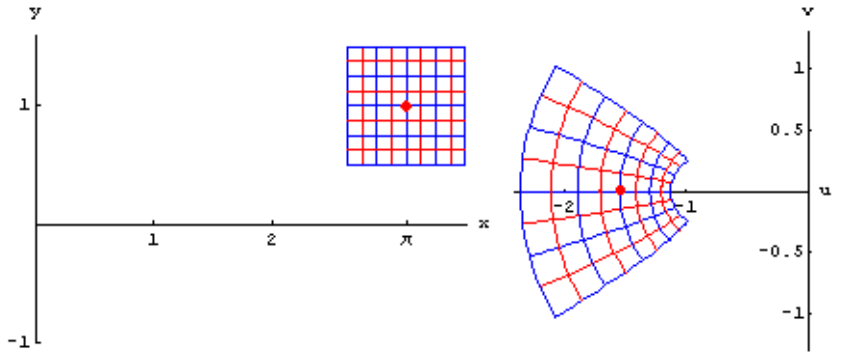


Figure 10.2 The analytic mapping $w = f(z)$ is conformal at the point z_0, where $f'(z_0) \neq 0$.

Proof.

Example 10.1. Show that the mapping $w = f(z) = \cos(z)$ is conformal at the points $z_1 = i$, $z_2 = 1$, $z_3 = \pi + i$ and , $z_4 = 1 - i$ and determine the angle of rotation given by $\alpha = \text{Arg}(f'(z))$ at the given points.



Solution. Because $f'(z) = -\sin(z)$, we conclude that the mapping $w = \cos(z)$ is conformal at all points except $z = n\pi$, where n is an integer. Calculation reveals that

$$f'(z_1) = f'(i) = -i \sinh 1,$$

$$f'(z_2) = f'(1) = -\sin 1,$$

$$f'(z_3) = f'(\pi + i) = i \sinh 1, \text{ and}$$

$$f'(z_4) = f'(1 - i) = -\cosh 1 \sin 1 + i \cos 1 \sinh 1.$$

Therefore the angle of rotation is given by

$$\alpha_1 = \text{Arg}(f'(z_1)) = f'(i) = -i \sinh 1,$$

$$\alpha_2 = \text{Arg}(f'(z_2)) = f'(1) = -\sin 1,$$

$$\alpha_3 = \text{Arg}(f'(z_3)) = f'(\pi + i) = i \sinh 1, \text{ and}$$

$$\alpha_4 = \text{Arg}(f'(z_4)) = f'(1 - i) = -\cosh 1 \sin 1 + i \cos 1 \sinh 1$$
$$= \pi - \arctan(\cot 1 \tanh 1).$$

Explore Solution 10.1.

Let $f(z)$ be a nonconstant analytic function. If $f'(z_0) = 0$, then z_0 is called a critical point of $f(z)$, and the mapping $w = f(z)$ is not conformal at z_0. The next result shows what happens at a critical point.

Theorem 10.2. Let $f(z)$ be analytic at the point z_0. If $f'(z_0) = 0$, $f''(z_0) = 0$, ..., $f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$, then the mapping $w = f(z)$ magnifies angles at the vertex z_0 by the factor k.

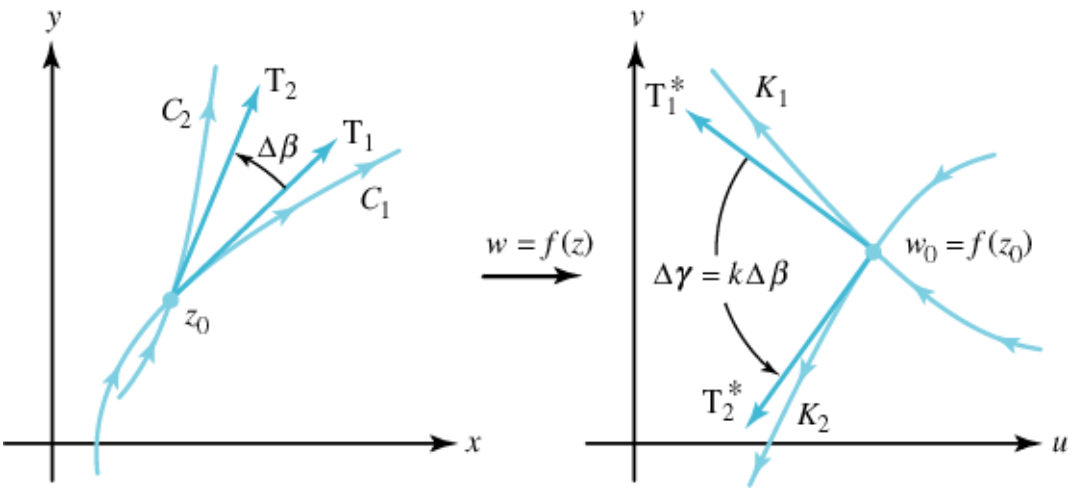


Figure 10.3 The analytic mapping $w = f(z)$ at point z_0, where $f'(z_0) = 0$, ..., $f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$.

Proof.

Example 10.2. Show that the mapping $w = f(z) = z^2$ maps the unit square $S = \{x + i\, y : 0 < x < 1, 0 < y < 1\}$ onto the region in the upper half-plane $\text{Im}(w) > 0$, which lies under the parabolas

$$u = 1 - \frac{1}{4} v^2 \quad \text{and} \quad u = -1 + \frac{1}{4} v^2$$

as shown in Figure 10.4.

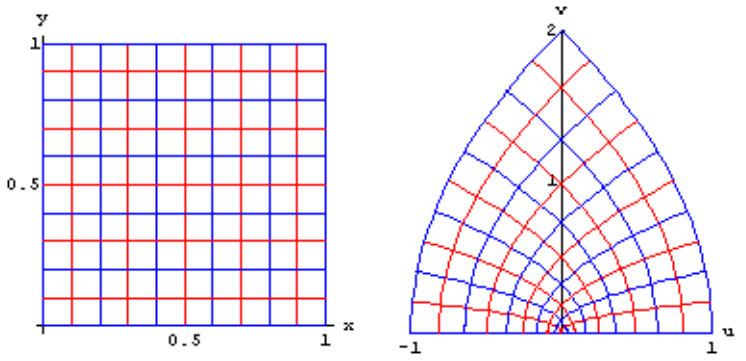


Figure 10.4 The mapping $w = z^2$.

Solution. The derivative is $f'(z) = 2z$, and we conclude that the mapping $w = z^2$ is conformal for all $z \neq 0$. Note that the right angles at the vertices $z_1 = 1$, $z_2 = 1 + i$, and $z_3 = i$ are mapped onto right angles at the vertices $w_1 = 1$, $w_2 = 2i$, and $w_3 = -1$, respectively. At the point $z_0 = 0$, we have $f'(z_0) = f'(0) = 0$ and $f''(z_0) = f''(0) = 2 \neq 0$. Hence angles at the vertex $z_0 = 0$ are magnified by the factor $k = 2$. In particular, the right angle at $z_0 = 0$ is mapped onto the straight angle at $w_0 = 0$.

Explore Solution 10.2.

Another property of a conformal mapping $w = f(z)$ is obtained by considering the modulus of $f'(z_0)$. If z_1 is near z_0, we can use the equation

$$f(z_1) = f(z_0) + f'(z_0)(z_1 - z_0) + \eta(z_1)(z_1 - z_0)$$

and neglect the term $\eta(z_1)(z_1 - z_0)$. We then have the approximation

(10-9) $$w_1 - w_0 = f(z_1) - f(z_0) \approx f'(z_0)(z_1 - z_0).$$

From Equation (10-9), the distance $|w_1 - w_0|$ between the images of the points z_1 and z_0 given approximately by $|f'(z_0)||z_1 - z_0|$. Therefore we say that the transformation $w = f(z)$ changes small distances near z_0 by the scale factor $|f'(z_0)|$. For example, the scale factor of the transformation $w = f(z) = z^2$ near the point $z_0 = 1 + i$ is $|f'(z_0)| = |2z_0| = |2(1 + i)| = 2\sqrt{2}$.

We also need to say a few things about the inverse transformation $z = g(w)$ of a conformal mapping $w = f(z)$ near a point z_0, where $f'(z_0) \neq 0$.
A complete justification of the following assertions relies on theorems studied in advanced calculus. (See, for instance, R. Creighton Buck, Advanced Calculus, 3rd ed. (New York, McGraw-Hill), pp. 358-361, 1978.)

We express the mapping $w = f(z)$ in the coordinate form

(10-10) $$u = u(x, y) \quad \text{and} \quad v = v(x, y).$$

The mapping in Equations (10-10) represents a transformation from the xy plane into the uv plane, and the Jacobian determinant, $J(x, y)$, is defined by

(10-11) $$J(x, y) = \begin{vmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{vmatrix}.$$

The transformation in Equations (10-10) has a local inverse, provided $J(x, y) \neq 0$. Expanding Equation (10-11) and using the Cauchy--Riemann equations, we obtain

$$J(x_0, y_0) = \begin{vmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{vmatrix}$$

$$= u_x(x_0, y_0)\, v_y(x_0, y_0) - v_x(x_0, y_0)\, u_y(x_0, y_0)$$

$$= u_x(x_0, y_0)\, u_x(x_0, y_0) - v_x(x_0, y_0)\, (-v_x(x_0, y_0))$$

$$= u_x^2(x_0, y_0) + v_x^2(x_0, y_0)$$

$$= |f'(z_0)|$$

$$\neq 0$$

Consequently, Equations (10-11) and (10-11) imply that a local inverse $z = g(w)$ exists in a neighborhood of the point $w_0 = f(z_0)$. The derivative of $g(w)$ at w_0 is given by the familiar expression

$$g'(w_0) = \lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{w \to w_0} \frac{z - z_0}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{w - w_0}$$

$$= \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = \lim_{z \to z_0} 1 \Big/ \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$$

$$= 1 \Big/ \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right)$$

$$= \frac{1}{f'(z_0)}$$

$$= \frac{1}{f'(g(w_0))}$$

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

by

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Jones and Bartlett Learning

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10.2 Bilinear Transformations - Mobius Transformations

Another important class of elementary mappings was studied by August Ferdinand Möbius (1790-1868). These mappings are conveniently expressed as the quotient of two linear expressions and are commonly known as linear fractional or bilinear transformations. They arise naturally in mapping problems involving the function arctan(z). In this section, we show how they are used to map a disk one-to-one and onto a half-plane.

Let $a, b, c,$ and d denote four complex constants with the restriction that $a\,d \neq b\,c$. Then the function

(10-13) $$w = S(z) = \frac{a\,z + b}{c\,z + d}$$

is called a bilinear transformation, a Möbius transformation, or a linear fractional transformation. If the expression for $S(z)$ in Equation (10-13) is multiplied through by the quantity $c\,z + d$, then the resulting expression has the bilinear form $c\,w\,z - a\,z + d\,w - b = 0$.
We collect terms involving z and write $z(c\,w - a) = -d\,w + b$. Then, for values of $w \neq \frac{a}{c}$ the inverse transformation is given by

(10-14) $$z = S^{-1}(w) = \frac{-d\,w + b}{c\,w - a}.$$

We can extend $S(z)$ and $S^{-1}(w)$ to mappings in the extended complex plane. The value $S(\infty)$ should be chosen to equal the limit of $S(z)$ as $z \to \infty$. Therefore we define

$$S(\infty) = \lim_{z \to \infty} S(z) = \lim_{z \to \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} = \frac{a}{c},$$

and the inverse is $S^{-1}\left(\frac{a}{c}\right) = \infty$. Similarly, the value $S^{-1}(\infty)$ is obtained by

$$S^{-1}(\infty) = \lim_{w\to\infty} S^{-1}(w) = \lim_{w\to\infty} \frac{-d+\frac{b}{w}}{c-\frac{a}{w}} == \frac{-d}{c},$$

and the inverse is $S\left(\frac{-d}{c}\right) = \infty$. With these extensions we conclude that the transformation $w = S(z)$ is a one-to-one mapping of the extended complex z-plane onto the extended complex w-plane.

We now show that a bilinear transformation carries the class of circles and lines onto itself. If $S(z)$ is an arbitrary bilinear transformation given by Equation (10-13) and $c = 0$, then $S(z)$ reduces to a linear transformation, which carries lines onto lines and circles onto circles. If $c \neq 0$, then we can write $S(z)$ in the form

(10-15)
$$S(z) = \frac{a z + b}{c z + d}$$

$$= \frac{c(a z + b)}{c(c z + d)}$$

$$= \frac{a c z + b c}{c(c z + d)}$$

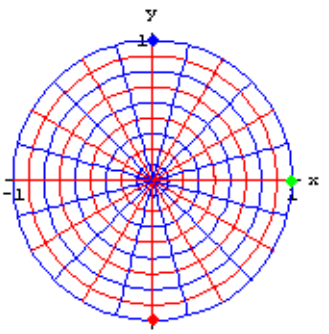
$$= \frac{a c z + a d - a d + b c}{c(c z + d)}$$

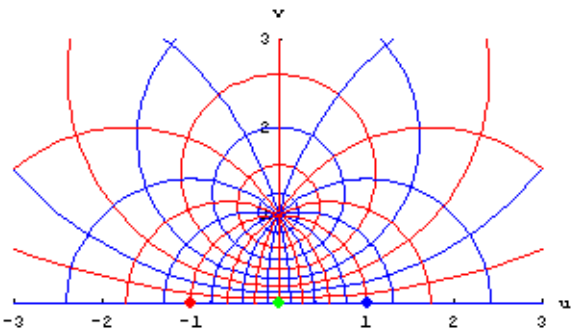
$$= \frac{a(c z + d) - a d + b c}{c(c z + d)}$$

$$= \frac{a}{c} + \frac{b c - a d}{c}\,\frac{1}{c z + d}$$

The condition $a d \neq b c$ precludes the possibility that $S(z)$ reduces to a constant. Equation (10-15) indicates that $S(z)$ can be considered as a composition of functions.
It is a linear mapping $\xi = c z + d$, followed by the reciprocal transformation $Z = \frac{1}{\xi}$, followed by $w = \frac{a}{c} + \frac{b c - a d}{c} Z$. In Chapter 2 we showed that each function in this composition maps the class of circles and lines onto itself; it follows that the bilinear transformation $S(z)$ has this property. A half-plane can be considered to be a family of parallel lines and a disk as a family of circles. Therefore we conclude that a bilinear transformation maps the class of half-planes and disks onto itself. Example 10.3 illustrates this idea.

Example 10.3. Show that $w = S(z) = \frac{i(1-z)}{1+z}$ maps the unit disk $D: |z| < 1$ one-to-one and onto the upper half-plane $\mathrm{Im}(w) > 0$.





Solution. We first consider the unit circle $C: |z| = 1$, which forms the boundary of the disk and find its image in the w plane.

If we write $S(z) = \frac{-i z + i}{z + 1}$, then we see that $a = -i$, $b = i$, $c = 1$, and $d = 1$.
Using Equation (10-14), we find that the inverse is given by

(10-16) $$z = S^{-1}(w) = \frac{-d w + b}{c w - a} = \frac{-(1) w + (i)}{(1) w - (-i)} = \frac{-w + i}{w + i}.$$

If $|z| = 1$, then Equation (10-16) implies that the images of points on the unit circle satisfy $\left|\frac{-w + i}{w + i}\right| = 1$ which yields the equation

(10-17) $$|w + i| = |-w + i|.$$

Squaring both sides of Equation (10-17), we obtain

$$|u + i v + i| = |-u - i v + i|$$

$$|u + i (1 + v)|^2 = |-u + i (1 - v)|^2$$

$$u^2 + (1 + v)^2 = (-u)^2 + (1 - v)^2$$

$$u^2 + (1 + v)^2 = u^2 + (1 - v)^2$$

$$(1 + v)^2 = (1 - v)^2$$

$$1 + 2 v + v^2 = 1 - 2 v + v^2$$

$$4 v = 0$$

$$v = 0$$

which is the equation of the u axis in the w plane.

The circle C divides the z plane into two portions, and its image is the u axis, which divides the w plane into two portions. The image of the point $z = 0$ is $w = S(0) = i$, so we expect that the interior of the circle C is mapped onto the portion of the w plane that lies above the u axis. To show that this outcome is true, we let $|z| < 1$. Then Equation (10-16) implies that the image values must satisfy the inequality $|-w + i| < |w + i|$, which we write as

$$d_1 = |w - i| < |w - (-i)| = d_2 .$$

If we interpret d_1 as the distance from w to i and d_2 as the distance from w to $-i$, then a geometric argument shows that the image point w must lie in the upper half-plane $\mathrm{Im}(w) > 0$, as shown in Figure 10.5. As $S(z)$ is one-to-one and onto in the extended complex plane, it follows that $S(z)$ maps the disk onto the half-plane.

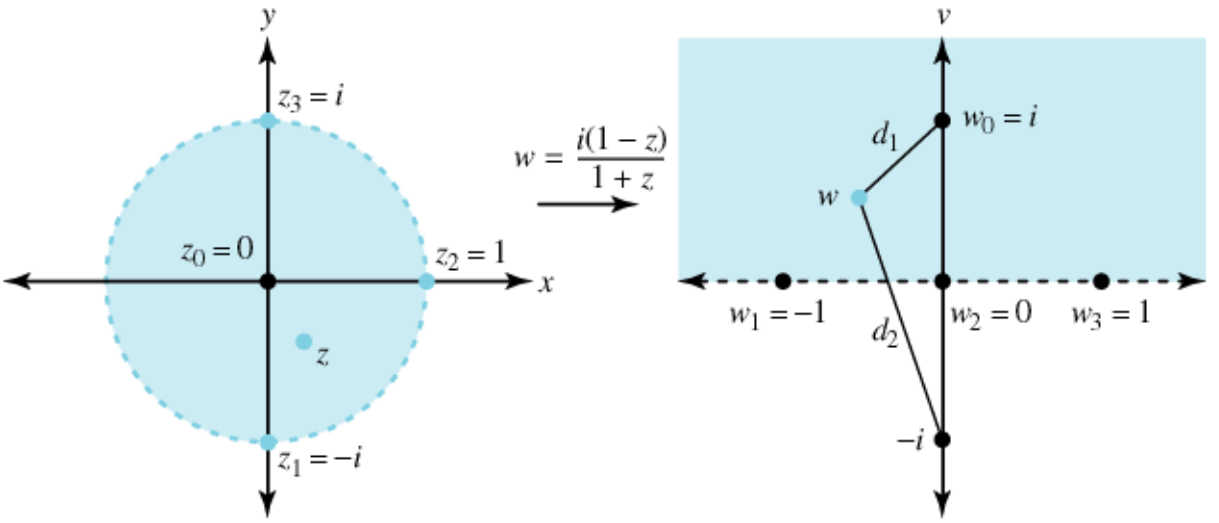


Figure 10.5 The image $|z| < 1$ under $w = \frac{i(1-z)}{1+z}$, the points $z_1 = -i$, $z_2 = 1$, $z_3 = i$ are mapped onto the points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$, respectively.

Explore Solution 10.3.

The general formula for a bilinear transformation (Equation (10-13)) appears to involve four independent coefficients: a, b, c, and d. But as $S(z)$ is not identically constant, either $a \neq 0$ or $c \neq 0$, we can express the transformation with three unknown coefficients and write either

$$S(z) = \frac{z + \frac{b}{a}}{\frac{c}{a} z + \frac{d}{a}} \quad \text{or} \quad S(z) = \frac{\frac{a}{c} z + \frac{b}{c}}{z + \frac{d}{c}},$$

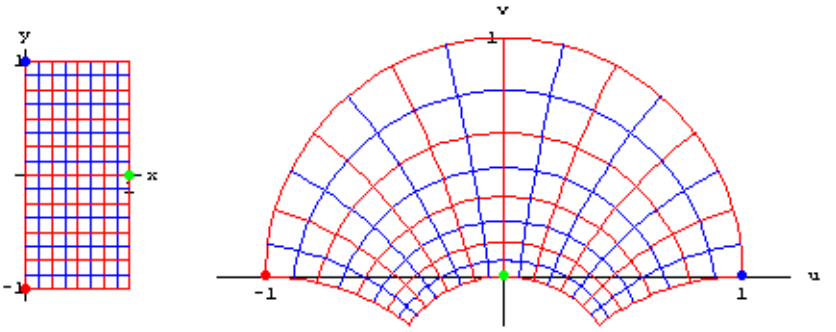
respectively. Doing so permits us to determine a unique a bilinear transformation if three distinct image values $S(z_1) = w_1$, $S(z_2) = w_2$, and $S(z_3) = w_3$ are specified. To determine such a mapping, we can conveniently use an implicit formula involving z and w.

Theorem 10.3 (The Implicit Formula). There exists a unique bilinear transformation that maps three distinct points z_1, z_2, and z_3 onto three distinct points w_1, w_2, and w_3, respectively. An implicit formula for the mapping is given by the equation

(10-18) $$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}.$$

Proof.

Example 10.4. Construct the bilinear transformation $w = S(z)$ that maps the points $z_1 = -i$, $z_2 = 1$, $z_3 = i$ onto the points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$, respectively.



Solution. We use the implicit formula, Equation (10-18), and write

$$\frac{(z-(-i))\,(1-i)}{(z-i)\,(1-(-i))} = \frac{(w-(-1))\,(0-1)}{(w-1)\,(0-(-1))}$$

$$\frac{(z+i)\,(1-i)}{(z-i)\,(1+i)} = \frac{(w+1)\,(0-1)}{(w-1)\,(0+1)}$$

$$\frac{(z+i)\,(1-i)}{(z-i)\,(1+i)} = \frac{w+1}{-w+1}.$$

Expanding this equation, collecting terms involving w and zw on the left and then simplify.

$$(z-i)\,(1+i)\,(w+1) = (z+i)\,(1-i)\,(-w+1)$$

$$(1+i)\,z\,w+(1-i)\,w+(1+i)\,z+(1-i)$$
$$=$$
$$(-1+i)\,z\,w+(-1-i)\,w+(1-i)\,z+(1+i)$$

$$z\,w+i\,z\,w+w-i\,w+z+i\,z+1-i$$
$$=$$
$$-z\,w+i\,z\,w-w-i\,w+z-i\,z+1+i$$

$$2\,z\,w+2\,w = -2\,i\,z+2\,i$$

$$z\,w+w = -i\,z+i$$

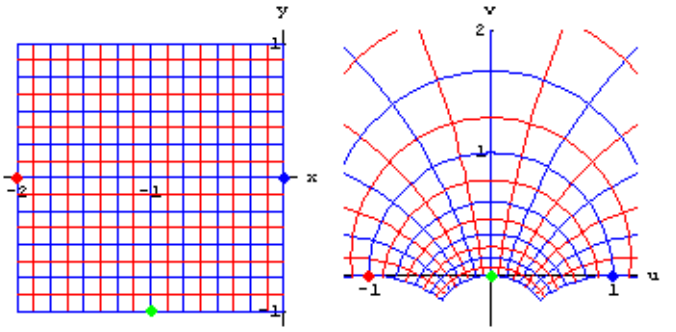
$$w\,(1+z) = i\,(1-z)$$

Therefore the desired bilinear transformation is

$$w = S(z) = \frac{i\,(1-z)}{1+z}.$$

Explore Solution 10.4.

Example 10.5. Find the bilinear transformation $w = S(z)$ that maps the points $z_1 = -2,\ z_2 = -1-i,\ z_3 = 0$ onto the points $w_1 = -1,\ w_2 = 0,\ w_3 = 1$, respectively.



Solution. Again, we use the implicit formula, Equation (10-18), and write

$$\frac{(z-(-2))\,((-1-i)-0)}{(z-0)\,((-1-i)-(-2))} = \frac{(w-(-1))\,(0-1)}{(w-1)\,(0-(-1))}$$

$$\frac{(z+2)\,(-1-i)}{(z)\,(-1-i+2)} = \frac{(w+1)\,(-1)}{(w-1)\,(1)}$$

$$\frac{z+2}{z}\,\frac{-1-i}{1-i} = \frac{1+w}{1-w}$$

Using the fact that $\frac{-1-i}{1-i} = \frac{1}{i}$, we rewrite this equation as

$$\frac{z+2}{i\,z} = \frac{1+w}{1-w}.$$

We now expand the equation and obtain

$$(z+2)\,(1-w) = i\,z\,(1+w)$$

$$z+2-z\,w-2\,w = i\,z+i\,z\,w$$

$$z-i\,z+2 = z\,w+i\,z\,w+2\,w$$

$$(1-i)\,z+2 = w\,(z+i\,z+2)$$

$$(1-i)\,z+2 = w\,((1+i)\,z+2)$$

which can be solved for w in terms of z, giving the desired solution

$$w = S(z) = \frac{(1-i)\,z+2}{(1+i)\,z+2}.$$

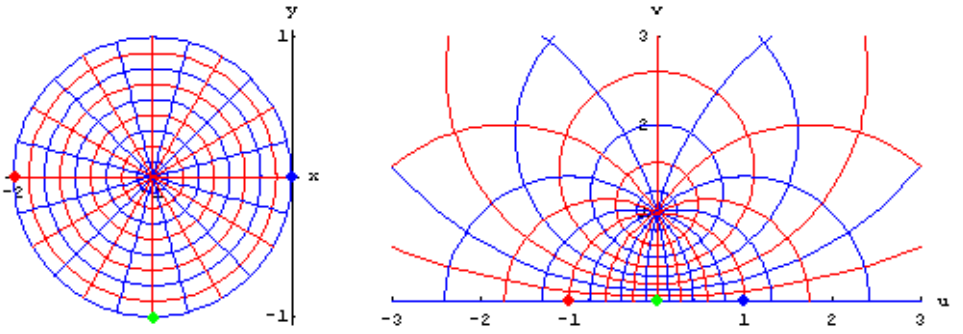
Explore Solution 10.5.

We let D be a region in the z plane that is bounded by either a circle or a straight line C. We further let z_1, z_2, and z_3 be three distinct points that lie on C and have the property that an observer moving along C from z_1 to z_3 through z_2 finds the region D to be on the left. If C is a circle and D is the interior of C, then we say that C is positively oriented. Conversely, the ordered triple (z_1, z_2, z_3) uniquely determines a region that lies to the left of C.

We let G be a region in the w plane that is bounded by either a circle of a straight line K. We further let w_1, w_2, and w_3 be three distinct points that lie on K such that an observer moving along K

from w_1 to w_3 through w_2 finds the region G to be on the left. Because a bilinear transformation is a conformal mapping that maps the class of circles and straight lines onto itself, we can use the implicit formula to construct a bilinear transformation $w = S(z)$ that is a one-to-one mapping of D onto G.

Example 10.6. Show that the mapping $w = S(z) = \frac{(1-i)z+2}{(1+i)z+2}$ maps the disk $D: |z+1| < 1$ one-to-one and onto the upper half plane $\mathrm{Im}(w) > 0$.



Solution. For convenience, we choose the ordered triple $z_1 = -2$, $z_2 = -1 - i$, $z_3 = 0$, which gives the circle $C: |z+1| = 1$ a positive orientation and the disk D a left orientation. From Example 10.5, the corresponding image points are

$$w_1 = S(z_1) = S(-2) = -1,$$

$$w_2 = S(z_2) = S(-1-i) = 0,$$

$$w_3 = S(z_3) = S(0) = 1.$$

Because the ordered triple of points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$, lie on the u axis, it follows that the image of circle C is the u axis. The points $w_1 = -1$, $w_2 = 0$, $w_3 = 1$ give the upper half-plane $G: \mathrm{Im}(w) > 0$ a left orientation. Therefore $w = S(z) = \frac{(1-i)z+2}{(1+i)z+2}$ maps the disk D onto the upper half-plane G. To check our work, we choose a point z_0 that lies in D and find the half-plane in which its image, w_0 lies. The choice $z_0 = -1$ yields $w_0 = S(z_0) = i$. Hence the upper half-plane is the correct image. This situation is illustrated in Figure 10.6.

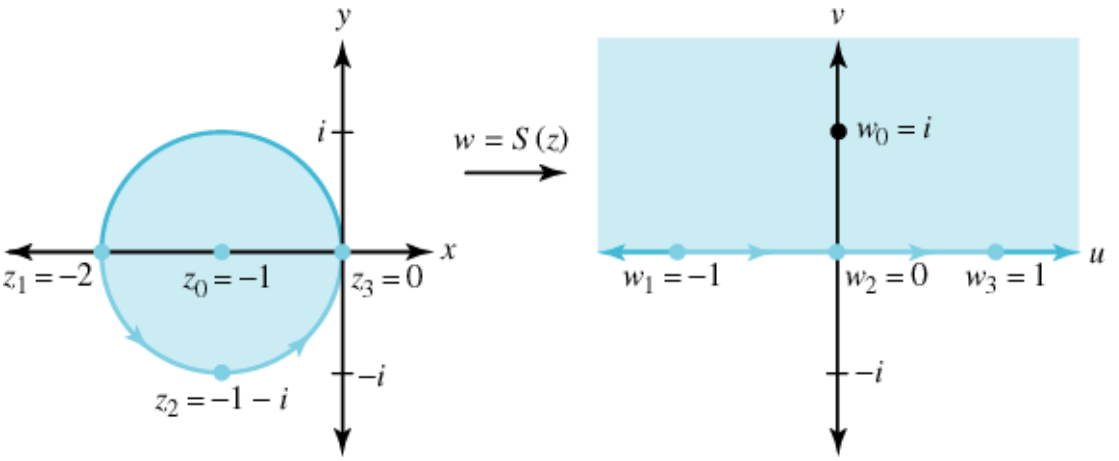


Figure 10.6 The bilinear mapping $w = S(z) = \frac{(1-i)z+2}{(1+i)z+2}$.

Explore Solution 10.6.

Corollary 10.1 (The Implicit Formula with a point at Infinity). In equation (10-18) the point at infinity can be introduced as one of the prescribed points in either the z plane or the w plane.

Proof.

Case 1. If $z_3 = \infty$, then we can write $\frac{(z_2 - z_3)}{(z - z_3)} = \frac{(z_2 - \infty)}{(z - \infty)} = 1$ and substitute this expression into Equation (10-18) to obtain $\frac{(z - z_1)(z_2 - \infty)}{(z - \infty)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$ which can be rewritten as $\frac{(z - z_1)(z_2 - \infty)}{(z_2 - z_1)(z - \infty)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$ and simplifies to obtain

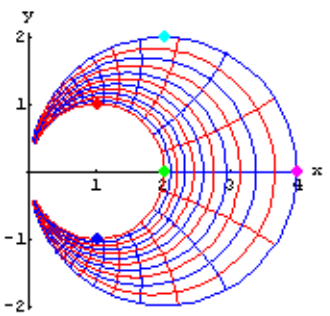
$$\frac{z - z_1}{z_2 - z_1} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}.$$

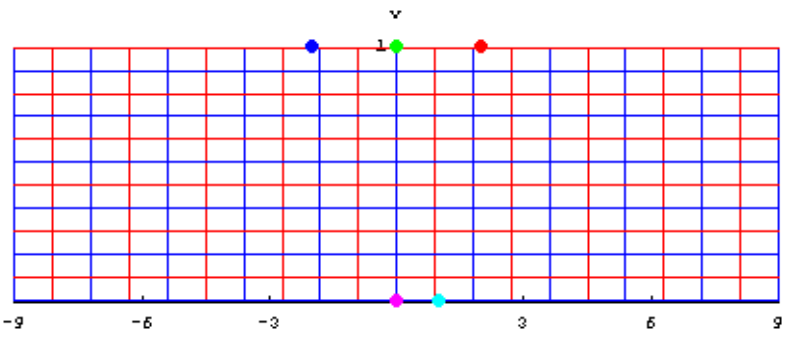
Case 2. If $w_3 = \infty$, then we can write $\frac{(w_2 - w_3)}{(w - w_3)} = \frac{(w_2 - \infty)}{(w - \infty)} = 1$ and substitute this expression into Equation (10-18) to obtain $\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w - \infty)(w_2 - w_1)}$ which can be rewritten as $\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - \infty)}{(w_2 - w_1)(w - \infty)}$ and simplifies to obtain

(10-21) $$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{w - w_1}{w_2 - w_1}.$$

Equation (10-21) is sometimes used to map the crescent-shaped region that lies between the tangent circles onto an infinite strip.

Example 10.7. Find the bilinear transformation $w = S(z)$ that maps the crescent-shaped region that lies inside the disk $D: |z - 2| < 2$ and outside the circle $|z - 1| = 1$ onto a horizontal strip.





Solution. For convenience we choose $z_1 = 4$, $z_2 = 2 + 2i$, $z_3 = 0$ and the image values $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$, respectively. The ordered triple $z_1 = -4$, $z_2 = 2 + 2i$, $z_3 = 0$ gives the circle $C: |z - 2| = 2$ a positive orientation and the disk $D: |z - 2| < 2$ has a left orientation. The image points $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ all lie on the extended u axis, and they determine a left orientation for the upper half-plane $\mathrm{Im}(w) > 0$. Therefore we can use the second implicit formula (Equation (10-21)) to write

$$\frac{(z - 4)(2 + 2i - 0)}{(z - 0)(2 + 2i - 4)} = \frac{w - 0}{1 - 0},$$

which determines a mapping of the disk $D: |z - 2| < 2$ onto the upper half-plane $\mathrm{Im}(w) > 0$. Use the fact that $\frac{2 + 2i}{-2 + 2i} = -i$ to simplify the preceding equation and get

$$\frac{z-4}{z}\,\frac{2+2i}{-2+2i} = \frac{z-4}{z}(-i) = \frac{w}{1}$$

which can be written in the form

$$w = S(z) = \frac{-iz+i4}{z}.$$

A straightforward calculation shows that the points $z_4 = 1-i$, $z_5 = 2$, $z_6 = 1+i$ are mapped onto the points

$$w_4 = S(z_4) = S(1-i) = -2+i,$$

$$w_5 = S(z_5) = S(2) = i,$$

$$w_6 = S(z_6) = S(1+i) = 2+i,$$

respectively. The points $w_4 = -2+i$, $w_5 = i$, $w_6 = 2+i$ lie on the horizontal line $\mathrm{Im}(w) > 1$ in the upper half-plane. Therefore the crescent-shaped region is mapped onto the horizontal strip $0 < \mathrm{Im}(w) < 1$, as shown in Figure 10.7.

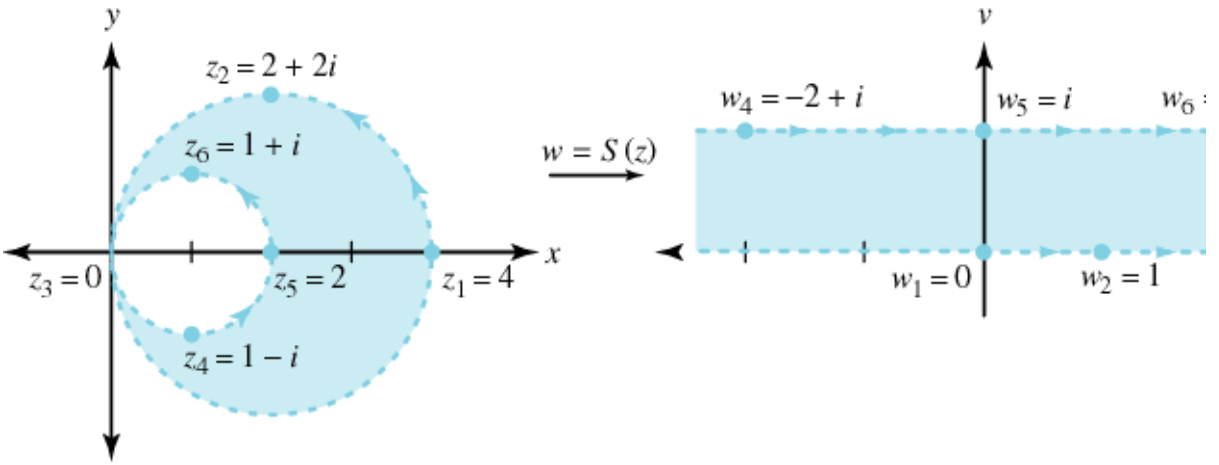


Figure 10.7 The mapping $w = S(z) = \frac{-iz+i4}{z}$.

Explore Solution 10.7.

Lines of Flux

In electronics, images of certain lines represent lines of electric flux, which comprise the trajectory of an electron placed in an electrical field. Consider the bilinear transformation

$$w = S(z) = \frac{z}{z-a} \quad \text{and} \quad z = S^{-1}(w) = \frac{aw}{w-1}.$$

The half rays $\{\mathrm{Arg}(w) = c\}$, where c is a constant, that meet at the origin $w = 0$ represent the lines of electric flux produced by a source located at $w = 0$ (and a sink at $w = \infty$). The preimage of this family of lines is a family of circles that pass through the points $z = 0$ and $z = a$. We visualize these circles as the lines of electric flux from one point charge to another. The limiting case as a is called a dipole and is discussed in Exercise 6, Section 10.11. The graphs for $a = 1$, $a = 0.5$, and

a = 0.1 are shown in Figure 10.8.

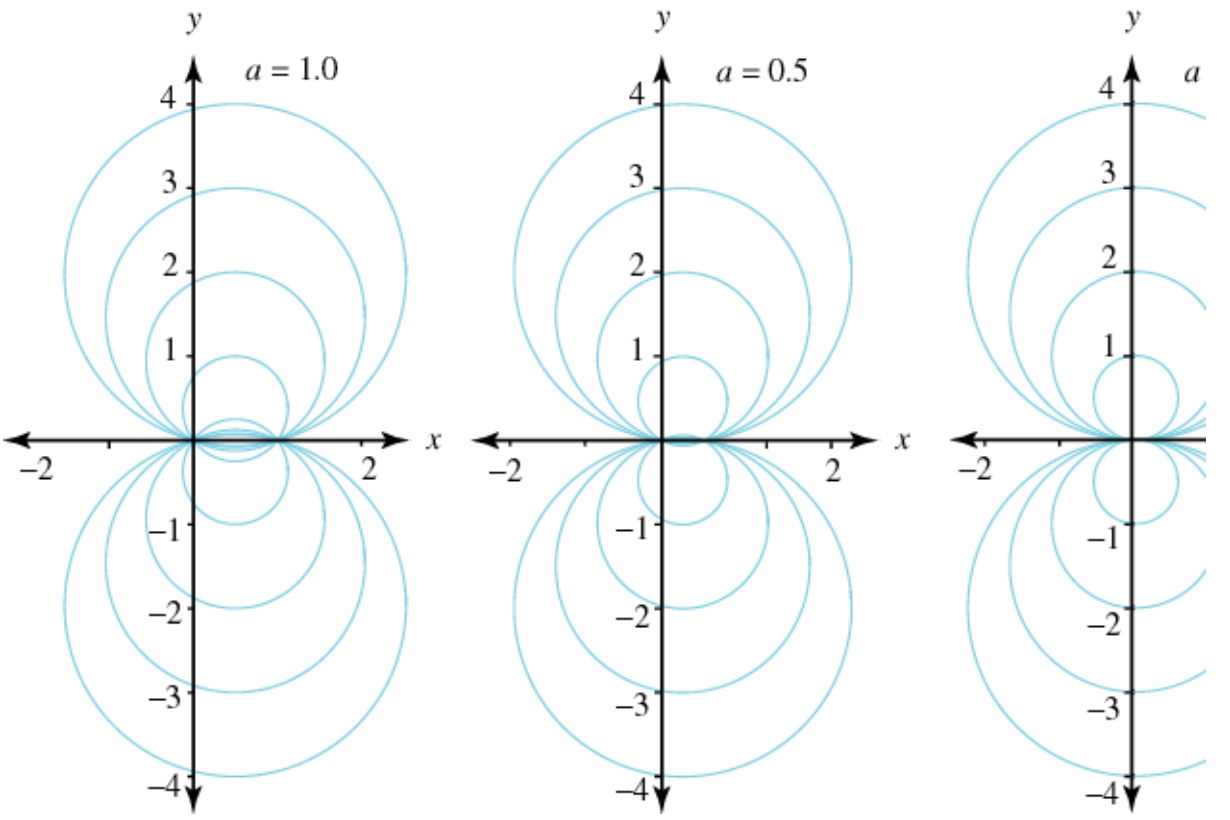


Figure 10.8 Images of Arg (w) = c under the mapping

$z = S^{-1}(w) = \frac{a w}{w - 1}$.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering
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10.3 Mapping Involving Elementary Functions

In Section 5.1 we saw that the function $w = f(z) = e^z$ is a one-to-one mapping of the fundamental period strip $-\pi < y \le \pi$ in the z-plane onto the w-plane with the point $w = 0$ deleted. Because $f'(z) \ne 0$, the mapping $w = e^z$ is a conformal mapping at each point z in the complex plane. The family of horizontal lines $y = c$ and $-\pi < c \le \pi$, and segments $x = a$ and $-\pi < y \le \pi$ form an orthogonal grid in the fundamental period strip. Their images under the mapping $w = e^z$ are the rays $\rho > 0$ and $\phi = c$ and the circles $|w| = e^a$, respectively. These images form an orthogonal curvilinear grid in the w-plane, as shown in Figure 10.9. If $-\pi < c < d \le \pi$, then the rectangle $R = \{x + iy : a < x < b, c < y < d\}$ is mapped one-to-one and onto the region $G = \{\rho e^{i\phi} : e^a < \rho < e^b, c < \phi < d\}$. The inverse mapping is the principal branch of the logarithm $z = \text{Log } w$.

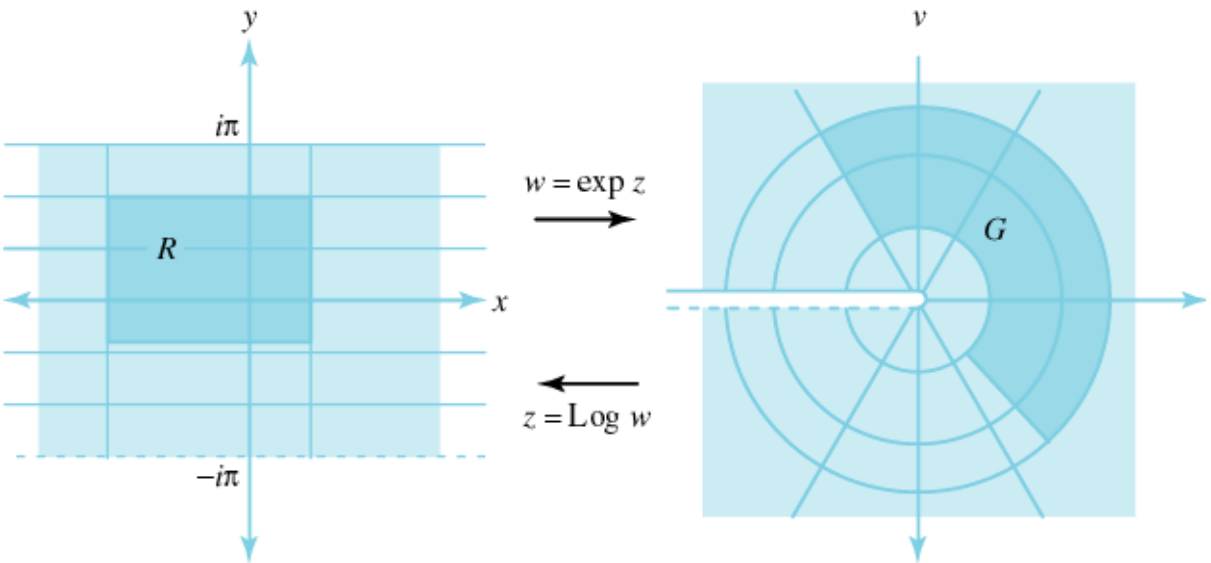
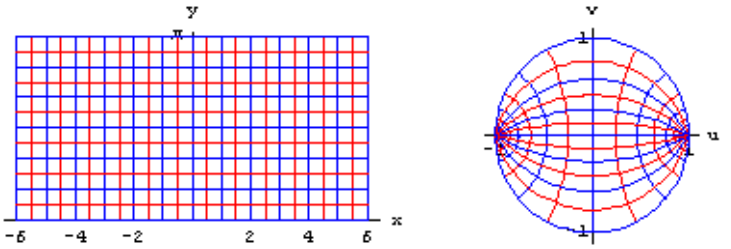


Figure 10.9 The conformal mapping $w = e^z$.

In this section we show how compositions of conformal transformations are used to construct mappings with specified characteristics.

Example 10.8. Show that the transformation $w = f(z) = \frac{e^z - i}{e^z + i}$ is a one-to-one conformal mapping of the horizontal strip $0 < y < \pi$ onto the disk $|w| < 1$. Furthermore, the x-axis is mapped onto the lower semicircle bounding the disk, and the line $y = \pi$ is mapped onto the upper semicircle.



Solution. The function $w = f(z) = \frac{e^z - i}{e^z + i}$ is the composition of $Z = e^z$ followed by $w = \frac{Z - i}{Z + i}$. The transformation $Z = e^z$ maps the horizontal strip $0 < y < \pi$ onto the upper half plane $\mathrm{Im}(Z) > 0$; the x axis is mapped on to the positive X axis; and the line $y = \pi$ is mapped onto the negative X axis. Then the bilinear transformation $w = \frac{Z - i}{Z + i}$ maps the upper half plane $\mathrm{Im}(Z) > 0$ onto the disk $|w| < 1$; the positive X axis is mapped onto the lower semicircle; and the negative X axis onto the upper semicircle. Figure 10.10 illustrates the composite mapping.

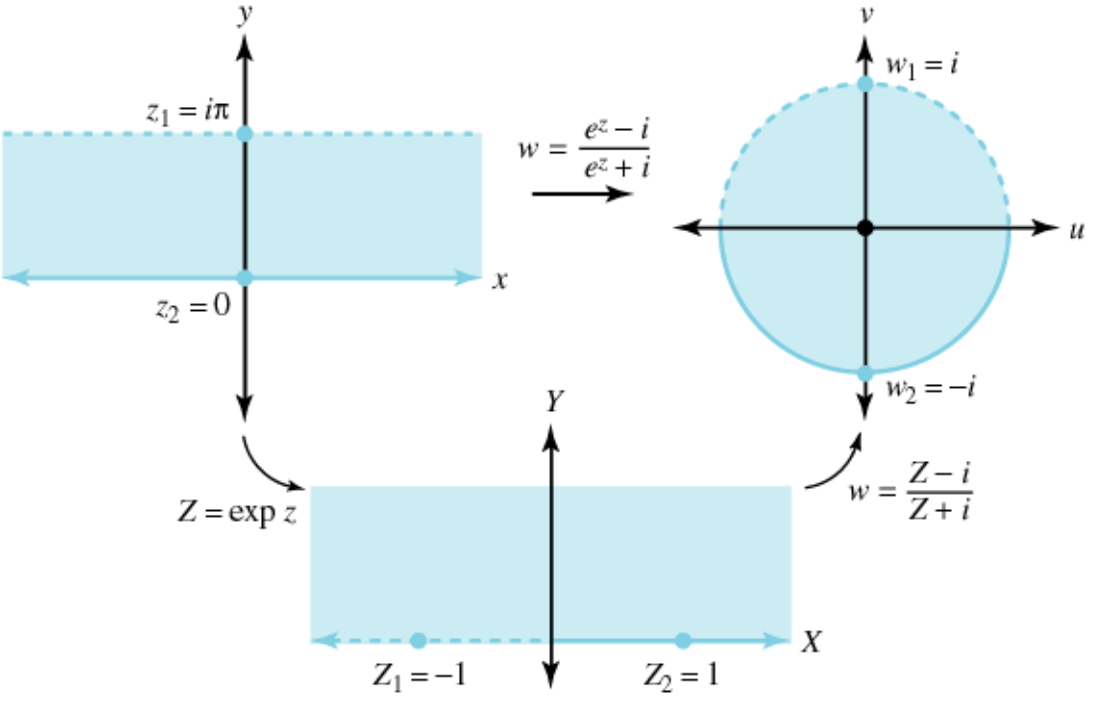
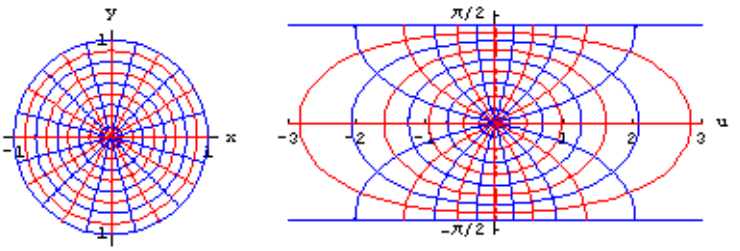


Figure 10.10 The composite transformation $w = \frac{e^z - i}{e^z + i}$.

Explore Solution 10.8.

Example10.9. Show that the transformation $w = f(z) = \mathrm{Log}\left(\frac{1 + z}{1 - z}\right)$ is a one-to-one conformal mapping of the unit disk $|z| < 1$ onto the horizontal strip $|v| < \frac{\pi}{2}$. Furthermore, the upper semicircle of the disk is mapped onto the line $v = \frac{\pi}{2}$ and the lower semicircle onto $v = \frac{-\pi}{2}$.



Solution. The function $w = f(z) = \text{Log}\left(\frac{1+z}{1-z}\right)$ is the composition of the bilinear transformation $Z = \frac{1+z}{1-z}$ followed by the logarithmic mapping $w = \text{Log}(Z)$. The image of the disk $|z| < 1$ under the bilinear mapping $Z = \frac{1+z}{1-z}$ is the right half-plane $\text{Re}(Z) > 0$; the upper semicircle is mapped onto the positive Y axis; and the lower semicircle is mapped onto the negative Y axis. The logarithmic function $w = \text{Log}(Z)$ then maps the right half-plane onto the horizontal strip; the image of the positive Y axis is the line $v = \frac{\pi}{2}$; and the image of the negative Y axis is the line $v = \frac{-\pi}{2}$. Figure 10.11 shows the composite mapping.

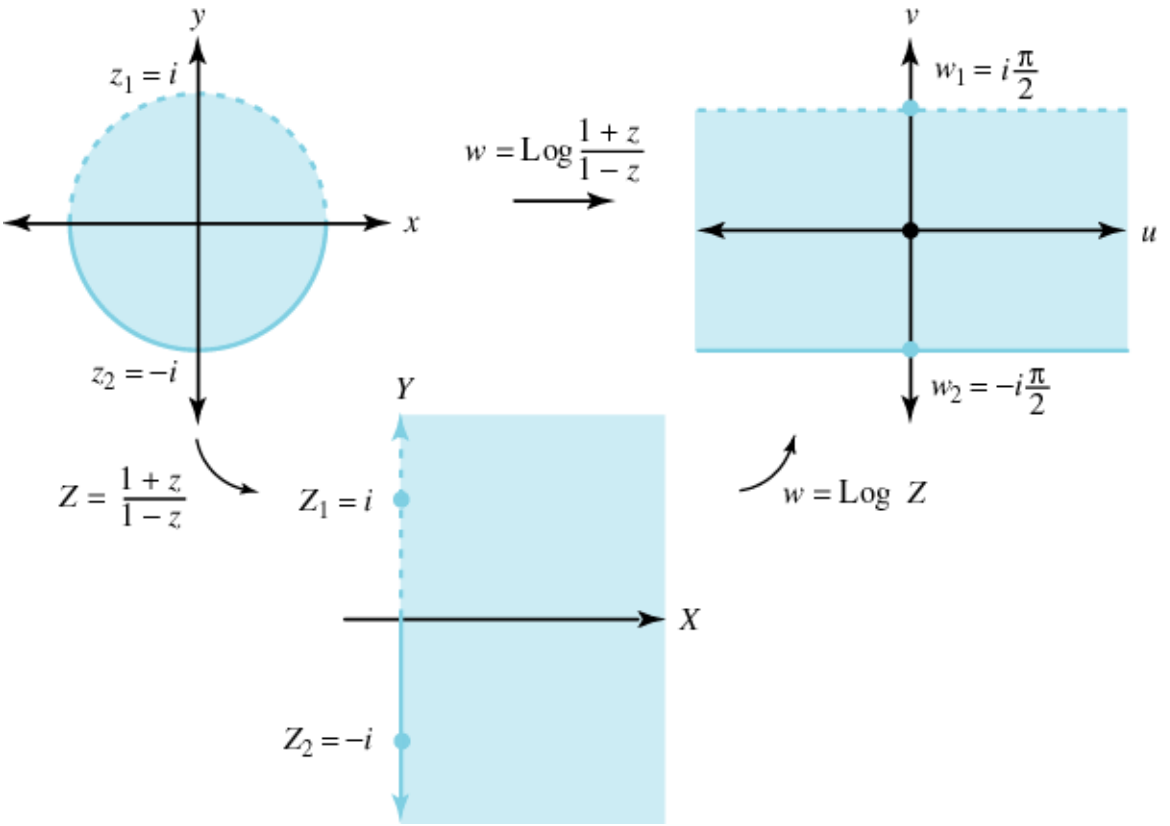
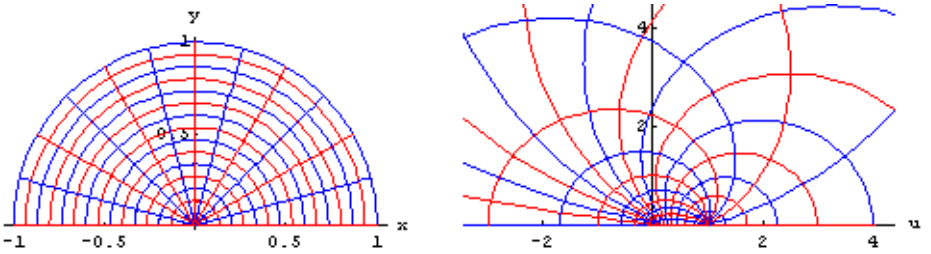


Figure 10.11 The composite transformation $w = \text{Log}\left(\frac{1+z}{1-z}\right)$.

Explore Solution 10.9.

Example 10.10. Show that the transformation $w = f(z) = \frac{(1+z)^2}{(1-z)^2} = \left(\frac{1+z}{1-z}\right)^2$ is a one-to-one conformal mapping of the portion of the unit disk $|z| < 1$ that lies in the upper half-plane $\text{Im}(z) > 0$ onto the upper half-plane $\text{Im}(w) > 0$. Furthermore, the upper semicircular portion of the boundary is mapped onto the line negative u-axis, and the segment $-1 < x < 1, y = 0$ is mapped onto the positive u-axis.



Solution. The function $w = f(z) = \frac{(1+z)^2}{(1-z)^2} = \left(\frac{1+z}{1-z}\right)^2$ is the composition of the bilinear transformation $Z = \frac{1+z}{1-z}$ followed by the mapping $w = Z^2$. The image of the half-disk under the bilinear mapping $Z = \frac{1+z}{1-z}$ is the first quadrant $X > 0, Y > 0$; the image of the segment $y = 0, -1 < x < 1$, is the positive X axis; and the image of the semicircle is the positive Y axis. The mapping $w = Z^2$ then maps the first quadrant in the Z plane onto the upper half-plane $\mathrm{Im}(w) > 0$, as shown in Figure 10.12.

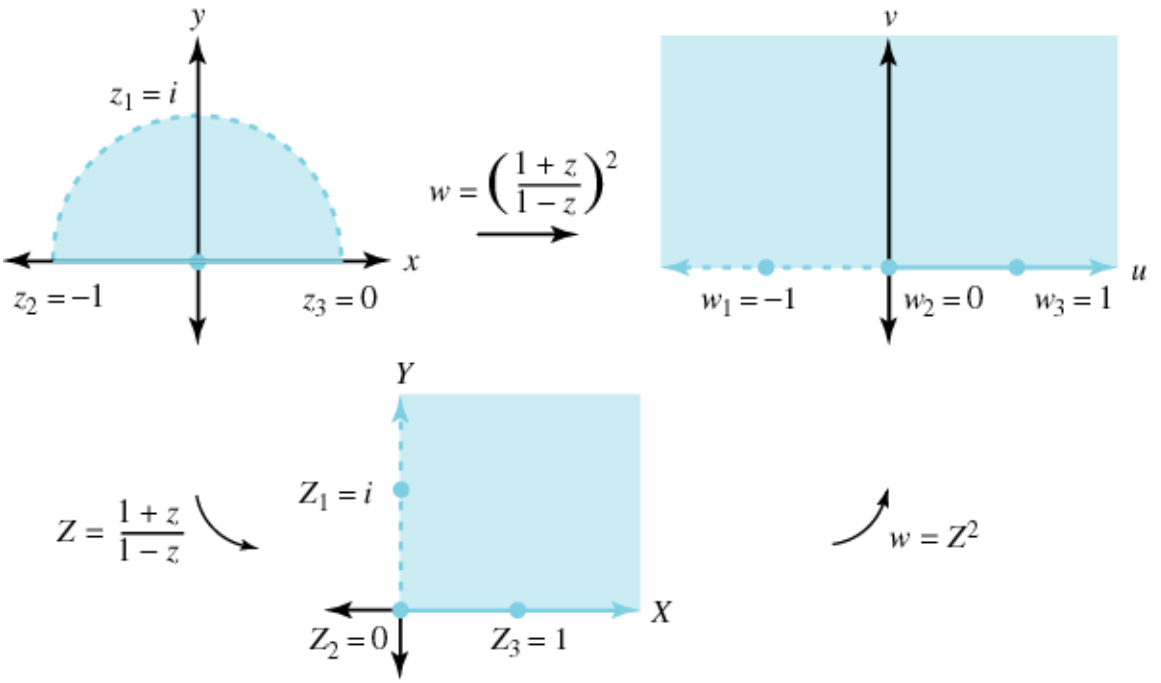
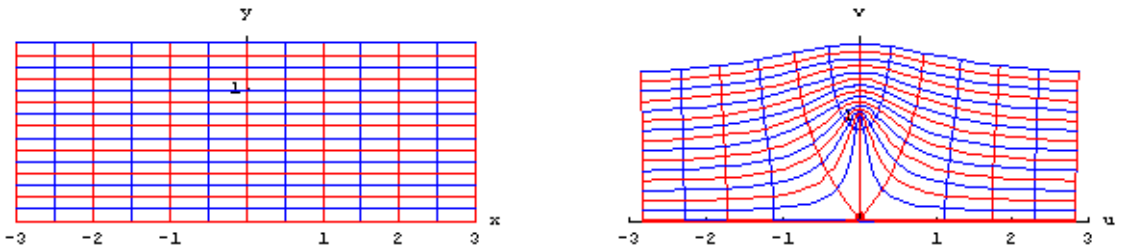


Figure 10.12 The composite transformation $w = \left(\frac{1+z}{1-z}\right)^2$.

Explore Solution 10.10.

Example 10.11. Consider the function $w = f(z) = (z^2 - 1)^{\frac{1}{2}}$, which is the composition of the functions $Z = z^2 - 1$ and $w = Z^{\frac{1}{2}}$ where the branch of the square root is $Z^{\frac{1}{2}} = R^{\frac{1}{2}}\left(\cos\left(\frac{\varphi}{2}\right) + i \sin\left(\frac{\varphi}{2}\right)\right)$, where $R = |Z|$, $\varphi = \arg(Z)$, and $0 \le \varphi < 2\pi$. Then the transformation $w = f(z)$ maps the upper half-plane $\mathrm{Im}(z) > 0$ one-to-one and onto the upper-half plane $\mathrm{Im}(w) > 0$ slit along the segment $u = 0, 0 < v \le 1$.



Solution. The function $Z = z^2 - 1$ maps the upper half-plane $\mathrm{Im}(z) > 0$ one-to-one and onto the Z-plane slit along the ray $Y = 0, X = -1$. Then the function $w = Z^{\frac{1}{2}}$ maps the slit plane onto the slit half-plane, as shown in Figure 10.13.

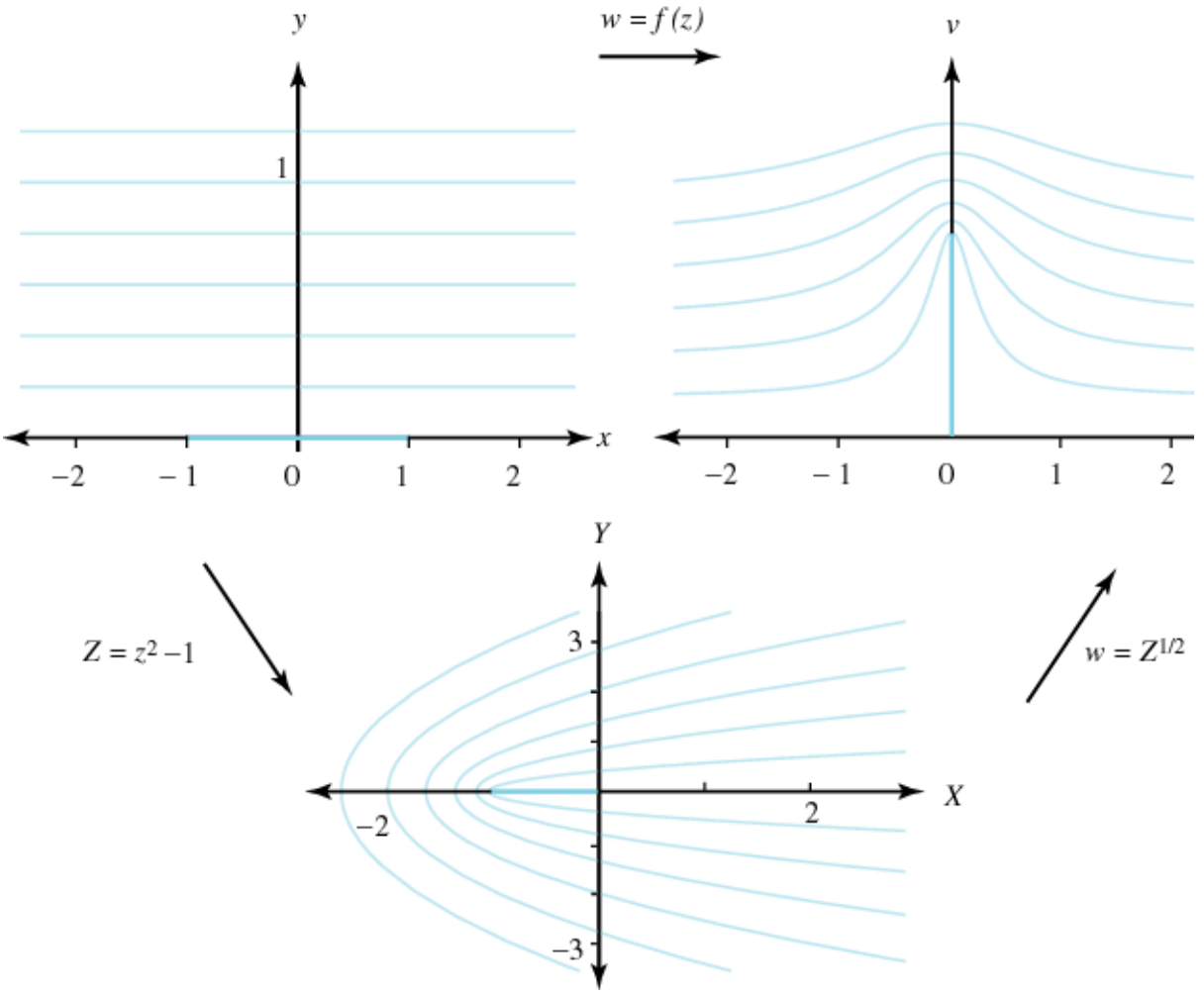


Figure 10.13 The composite transformation $w = (z^2 - 1)^{\frac{1}{2}}$ and the intermediate steps $Z = z^2 - 1$ and $w = Z^{\frac{1}{2}}$.

Explore Solution 10.11.

Remark 10.1. The images of the horizontal lines $y = b$ are curves in the w plane that bend around the segment from 0 to i. The curves represent the streamlines of a fluid flowing across the w plane. We discuss fluid flows in more detail in Section 10.7.

10.3.1 The Mapping $w = (z^2 - 1)^{1/2}$

The double-valued function $f(z) = (z^2 - 1)^{\frac{1}{2}}$ has a branch that is continuous for values of z distant from the origin. This feature is motivated by our desire for the approximation $(z^2 - 1)^{\frac{1}{2}} \approx z$ to hold for values of z distant from the origin. We begin by expressing $(z^2 - 1)^{\frac{1}{2}}$ as

(10-22) $$w = f_1(z) = (z - 1)^{\frac{1}{2}} (z + 1)^{\frac{1}{2}},$$

where the principal branch of the square root function is used in both factors. We claim that the mapping $w = f_1(z)$ is a one-to-one conformal mapping from the domain set D_1, consisting of the z plane slit along the segment $-1 \le x \le 1, y = 0$, onto the range set H_1, consisting of the w plane slit along the segment $u = 0, -1 \le v \le 1$.

To verify this claim, we investigate the two formulas on the right side of Equation (10-22) and express them in the form

$$(z - 1)^{\frac{1}{2}} = \sqrt{r_1}\, e^{i\theta_1/2}$$

where $r_1 = |z - 1|$ and $\theta_1 = \text{Arg}(z - 1)$, and

$$(z + 1)^{\frac{1}{2}} = \sqrt{r_2}\, e^{i\theta_2/2}$$

where $r_2 = |z + 1|$ and $\theta_2 = \text{Arg}(z + 1)$.

The discontinuities of θ_1 $\text{Arg}(z - 1)$ and $\text{Arg}(z + 1)$ are points on the real axis such that $x \le 1$ and $x \le -1$, respectively. We now show that $f_1(z)$ is continuous on the ray $x < -1, y = 0$.

We let $z_0 = x_0 + i\, y_0$ denote a point on the ray $x < -1, y = 0$ and then obtain the following limit as z approaches z_0 from the upper half-plane:

$$\lim_{\substack{x \to x_0 \\ \text{Im } z > 0}} f_1(z) = \lim_{\substack{x \to x_0 \\ \text{Im } z > 0}} \sqrt{r_1}\, e^{i\theta_1/2} \sqrt{r_2}\, e^{i\theta_2/2}$$

$$= \lim_{\substack{r \to |x_0 - 1| \\ \theta_1 \to \pi}} \sqrt{r_1}\, e^{i\theta_1/2} \lim_{\substack{r \to |x_0 + 1| \\ \theta_2 \to \pi}} \sqrt{r_2}\, e^{i\theta_2/2}$$

$$= \sqrt{|x_0 - 1|}\, e^{i\pi/2} \sqrt{|x_0 + 1|}\, e^{i\pi/2}$$

$$= \sqrt{|x_0 - 1|}\, (i) \sqrt{|x_0 + 1|}\, (i)$$

$$= i^2 \sqrt{|x_0 - 1|} \sqrt{|x_0 + 1|}$$

$$= -\sqrt{|x_0 - 1||x_0 + 1|}$$

$$= -\sqrt{|x_0^2 - 1|}$$

We let $z_0 = x_0 + i\, y_0$ denote a point on the ray $x < -1, y = 0$ and then obtain the following limits as z approaches z_0 from the lower half-plane:

$$\lim_{\substack{x\to x_0\\ \mathrm{Im}\,z<0}} f_1(z) = \lim_{\substack{x\to x_0\\ \mathrm{Im}\,z<0}} \sqrt{r_1}\, e^{i\theta_1/2}\sqrt{r_2}\, e^{i\theta_2/2}$$

$$= \lim_{\substack{r\to|x_0-1|\\ \theta_1\to-\pi}} \sqrt{r_1}\, e^{i\theta_1/2} \lim_{\substack{r\to|x_0+1|\\ \theta_2\to-\pi}} \sqrt{r_2}\, e^{i\theta_2/2}$$

$$= \sqrt{|x_0-1|}\, e^{-i\pi/2}\sqrt{|x_0+1|}\, e^{-i\pi/2}$$

$$= \sqrt{|x_0-1|}\,(-i)\sqrt{|x_0+1|}\,(-i)$$

$$= (-i)^2\sqrt{|x_0-1|}\sqrt{|x_0+1|}$$

$$= -\sqrt{|x_0-1||x_0+1|}$$

$$= -\sqrt{|x_0^2-1|}$$

We can easily find the inverse mapping and express it similarly:

$$z = g_1(w) = (w^2+1)^{\frac{1}{2}} = (w+i)^{\frac{1}{2}}(w-i)^{\frac{1}{2}},$$

where the branches of the square root function are given by

$$(w+i)^{\frac{1}{2}} = \sqrt{\rho_1}\, e^{i\phi_1/2}$$

where $\rho_1 = |w+i|$, $\phi_1 = \arg_{-\pi/2}(w+i)$, and $-\frac{\pi}{2} < \arg_{-\pi/2}(w+i) < \frac{3\pi}{2}$,

$$(w-i)^{\frac{1}{2}} = \sqrt{\rho_2}\, e^{i\phi_2/2}$$

where $\rho_2 = |w-i|$, $\phi_2 = \arg_{-\pi/2}(w-i)$, and $-\frac{\pi}{2} < \arg_{-\pi/2}(w-i) < \frac{3\pi}{2}$.

A similar argument shows that $g_1(w)$ is continuous for all w except those points that lie on the segment $u = 0,\ -1 \le v \le 1$. Verification that

$$g_1(f_1(z)) = z \quad \text{and} \quad f_1(g_1(w)) = w$$

hold for z in D_1 and w in H_1, respectively, is straightforward. Therefore we conclude that $w = f_1(z) = (z-1)^{\frac{1}{2}}(z+1)^{\frac{1}{2}}$ is a one-to-one mapping from D_1 onto H_1. Verifying that $f_1(z)$ is also analytic on the ray $x < -1,\ y = 0$, is tedious. We leave it as a challenging exercise.

10.3.2 The Riemann Surface for $w = (z^2-1)^{1/2}$

Using the other branch of the square root, we find that $w = f_2(z) = -f_1(z)$, is a one-to-one conformal mapping from the domain set D_2 consisting of the z-plane slit along the segment $-1 \le x \le 1,\ y = 0$, onto the range set H_2 consisting of the w-plane slit along the segment $u = 0,\ -1 \le v \le 1$. The sets D_1 and H_1 for $f_1(z)$ and D_2 and H_2 form the Riemann surface for the mapping, as shown in Figure 10.14.

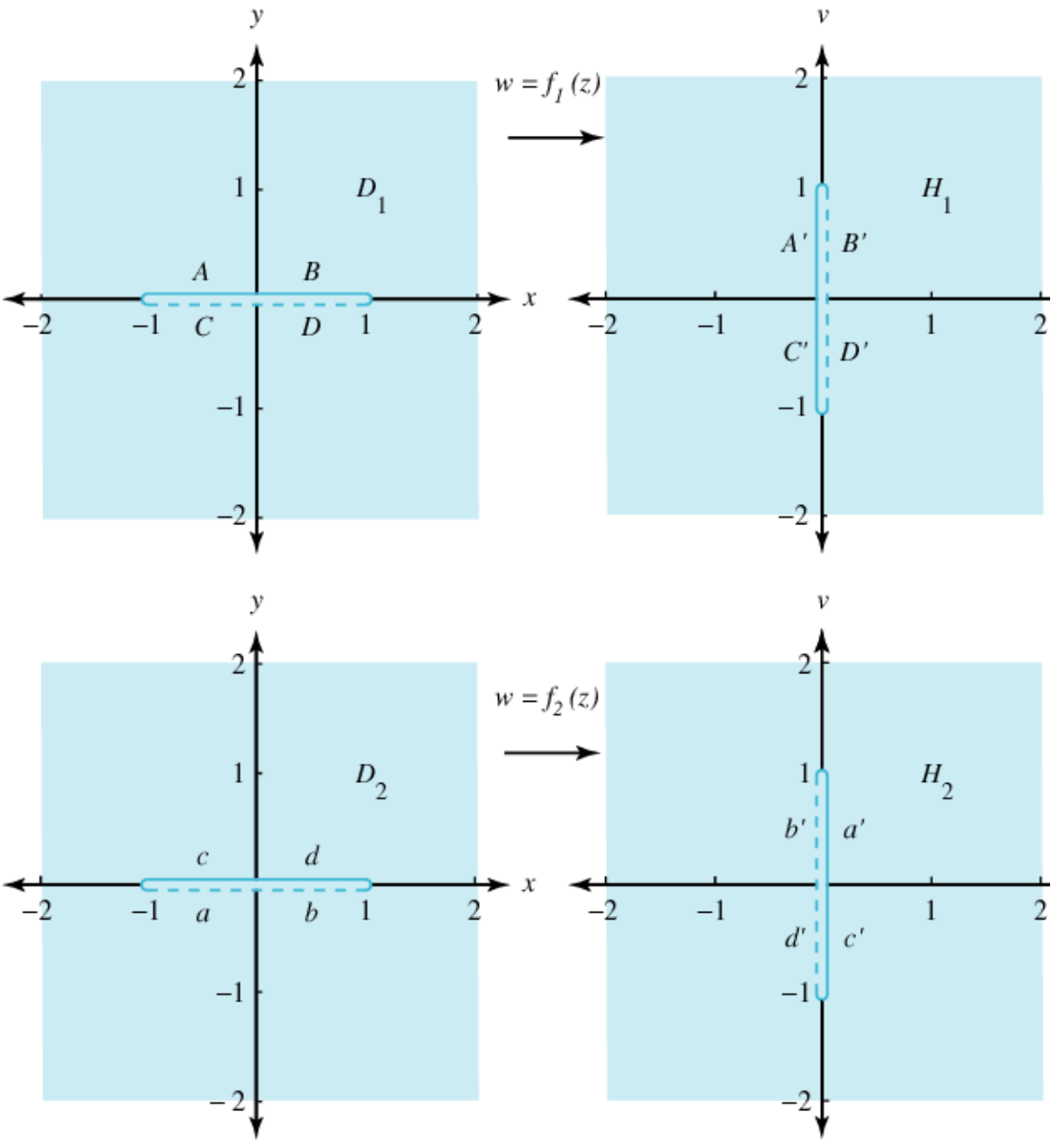


Figure 10.14 The mappings $w = f_1(z)$ and $w = f_2(z)$.

We obtain the Riemann surface for $w = (z^2 - 1)^{\frac{1}{2}}$ by gluing the edges of D_1 and D_2 together and the edges of H_1 and H_2 together. In the domain set, we glue edges A to a, B to b, C to c, and D to d. In the image set, we glue edges A′ to a′, B′ to b′, C′ to c′, and D′ to d′. The result is a Riemann domain surface and Riemann image surface for the mapping, as illustrated in Figures 10.15(a) and 10.15(b), respectively.

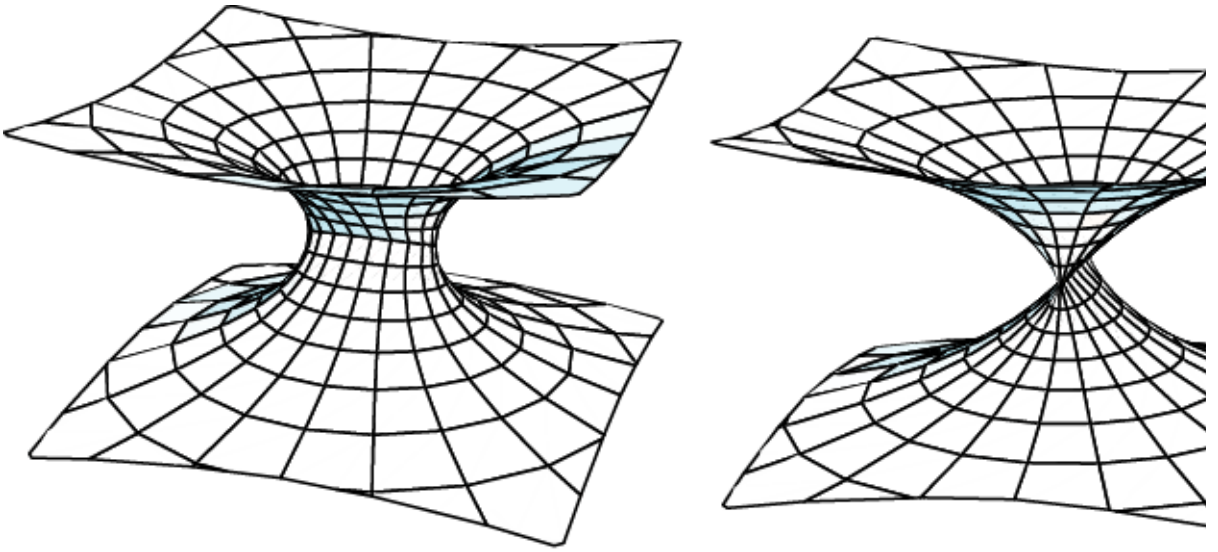


Figure 10.15 The Riemann surfaces for the mapping $w = (z^2 - 1)^{1/2}$.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

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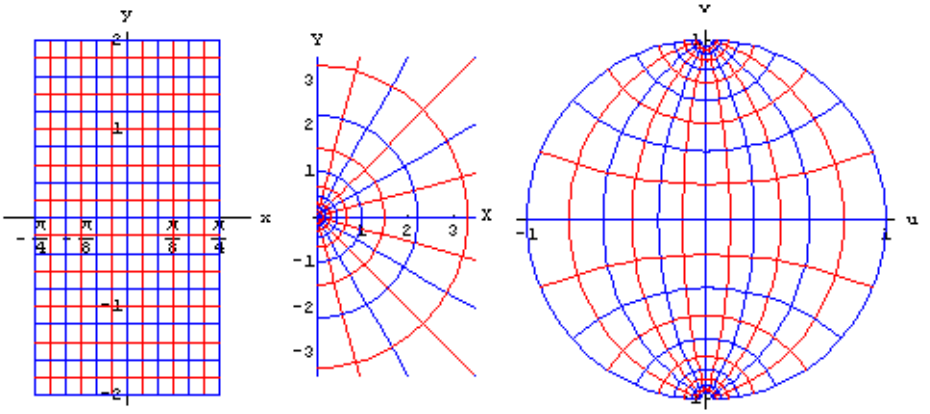
10.4 Mapping Involving Trigonometric Functions

The trigonometric functions can be expressed with compositions that involve the exponential function followed by a bilinear function. We can find images of certain regions by following the shapes of successive images in the composite mapping.

Example 10.12. The transformation $w = \tan z$ is a one-to-one conformal mapping of the vertical strip $|x| < \frac{\pi}{4}$ onto the disk $|w| < 1$.



Solution. Recall Equations (5-32) and (5-34), which are complex trigonometric identities:

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz}-e^{-iz}}{2i}.$$

Then we can write $w = \tan z = \frac{\sin z}{\cos z} = \frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = \frac{-i\,e^{i2z}+i}{e^{i2z}+1}$, and we use the formula

$$w = \tan z = \frac{-i\,e^{i2z}+i}{e^{i2z}+1}.$$

Then, mapping $w = \tan z$ can be considered to be the composition

$$w = \frac{-i\,Z + i}{Z + 1} \quad \text{and} \quad Z = e^{i2z}.$$

The transformation $Z = e^{i2z}$ maps the vertical strip $|x| < \frac{\pi}{4}$ one-to-one and onto the right half-plane $\mathrm{Re}(Z) > 0$. Then the bilinear transformation $w = \frac{-i\,Z + i}{Z + 1}$ maps the right half-plane one-to-one and onto the disk, as shown in Figure 10.16.

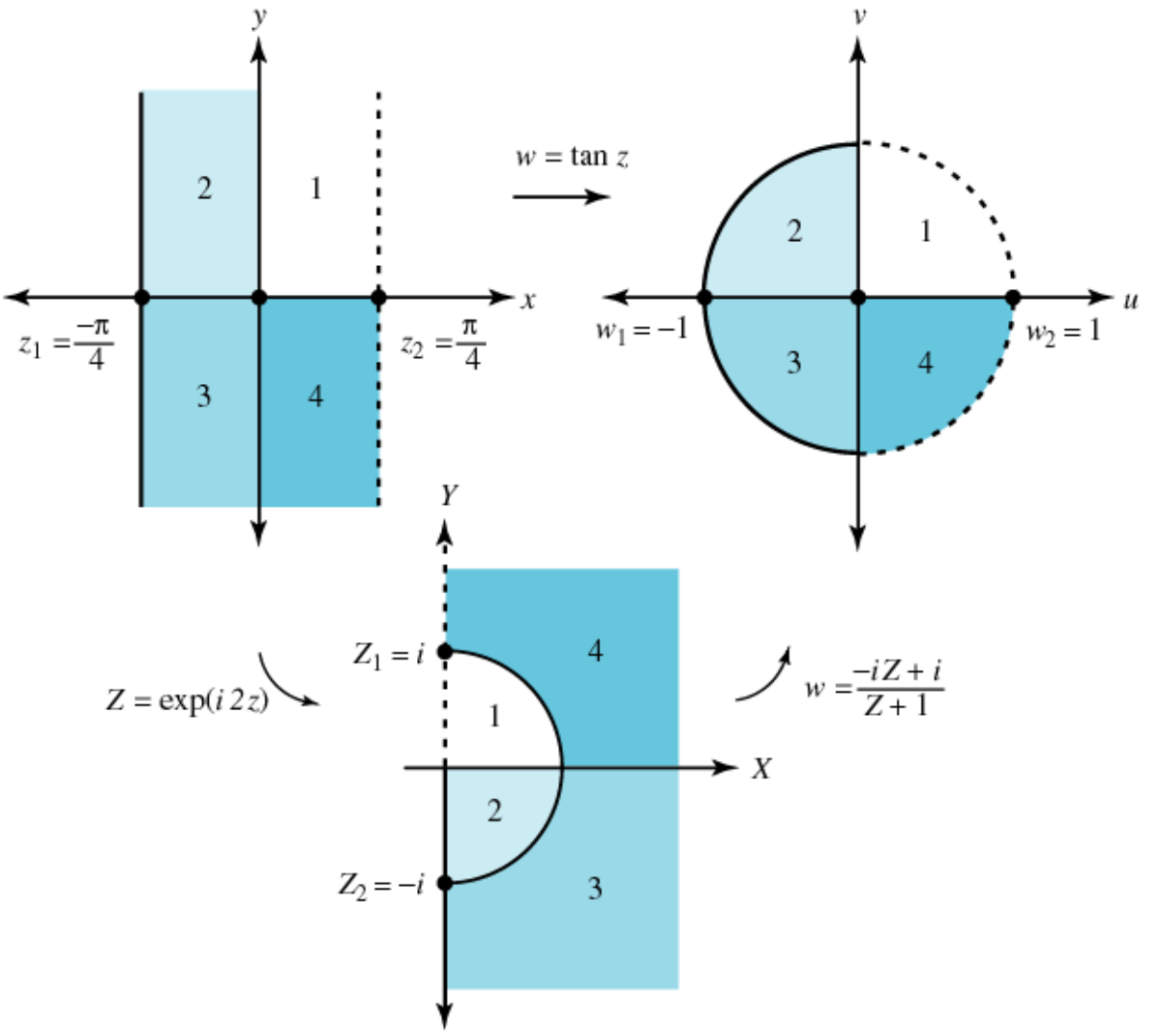
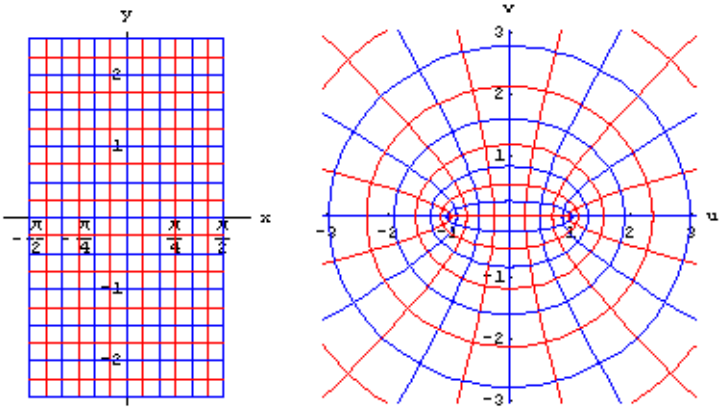


Figure 10.16 The composite transformation $w = \tan z$.

Explore Solution 10.12.

Example 10.13. Show that the transformation $w = f(z) = \sin z$ is a one-to-one conformal mapping of the vertical strip $|x| < \frac{\pi}{2}$ onto the w-plane slit along the rays $u \le -1,\ v = 0$ and $1 \le u,\ v = 0$.



Solution. Because $f'(z) = \cos z \neq 0$ for values of z satisfying the inequality $-\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}$, it follows that $w = \sin z$ is a conformal mapping. Using Equation (5-33), we write

$$u + i\,v = \sin z = \sin x \cosh y + i \cos x \sinh y.$$

If $|a| < \frac{\pi}{2}$, then the image of the vertical line $x = a$ is the curve in the w plane given by the parametric equations

$$u = \sin a \cosh y \quad \text{and} \quad v = \cos a \sinh y,$$

for $-\infty < y < \infty$. Next, we rewrite these equations as

$$\cosh y = \frac{u}{\sin a} \quad \text{and} \quad \sinh y = \frac{v}{\cos a}.$$

We now eliminate y from these equations by squaring and using the hyperbolic identity $\cosh^2 y - \sinh^2 y = 1$. The result is the single equation

(10-23) $$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1.$$

The curve given by Equation (10-23) is identified as a hyperbola in the uv plane that has foci at the points $(\pm 1, 0)$. Therefore the vertical line $x = a$ is mapped one-to-one onto the branch of the hyperbola given by Equation (10-23) that passes through the point $(\sin a, 0)$. If $0 < a < \frac{\pi}{2}$, then it is the right branch; if $-\frac{\pi}{2} < a < 0$, it is the left branch. The image of the y axis, which is the line $x = 0$, is the v axis. The images of several vertical lines are shown in Figure 10.17(a).

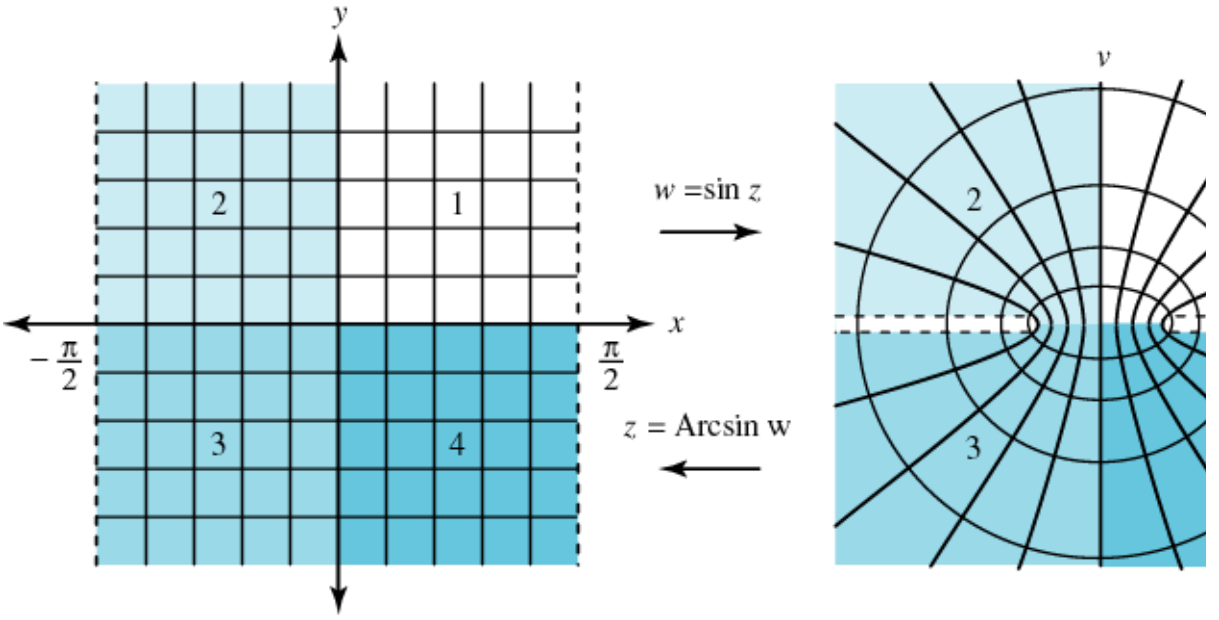


Figure 10.17 The transformation $w = \sin z$.

The image of the horizontal segment $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y = b$ is the curve in the w plane given by the parametric equations

$$u = \sin x \cosh b \quad \text{and} \quad v = \cos x \sinh b,$$

for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. We rewrite them as

$$\sin x = \frac{u}{\cosh b} \quad \text{and} \quad \cos x = \frac{v}{\sinh b}.$$

We now eliminate x from the equations by squaring and using the trigonometric identity $\cos^2 x + \sin^2 x = 1$. The result is the single equation

(10-24) $$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sinh^2 b} = 1.$$

The curve given by Equation (10-24) is identified as an ellipse in the uv plane that passes through the points $(\pm \cosh b, 0)$ and $(0, \pm \sinh b)$ and has foci at the points $(\pm 1, 0)$. Therefore, if $b > 0$, then $v = \cos x \sinh b > 0$, and the image of the horizontal segment is the portion of the ellipse given by Equation (10-24) that lies in the upper half-plane $\text{Im}(w) > 0$. If $b < 0$, then $v = \cos x \sinh b < 0$, and the image of the horizontal segment is the portion of the ellipse given by Equation (10-24) that lies in the lower half-plane $\text{Im}(w) < 0$. The images of several segments are shown in Figure 10.17(b).

Explore Solution 10.13.

10.4.1 The Complex Arcsine Function

We now develop explicit formulas for the real and imaginary parts of the principal value of the

arcsine function $w = f(z) = \text{Arcsin}\, z$. We use this mapping to solve problems involving steady temperatures and ideal fluid flow in Section 10.7. The mapping is found by solving the equation

(10-25) $$x + i\, y = \sin w = \sin u \cosh v + i \cos u \sinh v,$$

for u and v expressed as functions of x and y. To solve for u, we first equate the real and imaginary parts of Equation (10-25) and obtain the equations

$$\cosh v = \frac{x}{\sin u} \quad \text{and} \quad \sinh v = \frac{y}{\cos u}.$$

Then we eliminate v from these equations and obtain the single equation

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1.$$

If we treat u as a constant, this equation represents a hyperbola in the xy plane, the foci occur at the points $(\pm 1, 0)$, and the transverse axis is given by $2 \sin u$. Therefore a point (x, y) on the hyperbola must satisfy the equation

$$2 \sin u = \sqrt{(x+1)^2 + y^2} - \sqrt{(x-1)^2 + y^2}.$$

The quantity on the right side of this equation is the difference of the distances from (x, y) to $(-1, 0)$ and from (x, y) to $(1, 0)$. We now solve the equation for u to obtain the real part:

(10-26) $$u(x, y) = \text{Arcsin}\left(\frac{\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}}{2}\right).$$

The principal branch of the real function $\text{Arcsin}\, t$ is used in Equation (10-26), where the range values satisfy the inequality $-\frac{\pi}{2} < \text{Arcsin}\, t < \frac{\pi}{2}$.

Similarly, we can start with Equation (10-25) and obtain the equations

$$\sin u = \frac{x}{\cosh v} \quad \text{and} \quad \cos u = \frac{y}{\sinh v}$$

We then eliminate u from these equations and obtain the single equation

$$\frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1.$$

If we treat v as a constant, then this equation represents an ellipse in the xy plane, the foci occur at the points $(\pm 1, 0)$, and the major axis has length $2 \cosh v$. Therefore a point (x, y) on this ellipse must satisfy the equation

$$2 \cosh v = \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2}.$$

The quantity on the right side of this equation is the sum of the distances from (x, y) to $(-1, 0)$ and from (x, y) to $(1, 0)$.

The function $z = \sin w$ maps points in the upper half (lower half) of the vertical strip $-\frac{\pi}{2} < u < \frac{\pi}{2}$ onto the upper half-plane (lower half-plane), respectively. Hence we can solve the

preceding equation and obtain v as a function of x and y:

(10-27) $$v(x, y) = \text{sign}(y)\ \text{Arccosh}\left(\frac{\sqrt{(x+1)^2+y^2}+\sqrt{(x-1)^2+y^2}}{2}\right),$$

where $\text{sign}(y) = 1$, if $y > 0$, and $\text{sign}(y) = -1$, if $y < 0$. The real function given by $\text{Arccosh}\, t = \ln\left(t + \sqrt{(t^2 - 1)}\right)$ with t is used in Equation (10-27).

Therefore the mapping $w = \text{Arcsin}\, z$ is a one-to-one conformal mapping of the z plane cut along the rays $x \le -1,\ y = 0$, and $x \ge 1,\ y = 0$, onto the vertical strip $-\frac{\pi}{2} < u < \frac{\pi}{2}$ in the w plane, which can be construed from Figure 10.17 if we interchange the roles of the z and w planes. The image of the square $0 \le x \le 4,\ 0 \le y \le 4$, under $w = \text{Arcsin}\, z$, is shown in Figure 10.18. We obtained it by plotting the two families of curves $\{(u(c, t), v(c, t)) : 0 \le t \le 4\}$ and $\{(u(t, c), v(t, c)) : 0 \le t \le 4\}$, where $c = \frac{k}{5},\ k = 0, 1, \ldots, 20$.

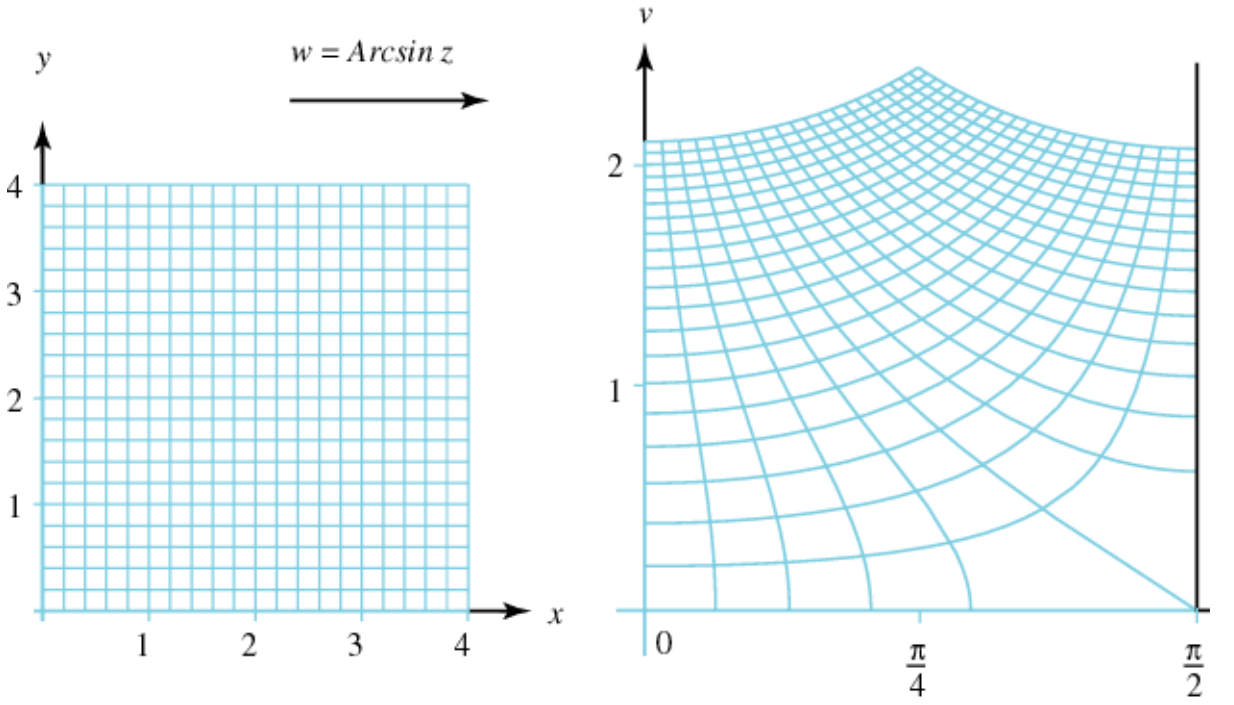
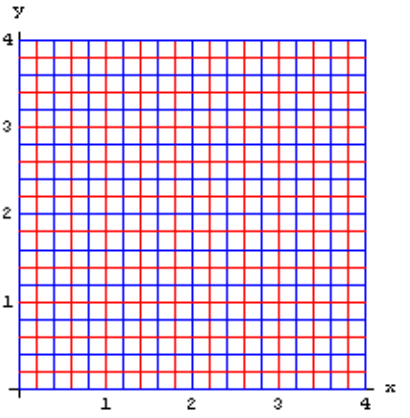
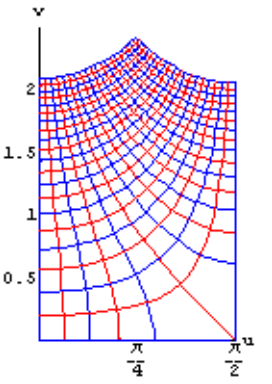


Figure 10.18 The mapping $w = \text{Arcsin}\, z$.

The formulas in Equations (10-26) and (10-27) are also convenient for evaluating $w = \text{Arcsin}\, z$, as shown in Example 10.14.

Extra Example 1. The transformation $w = f(z) = \text{Arcsin}\, z$ is a one-to-one conformal mapping of the first quadrant $x > 0,\ y > 0$ onto the semi-infinite strip $|u| < \frac{\pi}{2},\ v > 0$.





Explore Solution for Extra Example 1.

Example 10.14. Find the principal value of $\text{Arcsin}(1+i)$.

Solution. Using Formulas (10-26) and (10-27), we get

$$\text{Arcsin}(z) = \text{Arcsin}(x+iy) = u(x,y) + i\,v(x,y)$$

$$= \text{Arcsin}\left(\frac{\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}}{2}\right) + i\,\text{sign}(y)\,\text{Arccosh}\left(\frac{\sqrt{(x+1)^2+y^2} + \sqrt{(x-1)^2+y^2}}{2}\right)$$

Evaluating these equations at the point $z = 1+i$ or $(z, y) = (1, 1)$, we get

$$\text{Arcsin}(1+i) = u(1,1) + i\,v(1,1)$$

$$= \text{Arcsin}\left(\frac{\sqrt{2^2+1^2} - \sqrt{0^2+1^2}}{2}\right) + i\,\text{sign}(1)\,\text{Arccosh}\left(\frac{\sqrt{2^2+1^2} + \sqrt{0^2+1^2}}{2}\right)$$

$$= \text{Arcsin}\left(\frac{\sqrt{5}-1}{2}\right) + i\,\text{Arccosh}\left(\frac{\sqrt{5}+1}{2}\right)$$

$$= 0.6662394325 + i\,1.0612750619$$

Explore Solution 10.14.

Is there any reason to assume that there exists a conformal mapping for some specified domain D onto another domain G? The theorem concerning the existence of conformal mappings is attributed to Riemann and can be found in the book by Lars V. Ahlfors Complex Analysis (New York: McGraw-Hill Book Co.) Chapter 6, 1966.

Theorem 10.4 (Riemann Mapping Theorem). If D is any simply connected domain in the plane (other than the entire plane itself), then there exists a one-to-one conformal mapping w = f(z) that maps D onto the unit disk |w| < 1.

Proof.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

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Chapter 11 Applications of Harmonic Functions

11.1 Preliminaries

Overview

A wide variety of problems in engineering and physics involve harmonic functions, which are the real or imaginary part of an analytic function. The standard applications are two dimensional steady state temperatures, electrostatics, fluid flow and complex potentials. The techniques of conformal mapping and integral representation can be used to construct a harmonic function with prescribed boundary values. Noteworthy methods include Poisson's integral formulae; the Joukowski transformation; and Schwarz-Christoffel transformation. Modern computer software is capable of implemeting these complex analysis methods.

In most applications involving harmonic functions, a harmonic function that takes on prescribed values along certain contours must be found. In presenting the material in this chapter, we assume that you are familiar with the material covered in Sections 2.5, 3.3, 5.1, and 5.2. If you aren't, please review it before proceeding.

Example 11.1. Find the function $u(x,y)$ that is harmonic in the vertical strip $a < \mathrm{Re}(z) < b$ and takes on the boundary values

$$u(a,y) = U_1 \quad \text{for all } y, \text{ and}$$
$$u(b,y) = U_2 \quad \text{for all } y,$$

along the vertical lines $x = a$ and $x = b$, respectively.

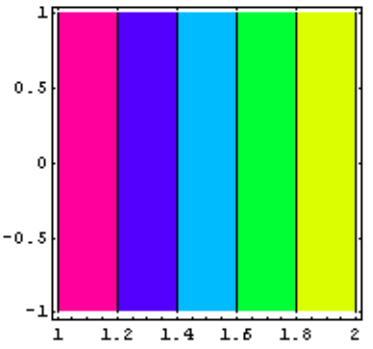

Figure 11.1 Level curves of the harmonic function

$u(x, y) = U_1 + \frac{U_2 - U_1}{b-a}(x - a)$.

Solution. Intuition suggests that we should seek a solution that takes on constant values along the vertical lines of the form $x = x_0$ and that $u(x, y)$ be a function of x alone; that is,

$$u(x, y) = P(x), \quad \text{for } a \le x \le b \text{ and for all } y.$$

Laplace's equation, $u_{xx}(x, y) + u_{yy}(x, y) = 0$, implies that $P''(x) = 0$, which implies $P(x) = m x + c$, where m and c are constants. The stated boundary conditions $u(a, y) = P(a) = U_1$ and $u(b, y) = P(b) = U_2$ lead to the solution

$$u(x, y) = U_1 + \frac{U_2 - U_1}{b-a}(x - a).$$

The level curves $u(x, y) = \text{constant}$ are vertical lines as indicated in Figure 11.1.

Explore Solution 11.1.

Example 11.2. Find the function $\Psi(x, y)$ that is harmonic in the sector $0 < \text{Arg}(z) < \alpha$ and takes on the boundary values

$$\Psi(x, 0) = C_1 \quad \text{for } x > 0,$$
$$\Psi(x, y) = C_2 \quad \text{for all points on the ray } r > 0, \theta = \alpha.$$

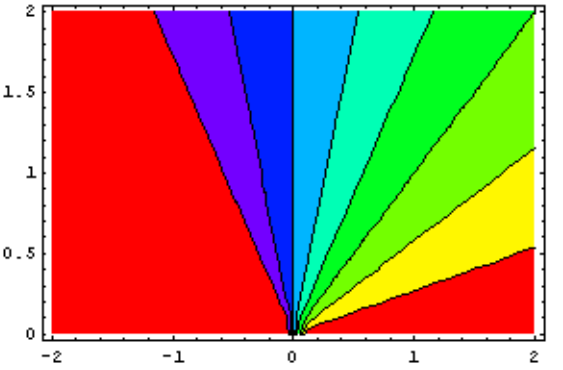

Figure 11.2 Level curves of the harmonic function

$\Psi(x, y) = C_1 + \frac{C_2 - C_1}{\alpha}\text{Arg } z$.

Solution. Recalling that the function $\text{Arg } z$ is harmonic and takes on constant values along rays emanating from the origin, we see that a solution has the form

$$\Psi(x, y) = a + b \operatorname{Arg} z,$$

where a and b are constants. The boundary conditions lead to

$$\Psi(x, y) = C_1 + \frac{C_2 - C_1}{\alpha} \operatorname{Arg} z.$$

The level curves $\Psi(x, y) = \text{constant}$ are rays emanating from the origin as indicated in Figure 11.2.

Explore Solution 11.2.

Example 11.3. Find the function $\Phi(x, y)$ that is harmonic in the annulus $1 < |z| < R$ and takes on the boundary values

$$\Phi(x, y) = K_1 \quad \text{when} \quad |z| = 1, \text{ and}$$
$$\Phi(x, y) = K_2 \quad \text{when} \quad |z| = R.$$

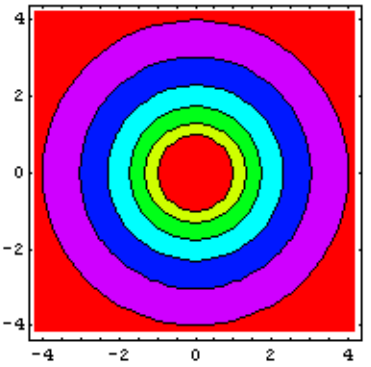

Figure 11.3 Level curves of the harmonic function $\Phi(x, y) = K_1 + \frac{K_2 - K_1}{\ln R} \ln |z|$.

Solution. This problem is a companion to the one in Example 11.2. Here we use the fact that $\ln |z|$ is a harmonic function, for all $z \neq 0$. The solution is

$$\Phi(x, y) = K_1 + \frac{K_2 - K_1}{\ln R} \ln |z|,$$

and the level curves $\Phi(x, y) = \text{constant}$ are concentric circles, as illustrated in Figure 11.3.

Explore Solution 11.3.

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11.2 Invariance of Laplace's Equation and the Dirichlet Problem

Let D be a domain whose boundary is made up of piecewise smooth contours joined end to end. The Dirichlet problem is to find a function $\phi(u, y)$ that is harmonic in D such that $\phi(u, y)$ takes on prescribed values at points on the boundary. Let us first study the problem in the upper half plane.

Theorem 11.1 (Invariance of Laplace's Equation). Let $\Phi(u, v)$ be harmonic in a domain G in the w-plane. Then $\Phi(u, v)$ satisfies Laplace's equation

(11-1) $$\Phi_{uu}(u, v) + \Phi_{vv}(u, v) = 0,$$

at each point $w = u + i v$ in G. If $w = f(z) = u(x, y) + i v(x, y)$ is a conformal mapping from a domain D in the z-plane onto G, then the composition

(11-2) $$\phi(x, y) = \Phi(u(x, y), v(x, y)),$$

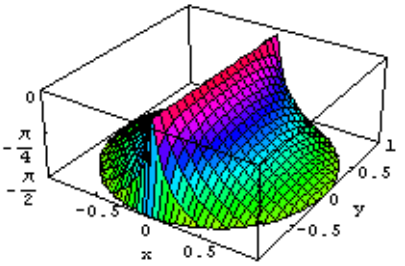
is harmonic in D, and $\phi(x, y)$ satisfies Laplace's equation

(11-3) $$\phi_{xx}(u, y) + \phi_{yy}(x, y) = 0,$$

at each point $z = x + i y$ in D.

Proof.

Example 11.4. Show that $\phi(x, y) = \text{Arctan}\left(\frac{2x}{x^2 + y^2 - 1}\right)$ is harmonic in the disk $|z| < 1$.



Solution. The results of Exercise 7(b), of Section 10.2, show that the function

$$f(z) = \frac{i+z}{i-z}$$

is a conformal mapping of the disk $|z| < 1$ onto the right half-plane $\mathrm{Re}(w) > 0$, it can be rewritten as

$$f(x+iy) = \frac{i+x+iy}{i-x-iy} = \frac{x+i(1+y)}{-x+i(1-y)}$$

$$= \frac{(x+i(1+y))(-x-i(1-y))}{(-x+i(1-y))(-x-i(1-y))}$$

$$= \frac{1-x^2-y^2-2ix}{x^2+(1-y)^2}$$

$$= \frac{1-x^2-y^2}{x^2+(1-y)^2} - i\frac{2x}{x^2+(1-y)^2}$$

The results from Exercise 7(b), Section 5.2, show that the function

$$\Phi(u,v) = \mathrm{Arctan}\left(\frac{v}{u}\right) = \mathrm{Arg}(u+iv)$$

is harmonic in the right half-plane $\mathrm{Re}(w) > 0$. Taking the real and imaginary parts of $f(x+iy)$, we have

$$u(x,y) = \frac{1-x^2-y^2}{x^2+(1-y)^2} \quad \text{and} \quad v(x,y) = \frac{-2x}{x^2+(1-y)^2}.$$

Substituting these equations into the formula for $\Phi(u,v)$ and using (11-2), we find that

$$\phi(x,y) = \mathrm{Arctan}\left(\frac{v(x,y)}{u(x,y)}\right) = \mathrm{Arctan}\left(\frac{2x}{x^2+y^2-1}\right) \text{ is harmonic for } |z| < 1.$$

Explore Solution 11.4.

Let D be a domain whose boundary is made up of piecewise smooth contours joined end to end. The Dirichlet problem is to find a function Φ that is harmonic in D such that takes on prescribed values at points on the boundary. Let's first look at this problem in the upper half-plane.

Example 11.5. Show that the function

(11-4) $$\Phi(u, v) = \frac{1}{\pi}\,\text{Arctan}\left(\frac{v}{u - u_0}\right) = \frac{1}{\pi}\,\text{Arg}(w - u_0)$$

is harmonic in the upper half plane $\text{Im}(w) > 0$ and has boundary values

$$\Phi(u, 0) = 0, \quad \text{for} \quad u > u_0;$$

$$\Phi(u, 0) = 1, \quad \text{for} \quad u < u_0.$$

Solution. The function

$$g(w) = \frac{1}{\pi}\,\text{Log}(w - u_0) = \frac{1}{\pi}\ln|w - u_0| + \frac{i}{\pi}\,\text{Arg}(w - u_0)$$

is analytic in the upper half-plane $\text{Im}(w) > 0$, and its imaginary part is the harmonic function $\Phi(u, v) = \frac{1}{\pi}\,\text{Arg}(w - u_0)$.

Explore Solution 11.5.

Remark 11.1 Let t be a real number. We shall use the convention $\text{Arctan}(\pm\infty) = \frac{\pi}{2}$ so that the function $\text{Arctan}(t)$ denotes the branch of the inverse tangent that lies in the range $0 < \text{Arctan}(t) < \pi$. Doing so permits us to write the solution in equation (11-4) as $\Phi(u, v) = \frac{1}{\pi}\,\text{Arctan}\left(\frac{v}{u - u_0}\right)$.

Theorem 11.2 (N-Value Dirichlet Problem for the Upper Half Plane). Let $u_1 < u_2 < \ldots < u_{N-1}$ denote $N - 1$ real constants. The function

(11-5) $$\Phi(u, v) = a_{N-1} + \frac{1}{\pi}\sum_{k=1}^{N-1}(a_{k-1} - a_k)\,\text{Arg}(w - u_k)$$

$$\Phi(u, v) = a_{N-1} + \frac{1}{\pi}\sum_{k=1}^{N-1}(a_{k-1} - a_k)\,\text{Arctan}\left(\frac{v}{u - u_k}\right)$$

is harmonic in the upper half plane $\text{Im}(w) > 0$ and takes on the boundary values

$$\Phi(u, 0) = a_0, \quad \text{for} \quad u < u_1;$$

$$\Phi(u, 0) = a_k, \quad \text{for} \quad u_k < u < u_{k+1} \text{ for } k = 1, 2, \ldots, N - 2;$$

$$\Phi(u, 0) = a_{N-1}, \quad \text{for} \quad u > u_{N-1}.$$

The situation is illustrated in Figure 11.4.

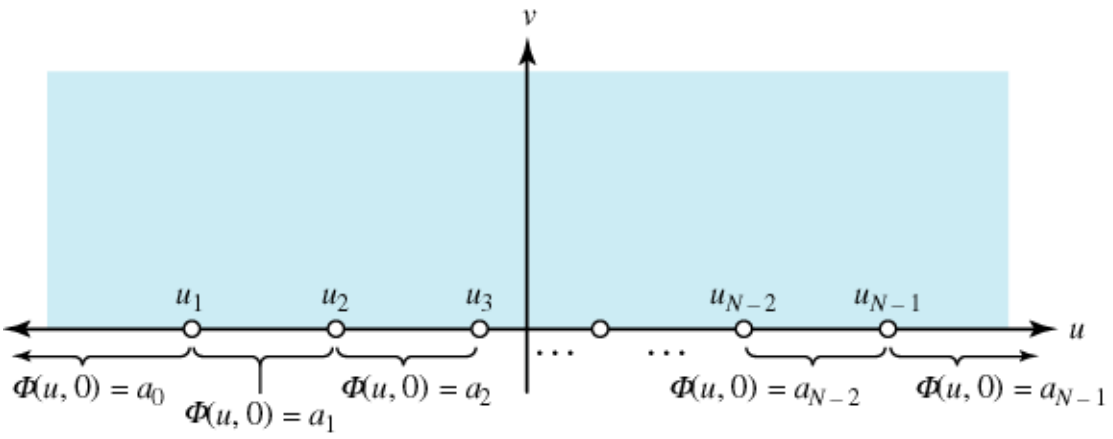


Figure 11.4 The boundary conditions for the harmonic function $\Phi(u, v)$.

Proof.

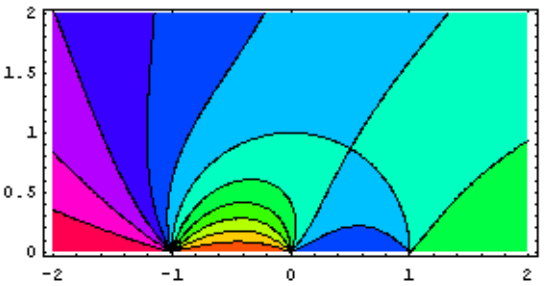
Example 11.6. Find a function $\phi(x, y)$ that is harmonic in the upper half-plane $\mathrm{Im}(z) > 0$ and takes on the boundary values indicated in Figure 11.5. That is

$$\phi(x, 0) = 4, \quad \text{for} \quad x < -1,$$

$$\phi(x, 0) = 1, \quad \text{for} \quad -1 < x < 0,$$

$$\phi(x, 0) = 3, \quad \text{for} \quad 0 < x < 1,$$

$$\phi(x, 0) = 2, \quad \text{for} \quad 1 < x.$$



Solution. This is a four-value Dirichlet problem in the upper half-plane defined by $\mathrm{Im}(z) > 0$. For the z plane, the solution in Equation (11-5) becomes

$$\phi(x, y) = a_3 + \frac{1}{\pi}\sum_{k=1}^{3}(a_{k-1} - a_k)\,\text{Arg}(z - x_k)$$

$$\phi(x, y) = a_3 + \frac{a_0 - a_1}{\pi}\text{Arg}(z - x_1) + \frac{a_1 - a_2}{\pi}\text{Arg}(z - x_2) + \frac{a_2 - a_3}{\pi}\text{Arg}(z - x_3)$$

Here we have $a_0 = 4$, $a_1 = 1$, $a_2 = 3$, and $a_3 = 2$ and $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$, which we substitute into equation for $\phi(x, y)$ to obtain

$$\phi(x, y) = 2 + \frac{4-1}{\pi}\text{Arg}(z - (-1)) + \frac{1-3}{\pi}\text{Arg}(z - 0) + \frac{3-2}{\pi}\text{Arg}(z - 1)$$

$$= 2 + \frac{3}{\pi}\text{Arg}(z + 1) - \frac{2}{\pi}\text{Arg}(z) + \frac{1}{\pi}\text{Arg}(z - 1)$$

$$= 2 + \frac{3}{\pi}\text{Arg}(x + 1 + i\,y) - \frac{2}{\pi}\text{Arg}(x + i\,y) + \frac{1}{\pi}\text{Arg}(x - 1 + i\,y)$$

$$= 2 + \frac{3}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right) - \frac{2}{\pi}\text{Arctan}\left(\frac{y}{x}\right) + \frac{1}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right)$$

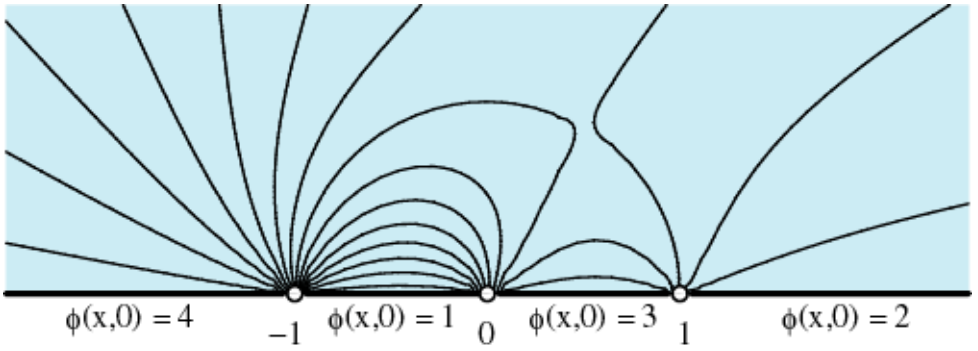
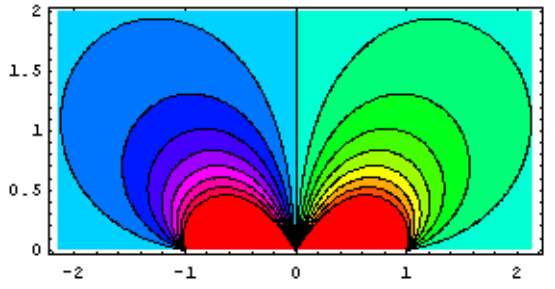


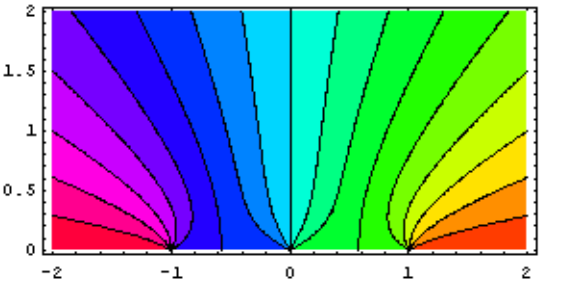
Figure 11.5 The boundary values for the Dirichlet problem.

Explore Solution 11.6.

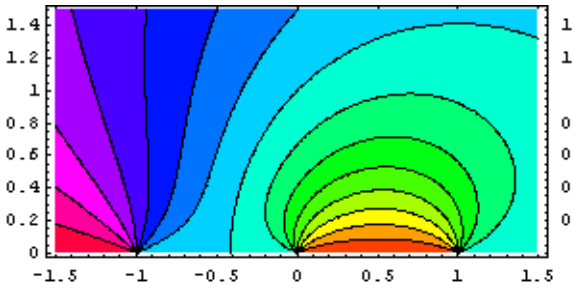
Extra Examples1 (a-d). Find a function $\phi(x, y)$ that is harmonic in the upper half-plane $\text{Im}(z) > 0$ and takes on the indicated boundary values.



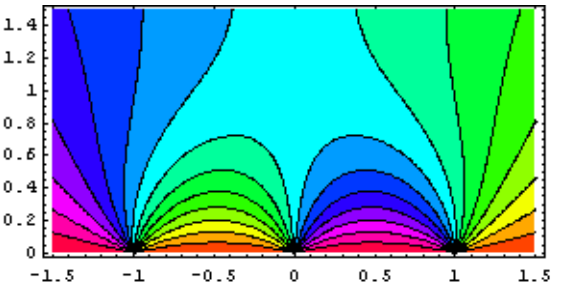
(a) $a_0 = 0$, $a_1 = 1$, $a_2 = -1$, $a_3 = 0$



(b) $a_0 = 3$, $a_1 = 2$, $a_2 = 1$, $a_3 = 0$



(c) $a_0 = 2,\ a_1 = 1,\ a_2 = 0,\ a_3 = 1$



(d) $a_0 = 1,\ a_1 = 0,\ a_2 = 1,\ a_3 = 0$

Explore Solution Extra Example 1 (a).

Explore Solution Extra Example 1 (b).

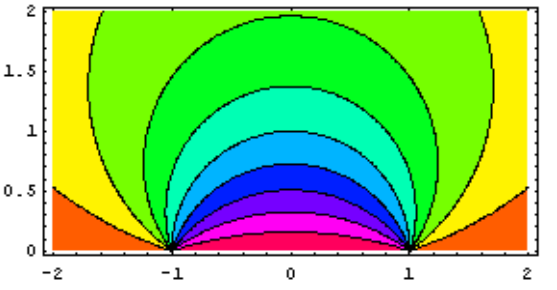
Explore Solution Extra Example 1 (c).

Explore Solution Extra Example 1 (d).

Example 11.7. Find a function $\phi(x, y)$ that is harmonic in the upper half-plane $\text{Im}[z] > 0$, which takes on the boundary values

$$\phi(x, 0) = 1, \quad \text{for} \quad |x| < 1;$$

$$\phi(x, 0) = 0, \quad \text{for} \quad |x| > 1.$$



Solution. This is a three-value Dirichlet problem in the upper half-plane defined by $\text{Im}(z) > 0$. For the z plane, the solution in Equation (11-5) becomes

$$\phi(x, y) = a_2 + \frac{1}{\pi}\sum_{k=1}^{2}(a_{k-1} - a_k)\,\text{Arg}(z - x_k)$$

$$\phi(x, y) = a_2 + \frac{a_0 - a_1}{\pi}\text{Arg}(z - x_1) + \frac{a_1 - a_2}{\pi}\text{Arg}(z - x_2)$$

Here we have $a_0 = 0$, $a_1 = 1$, and $a_2 = 0$ and $x_1 = -1$, and $x_2 = 1$, which we substitute into the above equation for $\phi(x, y)$ to obtain

$$\phi(x, y) = 0 + \frac{0-1}{\pi}\,\text{Arg}(z-(-1)) + \frac{1-0}{\pi}\,\text{Arg}(z-1)$$

$$= -\frac{1}{\pi}\,\text{Arg}(z+1) + \frac{1}{\pi}\,\text{Arg}(z-1)$$

$$= -\frac{1}{\pi}\,\text{Arg}(x+1+i\,y) + \frac{1}{\pi}\,\text{Arg}(x-1+i\,y)$$

$$= -\frac{1}{\pi}\,\text{Actan}\left(\frac{y}{x+1}\right) + \frac{1}{\pi}\,\text{Actan}\left(\frac{y}{x-1}\right)$$

A three-dimensional graph of $\phi(x, y)$ is shown in Figure 11.6.

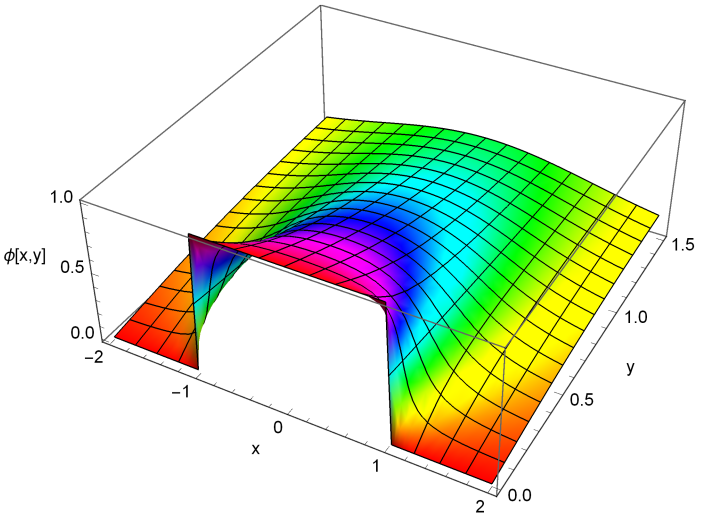


Figure 11.6 The graph of $\phi(x, y) = -\frac{1}{\pi}\,\text{Actan}\left(\frac{y}{x+1}\right) + \frac{1}{\pi}\,\text{Actan}\left(\frac{y}{x-1}\right)$ with the boundary values $\phi(x, 0) = 1$, for $|x| < 1$, and $\phi(x, 0) = 0$, for $|x| > 1$.

Explore Solution 11.7.

We now state the N-value Dirichlet problem for a simply connected domain. We let D be a simply connected domain bounded by the simple closed contour C and let $z_1, z_2, \ldots, z_{N-1}, z_N$ denote N points that lie along C in this specified order as C is traversed in the positive direction (counterclockwise). Then we let C_k denote the portion of C that lies strictly between z_k and z_{k+1}, for $k = 1, 2, \ldots, N-1$, and let C_N denote the portion that lies strictly between z_N and z_1. Finally, we let $a_1, a_2, \ldots, a_{N-1}, a_N$ be real constants.

We want to find a function $\phi(x, y)$ that is harmonic in D and continuous on $D \cup C_1 \cup C_2 \cup \ldots \cup C_{N-1} \cup C_N$ that takes on the boundary values

$$\phi(x, y) = a_1, \quad \text{for } z = x + i\,y \text{ on } C_1;$$

$$\phi(x, y) = a_2, \quad \text{for } z = x + i\,y \text{ on } C_2;$$

(11-6)
$$\phi(x, y) = a_k, \quad \text{for } z = x + i\,y \text{ on } C_k \text{ for } k = 1, 2, \ldots, N;$$

$$\phi(x, y) = a_N, \quad \text{for } z = x + i\,y \text{ on } C_N.$$

The situation is illustrated in Figure 11.7.

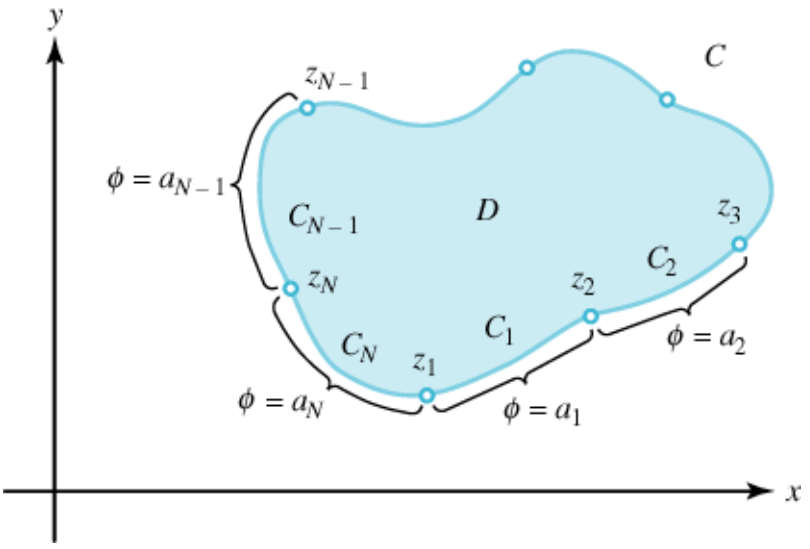


Figure 11.7 The boundary values for $\phi(x, y)$ for the Dirichlet problem in the simply connected domain D.

One method for finding $\phi(x, y)$ is to find a conformal mapping

(11-7) $$w = f(z) = u(x, y) + i\, v(x, y)$$

of D onto the upper half-plane $\mathrm{Im}(w) > 0$, such that the N points $z_1, z_2, \ldots, z_{N-1}, z_N$ are mapped onto the points $u_k = f(z_k)$, for $k = 1, \ldots, N-1$, and z_N is mapped onto $u_N = +\infty$ along the u axis in the w-plane.

When we use Theorem 11.1, the mapping in Equation (11-7) gives rise to a new N-value Dirichlet problem in the upper half-plane $\mathrm{Im}(w) > 0$ for which the solution is given by Theorem 11.2. If we set $a_0 = a_N$, then the solution to the Dirichlet problem in D with the boundary values from Equation (11-6) is

$$\phi(x, y) = a_{N-1} + \frac{1}{\pi}\sum_{k=1}^{N-1}(a_{k-1} - a_k)\,\mathrm{Arg}(f(z) - u_k)$$

$$\phi(x, y) = a_{N-1} + \frac{1}{\pi}\sum_{k=1}^{N-1}(a_{k-1} - a_k)\,\mathrm{Arctan}\left(\frac{v(x,y)}{u(x,y) - u_k}\right)$$

This method relies on our ability to construct a conformal mapping from D onto the upper half-plane $\mathrm{Im}(w) > 0$. Theorem 10.4 guarantees the existence of such a conformal mapping.

Example 11.8. Find a function $\phi(x, y)$ that is harmonic in the unit disk $|z| < 1$, which takes on the boundary values

(11-8) $$\phi(x, y) = 0, \quad \text{for } x + i\, y = e^{i\theta}, \quad 0 < \theta < \pi;$$

$$\phi(x, y) = 1, \quad \text{for } x + i\, y = e^{i\theta}, \quad \pi < \theta < 2\pi.$$

Solution. Example 10.3 showed that the transformation

$$w = u + i\,v = \frac{i\,(1-z)}{1+z} = \frac{i\,(1-x-i\,y)}{1+x+i\,y}$$

is a one-to-one conformal mapping of the unit disk $|z| < 1$ onto the upper half-plane $\mathrm{Im}(w) > 0$, which can be written as

(11-9)
$$u + i\,v = \frac{i\,(1-x-i\,y)\,(1+x-i\,y)}{(1+x+i\,y)\,(1+x-i\,y)}$$

$$= \frac{2\,y+i\,\left(1-x^2-y^2\right)}{1+2\,x+x^2+y^2}$$

$$= \frac{2\,y}{(x+1)^2+y^2} + i\,\frac{1-x^2-y^2}{(x+1)^2+y^2}$$

Equation (11-9) reveals that the points $z = x + i\,y$ lying on the upper semicircle $y > 0,\ 1 - x^2 - y^2 = 0$ are mapped onto the positive u axis. Similarly, the lower semicircle is mapped onto the negative u axis, as shown in Figure 11.8. The mapping given by Equation (11-9) gives rise to a new Dirichlet problem of finding a harmonic function $\Phi(u, v)$ that has the boundary values

$$\Phi(u, 0) = 0, \quad \text{for } u > 0, \quad \text{and}$$

$$\Phi(u, 0) = 1, \quad \text{for } u < 0,$$

as shown in Figure 11.8. Using the result of Example 11.5 and the functions $u(x, y) = \frac{2\,y}{(x+1)^2+y^2}$ and $v(x, y) = \frac{1-x^2-y^2}{(x+1)^2+y^2}$ from Equation (11-9), we get the solution to Equations (11-9):

$$\phi(x, y) = \frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{v(x,y)}{u(x,y)}\right) = \frac{1}{\pi}\,\mathrm{Arctan}\left(\frac{1-x^2-y^2}{2\,y}\right).$$

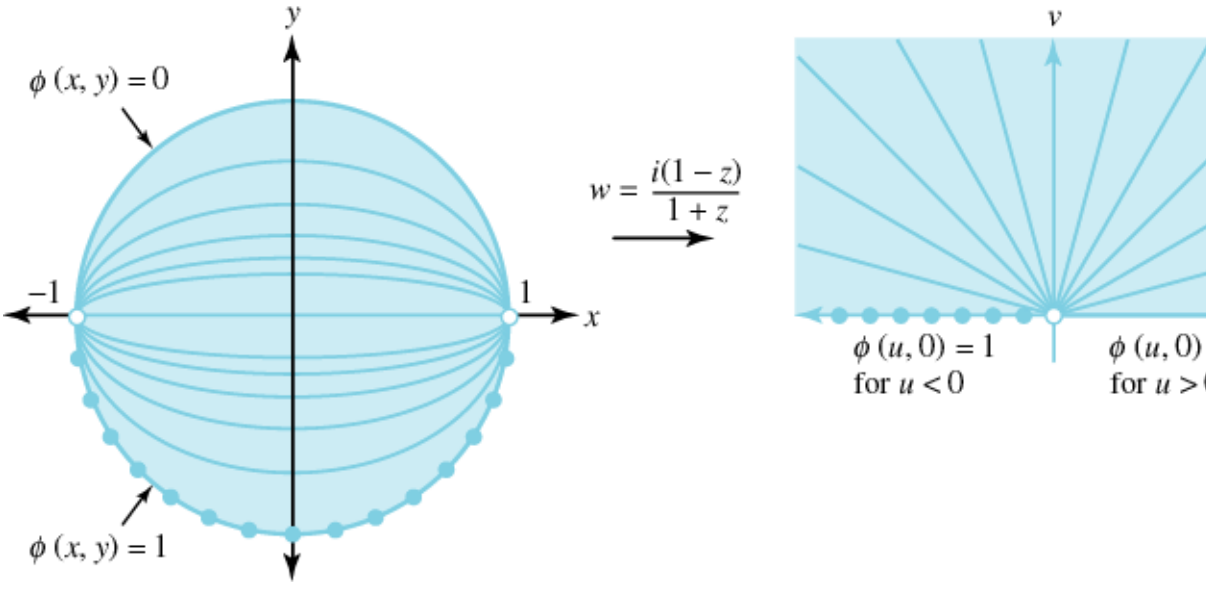


Figure 11.8 The Dirichlet problems for $|z| < 1$ and $\mathrm{Im}(w) > 0$.

Explore Solution 11.8.

Example 11.9. Find a function $\phi(x, y)$ that is harmonic in the upper half disk $H: y > 0, \ |z| < 1$, which takes on the boundary values

$$\phi(x, y) = 0, \quad \text{for} \quad x + i y = e^{i\theta}, \ 0 < \theta < \pi;$$

$$\phi(x, 0) = 1, \quad \text{for} \quad -1 < x < 1.$$

Solution. When we use the result of Exercise 4, Section 10.2, the transformation in Equation (11-9) maps the upper half-disk H onto the first quadrant $Q: u > 0, v > 0$. The conformal mapping given in Equation (11-9) maps the points $z = x + i y$ that lie on the segment $y = 0, \ -1 < x < 1$ onto the positive v axis.

Equation (11-9) gives rise to a new Dirichlet problem of finding a harmonic function $\Phi(u, v)$ in Q that has the boundary values

$$\Phi(u, 0) = 0, \quad \text{for } u > 0, \quad \text{and}$$

$$\Phi(0, v) = 1, \quad \text{for } v > 0,$$

as shown in Figure 11.9. In this case, the method in Example 11.2 can be used to show that $\Phi(u, v)$ is given by

$$\Phi(u, v) = 0 + \frac{1-0}{\pi/2} \operatorname{Arg}(w) = \frac{2}{\pi} \operatorname{Arg}(w) = \frac{2}{\pi} \operatorname{Arctan}\left(\frac{v}{u}\right)$$

Using the functions $u(x, y) = \frac{2y}{(x+1)^2+y^2}$ and $v(x, y) = \frac{1-x^2-y^2}{(x+1)^2+y^2}$ in Equation (11-9) in the preceding equation, we find the solution of the Dirichlet problem in H:

$$\phi(x, y) = \frac{2}{\pi} \operatorname{Arctan}\left(\frac{v(x,y)}{u(x,y)}\right) = \frac{2}{\pi} \operatorname{Arctan}\left(\frac{1-x^2-y^2}{2y}\right).$$

A three-dimensional graph of $\phi(x, y)$ in cylindrical coordinates is shown in Figure 11.10.

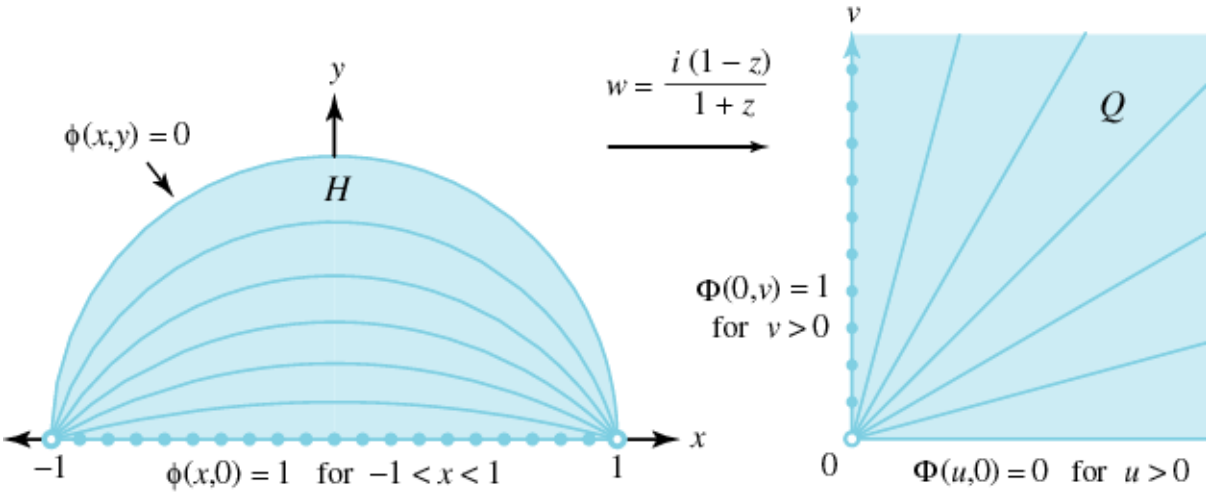


Figure 11.9 The Dirichlet problems for the domains H and Q.

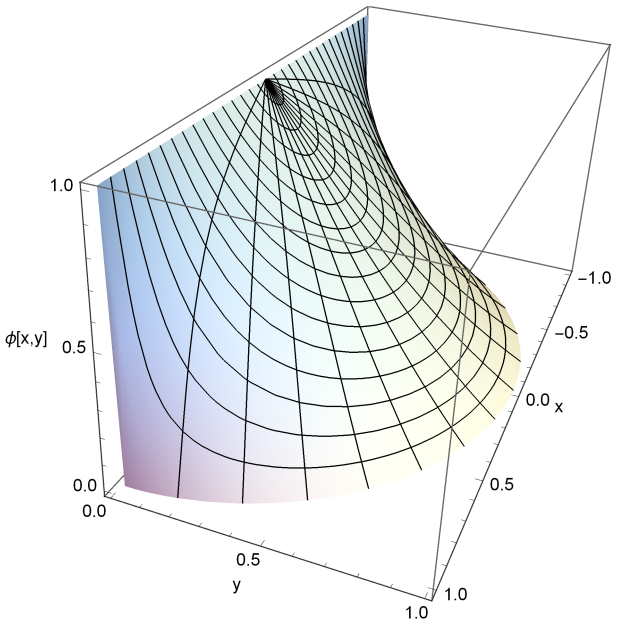


Figure 11.10 The graph $\phi(x, y) = \frac{2}{\pi}\,\text{Arctan}\,\frac{1-x^2-y^2}{2y} = \frac{2}{\pi}\,\text{Arctan}\,\frac{1-r^2}{2r\sin\theta}$.

Explore Solution 11.9.

Example 11.10. Find a function $\phi(x, y)$ that is harmonic in the quarter disk $G: x > 0,\ y > 0,\ |z| < 1$, which takes on the boundary values

$$\phi(x, y) = 0, \quad \text{for} \quad x + i\,y = e^{i\theta}, \quad 0 < \theta < \frac{\pi}{2};$$

$$\phi(x, 0) = 1, \quad \text{for} \quad 0 \le x < 1;$$

$$\phi(0, y) = 1, \quad \text{for} \quad 0 \le y < 1.$$

Solution. The function

(11-10) $$u + i\,v = z^2 = x^2 - y^2 + i\,2\,x\,y$$

maps the quarter-disk onto the upper half-disk $H: v > 0,\ |w| < 1$. The new Dirichlet problem in D is shown in Figure 11.11. From the result of Example 11.9 the solution $\Phi(u, v)$ in H is

(11-11) $$\Phi(u, v) = \frac{2}{\pi}\,\text{Arctan}\left(\frac{1-u^2-v^2}{2v}\right).$$

Using Equation (11-10), and $u = x^2 - y^2$ and $v = 2\,x\,y$ we obtain

$$u^2 + v^2 = (x^2 - y^2)^2 + (2\,x\,y)^2$$

$$= x^4 + 2\,x^2\,y^2 + y^4$$

$$= (x^2 + y^2)^2$$

which we use in Equation (11-11) to construct the solution in G:

$$\phi\,(x, y) = \frac{2}{\pi}\,\text{Arctan}\left(\frac{1-\left(x^2+y^2\right)^2}{4\,x\,y}\right).$$

A three-dimensional graph of $\phi\,(x, y) = \frac{2}{\pi}\,\text{Arctan}\left(\frac{1-\left(x^2+y^2\right)^2}{4\,x\,y}\right)$ in cylindrical coordinates is shown in Figure 11.12.

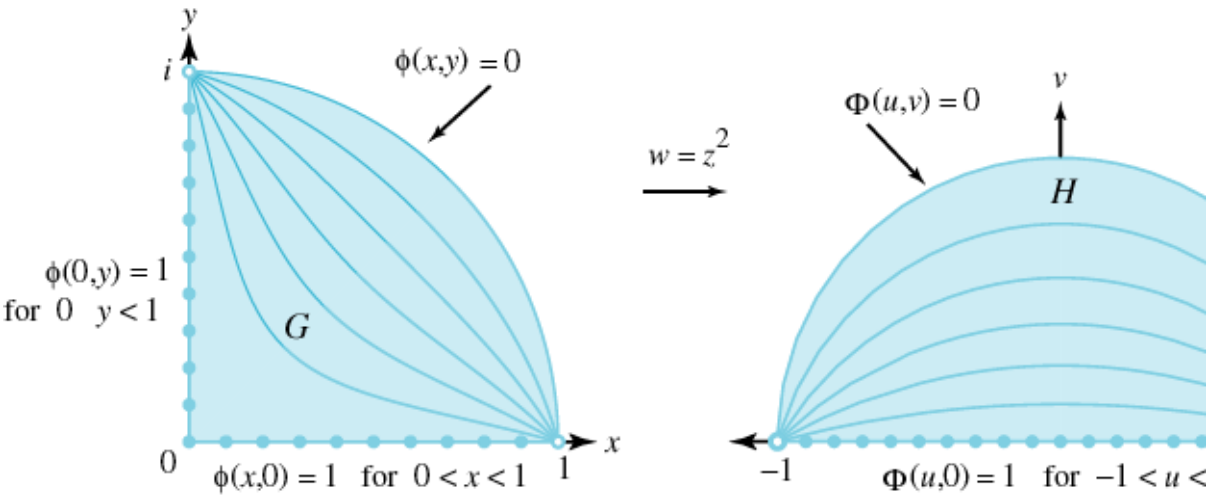


Figure 11.11 The Dirichlet problems for the domains G and H.

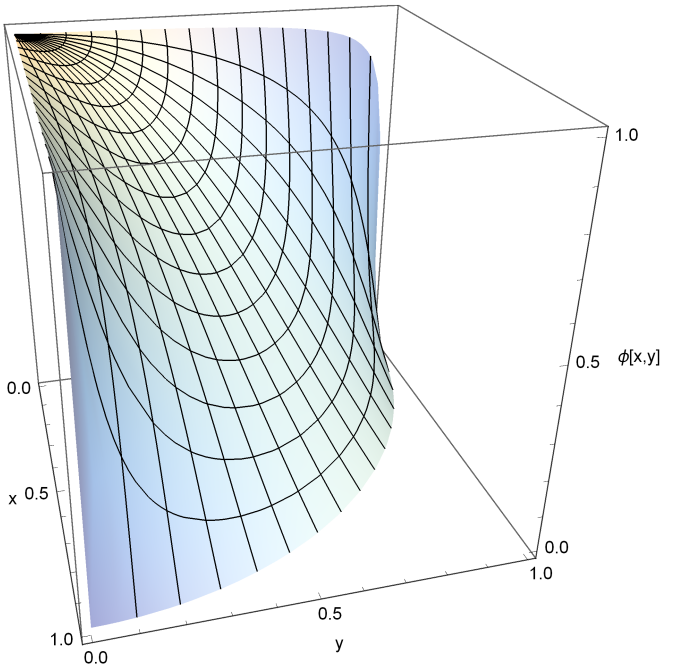


Figure 11.12 The graph

$$\phi\,(x, y) = \frac{2}{\pi}\,\text{Arctan}\left(\frac{1-\left(x^2+y^2\right)^2}{4\,x\,y}\right) = \frac{2}{\pi}\,\text{Arctan}\left(\frac{1-r^4}{4\,r^2\cos\theta\sin\theta}\right).$$

Explore Solution 11.10.

Complex Analysis for Mathematics and Engineering
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11.3 Poisson's Integral Formula for the Upper Half-Plane

The Dirichlet problem for the upper half-plane $\text{Im}(z) > 0$ is to find a function $\phi(x, y)$ that is harmonic in the upper half plane and has the boundary values $\phi(x, 0) = U(x)$, where $U(x)$, is a real-valued function of the real variable x. An important method for solving this problem is our next result which is attributed to the French mathematician Siméon Poisson.

Theorem 11.3 (Poisson's Integral Formula). Let U be a real-valued function that is piecewise continuous and bounded for all real t. The function

(11-12) $$\phi(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{U(t)}{(x-t)^2 + y^2}\, dt$$

is harmonic in the upper half plane $\text{Im}(z) > 0$ and has the boundary values

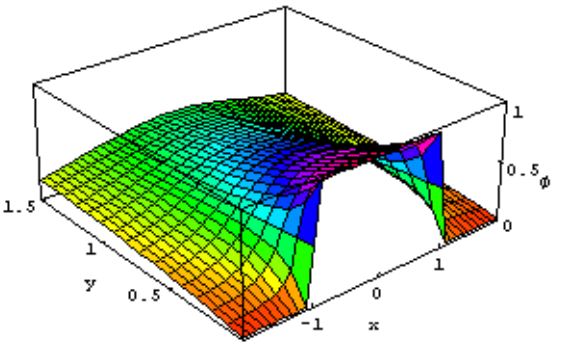
$$\phi(x, 0) = U(x)$$

wherever $U(x)$ is continuous.

Proof.

Example 11.11. Find the function $\phi(x, y)$ that is harmonic in the upper half-plane $\text{Im}(z) > 0$, which takes on the boundary values

$$\phi(x, 0) = 1, \quad \text{for} \quad -1 \le x \le 1;$$
$$\phi(x, 0) = 0, \quad \text{for} \quad |x| > 1.$$



Solution. Using Equation (11-12), we obtain

$$\phi(x, y) = \frac{y}{\pi}\int_{-1}^{1}\frac{1}{(x-t)^2+y^2}\,dt$$

Using techniques from calculus we have the integration formula $\int\frac{y}{(x-t)^2+y^2}\,dt = \text{Arctan}\left(\frac{y}{x-t}\right)$. We obtain the solution as follows

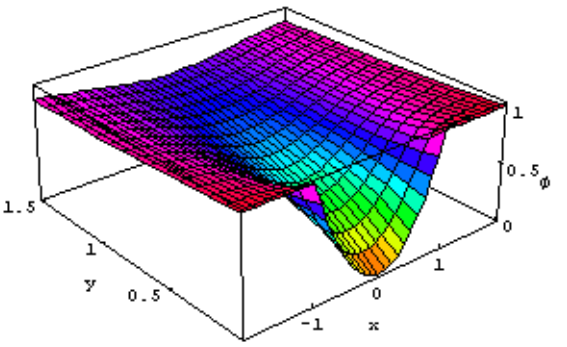
$$\phi(x, y) = \frac{1}{\pi}\left(\text{Arctan}\,\frac{y}{x-t}\right)\Big|_{t=-1}^{t=1}$$

$$= \frac{1}{\pi}\,\text{Arctan}\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\,\text{Arctan}\left(\frac{y}{x+1}\right)$$

Explore Solution 11.11.

Extra Example 1. Find the function $\phi(x, y)$ that is harmonic in the upper half-plane $\text{Im}(z) > 0$, which takes on the boundary values

$$\phi(x, 0) = x^2, \quad \text{for} \quad -1 \le x \le 1;$$
$$\phi(x, 0) = 0, \quad \text{for} \quad |x| \ge 1.$$



Explore Extra Solution 1.

Example 11.12. Find the function $\phi(x, y)$ that is harmonic in the upper half-plane $\text{Im}(z) > 0$, which takes on the boundary values

$$\phi(x, 0) = x, \quad \text{for} \quad -1 < x < 1;$$
$$\phi(x, 0) = 0, \quad \text{for} \quad |x| > 1.$$

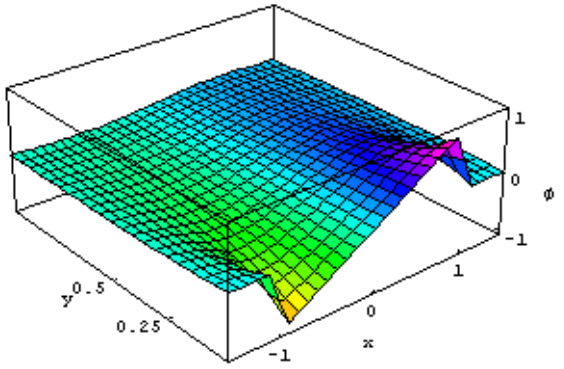


Figure 11.14 The graph of $\phi(x, y)$ with the boundary values $\phi(x, 0) = \begin{cases} x, & \text{for} \quad -1 < x < 1; \\ 0, & \text{for} \quad |x| > 1 \end{cases}$.

Solution. Using Equation (11-12), we obtain

$$\phi(x, y) = \frac{y}{\pi}\int_{-1}^{1}\frac{t}{(x-t)^2+y^2}\,dt$$

$$= \frac{y}{\pi}\int_{-1}^{1}\frac{(x-t)(-1)}{(x-t)^2+y^2}\,dt + \frac{x}{\pi}\int_{-1}^{1}\frac{y}{(x-t)^2+y^2}\,dt$$

Using techniques from calculus we have the integration formulas

$$\int\frac{(x-t)(-1)}{(x-t)^2+y^2}\,dt = \frac{1}{2}\ln\left((x-t)^2+y^2\right), \text{ and}$$
$$\int\frac{y}{(x-t)^2+y^2}\,dt = \text{Arctan}\left(\frac{y}{x-t}\right).$$

We obtain the solution as follows

$$\phi(x, y) = \frac{y}{\pi}\left(\frac{1}{2}\ln\left((x-t)^2+y^2\right)\right)\Bigg|_{t=-1}^{t=1} + \frac{x}{\pi}\left(\text{Arctan}\left(\frac{y}{x-t}\right)\right)\Bigg|_{t=-1}^{t=1}$$

$$= \frac{y}{2\pi}\ln\left(\frac{(x-1)^2+y^2}{(x+1)^2+y^2}\right) + \frac{x}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right) - \frac{x}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right)$$

The function $\phi(x, y)$ is continuous in the upper half-plane, and on the boundary $\phi(x, 0)$, except at the discontinuities $x = \pm 1$ on the real axis. The graph in Figure 11.14 shows this phenomenon.

Explore Solution 11.12.

Example 11.13. Use Poisson's Integral formula to find the harmonic function $\phi(x, y)$ that is harmonic in the upper half-plane $\text{Im}(z) > 0$, that takes on the boundary values

$$\phi(x, 0) = -1, \quad \text{for} \quad x \le -1;$$
$$\phi(x, 0) = x, \quad \text{for} \quad -1 \le x \le 1;$$
$$\phi(x, 0) = 1, \quad \text{for} \quad 1 \le x.$$

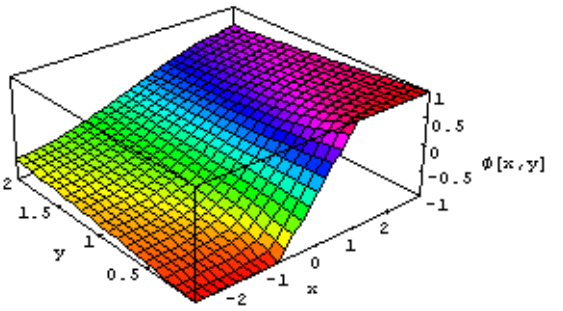


Figure 11.15 The graph of $\phi(x, y)$.

Solution. Using techniques from Section 11.2, we find that the function

$$v(x, y) = 1 - \frac{1}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right) - \frac{1}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right)$$

is harmonic in the upper half-plane and has the boundary values $v(x, 0) = 0$, for $|x| < 1$; $v(x, 0) = -1$, for $x < -1$; $v(x, 0) = 1$, for $1 < x > 1$. This function can be added to the one in Example 11.12 to obtain the desired result:

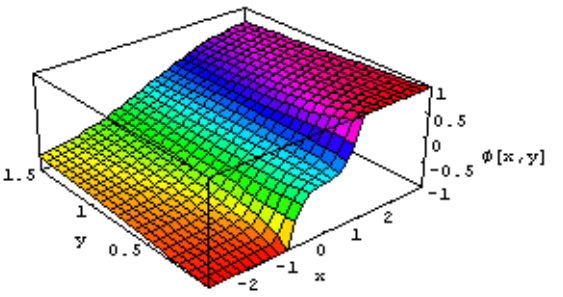
$$\begin{aligned}\phi(x, y) &= 1 - \frac{1}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right) + \frac{y}{2\pi}\ln\left(\frac{(x-1)^2+y^2}{(x+1)^2+y^2}\right) \\ &\quad + \frac{x}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right) - \frac{x}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right) \\ &= 1 + \frac{x-1}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right) - \frac{x+1}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right) + \frac{y}{2\pi}\ln\left(\frac{(x-1)^2+y^2}{(x+1)^2+y^2}\right)\end{aligned}$$

Figure 11.15 shows the graph of $\phi(x, y)$.

Explore Solution 11.13.

Extra Example 2. Use Poisson's Integral formula to find the harmonic function $\phi(x, y)$ that is harmonic in the upper half-plane $\text{Im}(z) > 0$, that takes on the boundary values

$$\begin{aligned}&\phi(x, 0) = -1, \quad \text{for} \quad x \le -1; \\ &\phi(x, 0) = x^3, \quad \text{for} \quad -1 \le x \le 1; \\ &\phi(x, 0) = 1, \quad \text{for} \quad 1 \le x.\end{aligned}$$



Explore Extra Solution 2.

Complex Analysis for Mathematics and Engineering

by

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11.4 Two-Dimensional Mathematical Models

We now consider problems involving steady state heat flow, electrostatics, and ideal fluid flow that can be solved with conformal mapping techniques. Conformal mapping transforms a region in which the problem is posed to one in which the solution is easy to obtain. As our solutions involve only two independent variables, x and y, we first mention a basic assumption needed for the validity of the model.

The physical problems we just mentioned are real-world applications and involve solutions in three-dimensional Cartesian space. Such problems generally would involve the Laplacian in three variables and the divergence and curl of three-dimensional vector functions. Since complex analysis involves only x and y, we consider the special case in which the solution does not vary with the coordinate along the axis perpendicular to the xy plane. For steady state heat flow and electrostatics, this assumption means that the temperature, T, or the potential, V, varies only with x, and y. Thus for the flow of ideal fluids, the fluid motion is the same in any plane that is parallel to the z plane. Curves drawn in the z plane are to be interpreted as cross sections that correspond to infinite cylinders perpendicular to the z plane. An infinite cylinder is the limiting case of a "long" physical cylinder, so the mathematical model that we present is valid provided the three-dimensional problem involves a physical cylinder long enough that the effects at the ends can be reasonably neglected.

In Section 11.1 and Section 11.2 we learned how to obtain solutions $\phi(x, y)$ for harmonic functions. For applications it is important to consider the family of level curves $\{\phi(x, y) = K_1 : K_1 \text{ is a real constant}\}$ and the conjugate harmonic function $\psi(x, y)$ and its family of level curves $\{\psi(x, y) = K_2 : K_2 \text{ is a real constant}\}$. It is convenient to introduce the terminology complex potential for the analytic function $F(z) = \phi(x, y) + i\,\psi(x, y)$. The following result regarding the orthogonality of the above mentioned families of level curves will be use in developing the ideas concerning the physical applications.

In Sections 11.1 and 11.2, we showed how to obtain solutions $\phi(x, y)$ for harmonic functions. For applications, we need to consider the family of level curves

(10-16) $\{\phi(x, y) = K_1 : K_1 \text{ is a real constant}\}$

and the conjugate harmonic function $\psi(x, y)$ and its family of level curves

(10-17) $\{\psi(x, y) = K_2 : K_2 \text{ is a real constant}\}$

For convenience, we introduce the term complex potential for the analytic function

$$F(z) = \phi(x, y) + i\,\psi(x, y).$$

We use Theorem 11.4, regarding the orthogonality of the families of level curves (10-16) and (10-17), to develop ideas concerning the physical applications that we will consider.

Theorem 11.4 (Orthogonal Families of Level Curves). Let $\phi(x, y)$ be harmonic in a domain D. Let $\psi(x, y)$ be the harmonic conjugate and let $F(z) = \phi(x, y) + i\,\psi(x, y)$ be the complex potential. Then the two families of level curves given in Equations (10-16) and (10-17), respectively, are orthogonal in the sense that if (a, b) is a point in common to the two curves $\phi(x, y) = K_1$ and $\psi(x, y) = K_2$, and if $F'(a + i\,b) \neq 0$, then these two curves intersect orthogonally.

Proof.

Exploration for Theorem 11.4.

The complex potential $F(z) = \phi(x, y) + i\,\psi(x, y)$ has many physical interpretations. Suppose, for example, that we have solved a problem in steady state temperatures. Then we can obtain the solution to a similar problem with the same boundary conditions in electrostatics by interpreting the isothermals as equipotential curves and the heat flow lines as flux lines. This implies that heat flow and electrostatics correspond directly.

Or suppose that we have solved a fluid flow problem. Then we can obtain a solution to an analogous problem in heat flow by interpreting the equipotentials as isothermals and streamlines as heat flow lines. Various interpretations of the families of level curves given in Equations (10-16) and (10-17) and correspondences between families are summarized in Table 11.1.

Physical Phenomenon	$\phi(x, y) = \text{constant}$	$\psi(x, y) = \text{constant}$
Heat flow	Isothermals	Heat flow lines
Electrostatics	Equipotential curves	Flux lines
Fluid flow	Equipotentials	Streamlines
Gravitational field	Gravitational potential	Lines of force
Magnetism	Potential	Lines of force
Diffusion	Concentration	Lines of flow
Elasticity	Strain function	Stress lines
Current flow	Potential	Lines of flow

Table 11.1 Interpretations for the level curves of $\phi(x, y)$ and $\psi(x, y)$.

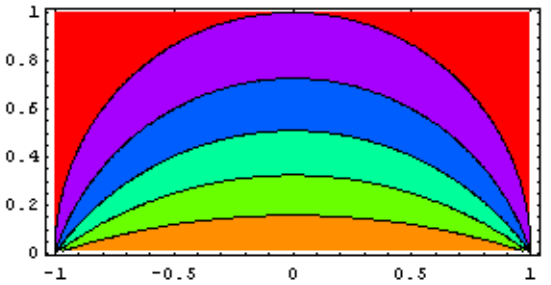
Preview of Applications

The following is a sample of applications taken from Section 11-5 to Section11-11.

Example 11.16. Find the temperature $T(x, y)$ at each point in the upper half-disk $H: y > 0, |z| < 1$, if the temperature at points on the boundary satisfies

$$T(x, y) = 100, \quad \text{for} \quad x + i\,y = e^{i\theta}, \ 0 < \theta < \pi \quad \text{on the upper semi-circle;}$$

$$T(x, 0) = 50, \quad \text{for} \quad -1 < x < 1 \quad \text{on the diameter.}$$



Solution. As discussed in Example 11.9, the transformation

$$w = u + i\,v = \frac{i\,(1-z)}{1+z} = \frac{i\,(1-x-i\,y)}{1+x+i\,y}$$

is a one-to-one conformal mapping of the half-disk H onto the first quadrant $Q: u > 0, v > 0$, and can be written as

(11-24)
$$u + i\,v = \frac{i\,(1-x-i\,y)\,(1+x-i\,y)}{(1+x+i\,y)\,(1+x-i\,y)}$$

$$= \frac{2\,y+i\,\left(1-x^2-y^2\right)}{1+2\,x+x^2+y^2}$$

$$= \frac{2\,y}{(x+1)^2+y^2} + i\,\frac{1-x^2-y^2}{(x+1)^2+y^2}$$

The conformal map given by Equation (11-24) gives rise to a new problem of finding the temperature $T^*(u, v)$ that satisfies the boundary conditions

$$T^*(u, 0) = 100, \quad \text{for} \quad u > 0;$$

$$T^*(0, v) = 50, \quad \text{for} \quad v > 0.$$

If we use Example 11.2, the harmonic function $T^*(u, v)$ is given by

(11-25)
$$T^*(u, v) = 100 + \frac{50-100}{\pi/2}\,\text{Arg}\,(u + i\,v) = 100 - \frac{100}{\pi}\,\text{Arctan}\left(\frac{v}{u}\right).$$

Substituting the expressions for u and v from Equation (11-24) into Equation (11-25) yields the desired solution:

$$T(x, y) = 100 - \frac{100}{\pi}\,\text{Arctan}\left(\frac{1-x^2-y^2}{2\,y}\right).$$

The isothermals $T(x, y) = \text{constant}$ are circles that pass through the points ±1, as shown in Figure 11.19.

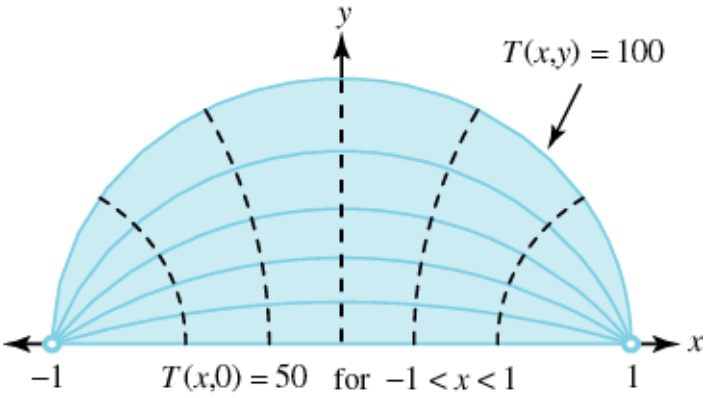


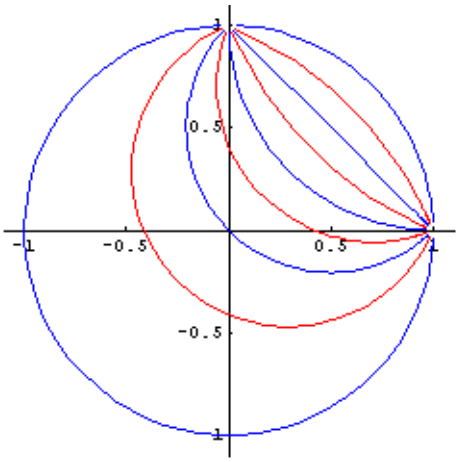
Figure 11.19 The temperature $T(x, y) = 100 - \frac{100}{\pi}\,\text{Arctan}\left(\frac{1-x^2-y^2}{2\,y}\right)$ in a half-disk.

Explore Solution 11.16.

Example 11.21. Find the electrical potential $\phi(x, y)$ in the disk $D: |z| < 1$ that satisfies the boundary values

$$\phi(x, y) = 80, \quad \text{for} \quad x + i\,y = z \text{ on } C_1 = \left\{z = e^{i\,\theta} : 0 < \theta < \frac{\pi}{2}\right\};$$

$$\phi(x, y) = 0, \quad \text{for} \quad x + i\,y = z \text{ on } C_2 = \left\{z = e^{i\,\theta} : \frac{\pi}{2} < \theta < 2\,\pi\right\}.$$



Solution. The mapping $w = S(z) = \frac{(1-i)(z-i)}{(z-1)}$ is a one-to-one conformal mapping of D onto the upper half-plane $\mathrm{Im}(w) > 0$ with the property that C_1 is mapped onto the negative u axis and C_2 is mapped onto the positive u axis. The potential $\Phi(u, v)$ in the upper half-plane that satisfies the new boundary values

$$\Phi(u, 0) = 80, \quad \text{for } u < 0,$$
$$\Phi(u, 0) = 0, \quad \text{for } u > 0,$$

is given by

(11-29) $$\Phi(u, v) = \frac{80}{\pi}\,\mathrm{Arg}(w) = \frac{80}{\pi}\,\mathrm{Arctan}\left(\frac{v}{u}\right).$$

A straightforward calculation shows that

$$w = u + i\,v = S(z) = \frac{(1-i)(x+i\,y-i)}{(x+i\,y-1)}$$

$$= \frac{(1-i)(x+i\,(y-1))}{(x-1+i\,y)}$$

$$= \frac{(1-i)(x+i\,(y-1))(x-1-i\,y)}{(x-1+i\,y)(x-1-i\,y)}$$

$$= \frac{1-2\,x+x^2-2\,y+y^2+i\left(1-x^2-y^2\right)}{1-2\,x+x^2+y^2}$$

$$= \frac{(x-1)^2+(y-1)^2-1+i\left(1-x^2-y^2\right)}{1-2\,x+x^2+y^2}$$

$$= \frac{(x-1)^2+(y-1)^2-1}{(x-1)^2+y^2} + i\,\frac{1-x^2-y^2}{(x-1)^2+y^2}$$

We substitute the real and imaginary parts, $u = u(x, y) = \frac{(x-1)^2+(y-1)^2-1}{(x-1)^2+y^2}$ and $v = v(x, y) = \frac{1-x^2-y^2}{(x-1)^2+y^2}$ from this equation, into Equation (11-29) to obtain the desired solution:

$$\phi(x, y) = \frac{80}{\pi}\,\mathrm{Arctan}\left(\frac{1-x^2-y^2}{(x-1)^2+(y-1)^2-1}\right).$$

The level curve $\Phi(u, v) = \alpha$ in the upper half-plane is a ray emanating from the origin, and the preimage $\phi(x, y) = \alpha$ in the unit disk is an arc of a circle that passes through the points $z = 1$ and $z = i$. Several level curves are illustrated in Figure 11.38.

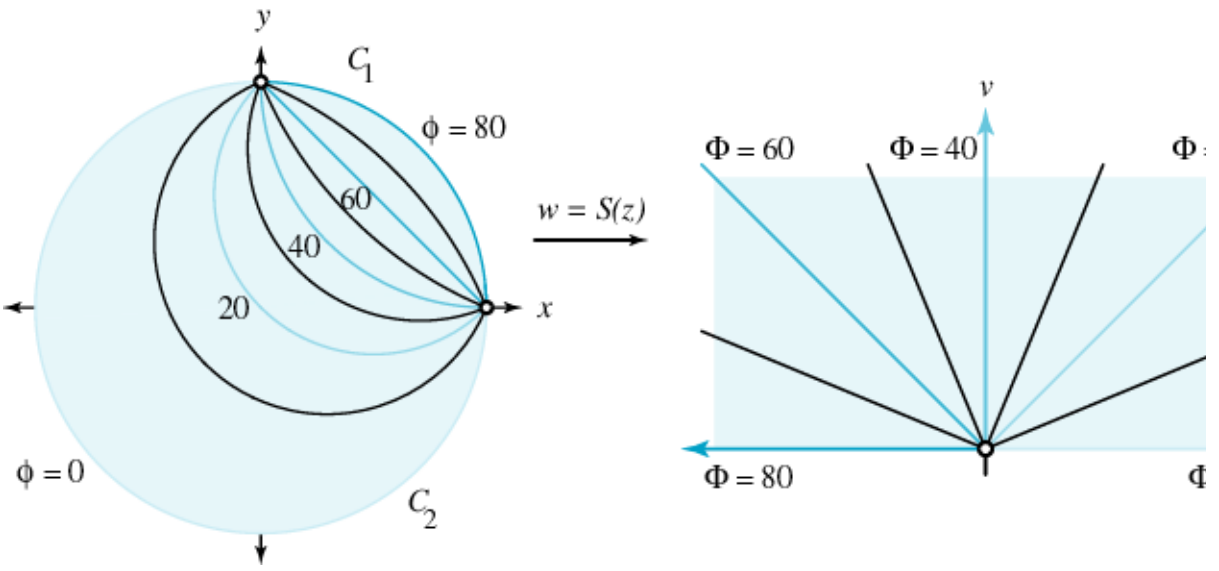
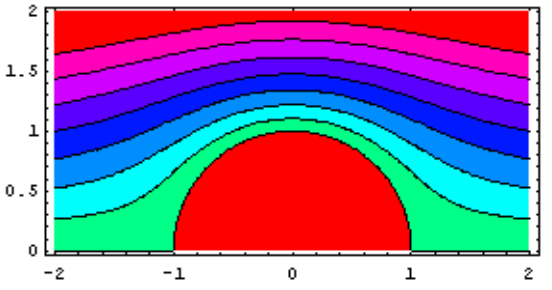


Figure 11.38 The potentials $\phi(x, y)$ and $\Phi(u, v)$.

Explore Solution 11.21.

Example 11.24. Find the complex potential for an ideal fluid flowing from left to right across the complex plane and around the unit circle $|z| = 1$.



Solution. We use the fact that the conformal mapping $w = S(z) = z + \frac{1}{z}$ maps the domain $D = \{z : |z| < 1\}$ one-to-one and onto the w plane slit along the segment $-2 \le u \le 2, v = 0$. The complex potential for a uniform horizontal flow parallel to this slit in the w plane is

$$F_1(w) = A w,$$

where A is a positive real number. The stream function for the flow in the w plane is $\Psi(u, v) = A v$ so that the slit lies along the streamline $\Psi(x, y) = 0$.

The composite function $F_2(z) = F_1(S(z))$ determines the fluid flow in the domain D, where the complex potential is

$$F_2(z) = F_1(S(z)) = A\left(z + \frac{1}{z}\right),$$

where $A > 0$. We can use polar coordinates to express $F_2(z)$ as

$$F_2(z) = F_2(r e^{i\theta}) = A\left(r e^{i\theta} + \frac{1}{r e^{i\theta}}\right)$$

$$= A\left(r\cos\theta + i\, r\sin\theta + \frac{1}{r\cos\theta + i\, r\sin\theta}\right)$$

$$= A\left(r\cos\theta + i\, r\sin\theta + \frac{r\cos\theta - i\, r\sin\theta}{(r\cos\theta + i\, r\,\mathrm{Sin}\,\theta)\,(r\cos\theta - i\, r\sin\theta)}\right)$$

$$= A\left(r\cos\theta + i\, r\sin\theta + \frac{r(\cos\theta - i\, r\sin\theta)}{r^2}\right)$$

$$= A\left(r + \frac{1}{r}\right)\cos\theta + i\, A\left(r - \frac{1}{r}\right)\sin\theta$$

The streamline $\psi(r,\theta) = A\left(r - \frac{1}{r}\right)\sin\theta = 0$ consists of the rays

$$r > 1,\ \theta = 0 \quad \text{and} \quad r > 1,\ \theta = \pi$$

along the x axis and the curve $r - \frac{1}{r} = 0$, which is the unit circle $|z| = r = 1$. Thus the unit circle can be considered as a boundary curve for the fluid flow.

The approximation $F_2(z) = A\left(z + \frac{1}{z}\right) \approx A\,z$ is valid for large values of z, so we can approximate the flow with a uniform horizontal flow having speed $|\mathbf{V}(x,y)| = A$ at points that are distant from the origin. The streamlines $\psi(x,y) = \text{constant}$ and their images $\Psi(u,v) = \text{constant}$ under the mapping $w = S(z) = z + \frac{1}{z}$ are illustrated in Figure 11.51.

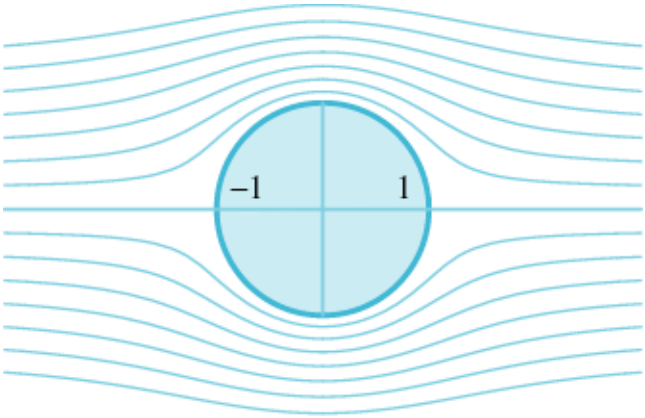


Figure 11.51 Fluid flow around a circle.

Explore Solution 11.24.

Example 11.27. Use the Schwarz Christoffel formula to verify that $w = f(z) = (z^2 - 1)^{\frac{1}{2}}$ maps the upper half plane $\mathrm{Im}(z) > 0$ onto the upper half plane $\mathrm{Im}(w) > 0$ slit along the line segment from $w = 0$ to $w_2 = i$. (Use the principal square root throughout.)

Solution. If we choose $x_1 = -1, x_2 = 0, x_3 = 1$ and $w_1 = -d, w_2 = i, w_3 = d$, then the formula

$$g'(z) = A(z-x_1)^{-\alpha_1/\pi}(z-x_2)^{-\alpha_2/\pi}(z-x_3)^{-\alpha_3/\pi}$$

$$= A(z-(-1))^{-\alpha_1/\pi}(z-0)^{-\alpha_2/\pi}(z-1)^{-\alpha_3/\pi}$$

$$= A(z+1)^{-\alpha_1/\pi}(z)^{-\alpha_2/\pi}(z-1)^{-\alpha_3/\pi}$$

will determine a mapping $w = g(z)$ from the upper half-plane $\mathrm{Im}(z) > 0$ onto the portion of the upper half-plane $\mathrm{Im}(w) > 0$ that lies outside the triangle with vertices $w_1 = -d, w_2 = i, w_3 = d$ as indicated in Figure 11.73(a). If we let $d \to 0$, then $w_1 \to 0$ and $w_3 \to 0$, and $\alpha_1 \to \frac{\pi}{2}, \alpha_2 \to -\pi, \alpha_3 \to \frac{\pi}{2}$.

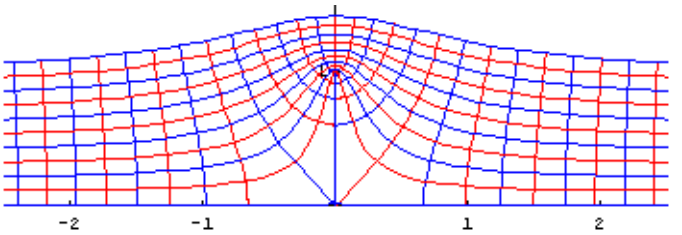


Figure 11.73 The region with $w_1 \to 0, w_2 = i, w_3 \to 0$ and $\alpha_1 \to \frac{\pi}{2}, \alpha_2 \to -\pi, \alpha_3 \to \frac{\pi}{2}$.

The limiting formula for the derivative $g'(z)$ becomes

$$f'(z) = A(z+1)^{-\alpha_1/\pi}(z)^{-\alpha_2/\pi}(z-1)^{-\alpha_3/\pi}$$

$$= A(z+1)^{-\frac{\pi}{2}/\pi}(z)^{-(-\pi)/\pi}(z-1)^{-\frac{\pi}{2}/\pi}$$

$$= A(z+1)^{-\frac{1}{2}}(z)^{1}(z-1)^{-\frac{1}{2}}$$

$$= A\,z\,(z^2-1)^{-\frac{1}{2}}$$

$$= A\frac{z}{(z^2-1)^{\frac{1}{2}}}$$

which will determine a mapping $w = f(z)$ from the upper half-plane $\mathrm{Im}(z) > 0$ onto the upper half plane $\mathrm{Im}(w) > 0$ slit along the line segment from $w = 0$ to $w_2 = i$ as indicated in Figure 11.73(b). An easy integration reveals that $f(z)$ is given by

$$f(z) = A\int \frac{z}{(z^2-1)^{\frac{1}{2}}}\,dz + B$$

$$= A(z^2-1)^{\frac{1}{2}} + B$$

and the boundary values $f(\pm 1) = 0$ and $f(0) = i$ lead to system of equations

$$A\left((\pm 1)^2-1\right)^{\frac{1}{2}} + B = 0 \quad \text{and} \quad A(0-1)^{\frac{1}{2}} + B = i$$

the solution is easily found to be $A = 1$ and $B = 0$ and the desired function is

$$f(z) = (z^2 - 1)^{\frac{1}{2}}.$$

Explore Solution 11.27.

Example 11.29. Show that the mapping $w = \frac{1}{\pi}\left((z^2 - 1)^{\frac{1}{2}} + \text{Log}\left(z + (z^2 - 1)^{\frac{1}{2}}\right)\right)$ maps the upper half plane $\text{Im}(z) > 0$ onto the domain in the w-plane that lies above the boundary curve consisting of the rays $u \le 0, v = 1$ and $u \ge 0, v = 0$ and the segment $u = 0, -1 \le v \le 0$ (see Figure 11.87).

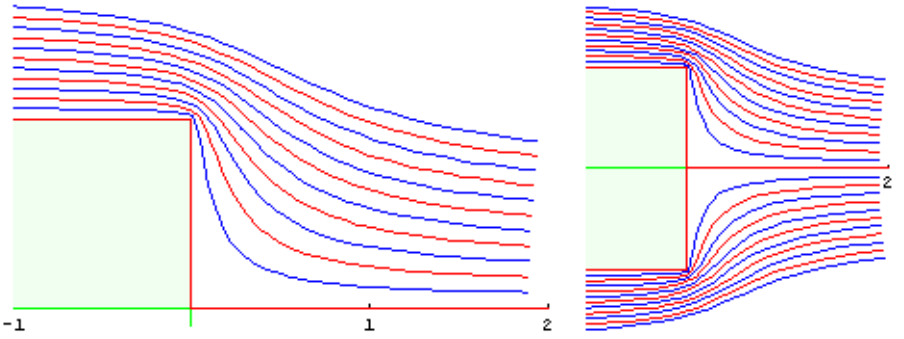


Figure 11.87 (a) Flow over a step. (b) Flow around a blunt object.

Hint: Set $x_1 = -1, x_2 = 1$ and $w_1 = 0, w_2 = -i$. In the w-plane $w_1 = i, w_2 = 0$, and the exterior angles are $\alpha_1 = -\frac{\pi}{2}, \alpha_2 = \frac{\pi}{2}$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A(z - x_1)^{-\alpha_1/\pi}(z - x_2)^{-\alpha_2/\pi}$$

$$= A(z - (-1))^{-\left(-\frac{\pi}{2}\right)/\pi}(z - 1)^{-\left(\frac{\pi}{2}\right)/\pi}$$

$$= A(z + 1)^{\frac{1}{2}}(z - 1)^{-\frac{1}{2}}$$

$$= A\frac{(z+1)^{\frac{1}{2}}}{(z-1)^{\frac{1}{2}}}$$

Integrate and get

$$f(z) = B + A\int \frac{(z+1)^{\frac{1}{2}}}{(z-1)^{\frac{1}{2}}}\,dz$$

$$= B + A\left((z^2 - 1)^{\frac{1}{2}} + \text{Log}\left(z + (z^2 - 1)^{\frac{1}{2}}\right)\right)$$

Solve for the coefficients A and B and obtain

$$w = f(z) = \frac{1}{\pi}\left((z^2 - 1)^{\frac{1}{2}} + \text{Log}\left(z + (z^2 - 1)^{\frac{1}{2}}\right)\right)$$

It maps the upper half-plane $\mathrm{Im}(z) > 0$ onto the domain in the w plane that lies above the boundary curve consisting of the rays $u \le 0, v = 1$ and $u \ge 0, v = 0$ and the segment $u = 0, -1 \le v \le 0$. Furthermore, the image of horizontal streamlines in the z plane are curves in the w plane given by the parametric equation

$$w = u + i\,v = f(t + i\,c)$$

$$= \frac{1}{\pi}\left(\left((t + i\,c)^2 - 1\right)^{\frac{1}{2}} + \mathrm{Log}\left(t + i\,c + \left((t + i\,c)^2 - 1\right)^{\frac{1}{2}}\right)\right)$$

$$= \frac{1}{\pi}\left(\left(t^2 - c^2 - 1 + i\,2\,c\,t\right)^{\frac{1}{2}} + \mathrm{Log}\left(t + i\,c + \left(t^2 - c^2 - 1 + i\,2\,c\,t\right)^{\frac{1}{2}}\right)\right)$$

for $-\infty < t < \infty$. The new flow is that of a step in the bed of a deep stream and is illustrated in Figure 11.87(a). The function $w = f(z)$ is also defined for values of z in the lower half-plane, and the images of horizontal streamlines that lie above or below the x axis are mapped onto streamlines that flow past a long rectangular obstacle, which is illustrated in Figure 11.87(b).

Explore Solution 11.29.

Example 11.30. (Source and sink of equal strength) Let a source and sink of unit strength be located at the points $z_1 = +1$ and $z_2 = -1$, respectively. The complex potential for a fluid flowing from the source at $z_1 = +1$ to the sink at $z_2 = -1$ is

$$F(z) = \log(z - 1) - \log(z + 1) = \log\left(\frac{z-1}{z+1}\right).$$

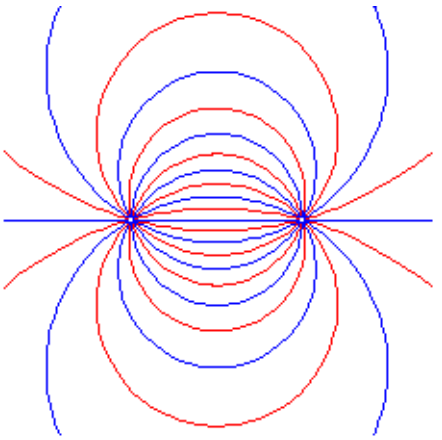


Figure 11.95 (a) Source and sink of unit strength.

The velocity potential and stream function are

$$\phi(x, y) = \ln\left|\frac{z-1}{z+1}\right|, \text{ and}$$

$$\psi(x, y) = \arg\left(\frac{z-1}{z+1}\right),$$

respectively. Solving for the streamline $\psi(x, y) = c$, we start with

$$c = \arg\left(\frac{z-1}{z+1}\right) = \arg\left(\frac{(x+i\,y-1)\,(x-i\,y+1)}{(x+i\,y+1)\,(x-i\,y+1)}\right)$$

$$= \arg\left(\frac{x^2+y^2-1+i\,2\,y}{x^2+2\,x+1+y^2}\right) = \arg\left(\frac{x^2+y^2-1+i\,2\,y}{(x+1)^2+y^2}\right)$$

$$= \arctan\left(\frac{2\,y}{x^2+y^2-1}\right)$$

and obtain the equation

$$(\tan c)\,(x^2 + y^2 - 1) = 2\,y.$$

A straightforward calculation shows that points on the streamline must satisfy the equation

$$x^2 + (y - \cot c)^2 = 1 + \cot^2 c,$$

which is the equation of a circle with center at $(0, \cot c)$ that passes through the points $(\pm 1, 0)$. Several streamlines are indicated in Figure 11.95(a).

Explore Solution 11.30.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering
by
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11.5 Steady State Temperatures

In the theory of heat conduction, an assumption is made that heat flows in the direction of decreasing temperature. Another assumption is that the time rate at which heat flows across a surface area is proportional to the component of the temperature gradient in the direction perpendicular to the surface area. If the temperature $T(x, y)$ does not depend on time, then the heat flow at the point (x, y) is given by the vector

$$\vec{\mathbf{V}}(x, y) = -K \operatorname{grad} T(x, y) = -K\left(T_x(x, y) + i\, T_y(x, y)\right),$$

where K is the thermal conductivity of the medium and is assumed to be constant. If $\triangle z$ denotes a straight-line segment of length $\triangle s$, then the amount of heat flowing across the segment per unit of time is

(11-19) $$\vec{\mathbf{V}} \bullet \vec{\mathbf{N}}\, \triangle s$$

where $\vec{\mathbf{N}}$ is a unit vector perpendicular to the segment.

If we assume that no thermal energy is created or destroyed within the region, then the net amount of heat flowing through any small rectangle with sides of length $\triangle x$ and $\triangle y$ is identically zero (see Figure 11.16(a)). This leads to the conclusion that $T(x, y)$ is a harmonic function. The following heuristic argument is often used to suggest that $T(x, y)$ satisfies Laplace's equation. Using Expression (11-19) we find that the amount of heat flowing out of the right edge of the rectangle in Figure 11.16(a) is approximately

(11-20) $$\vec{\mathbf{V_1}} \bullet \vec{\mathbf{N_1}}\, \triangle s = -K\left(T_x(x + \triangle x, y) + i\, T_x(x + \triangle x, y)\right) \bullet (1 + 0\, i)\, \triangle y$$

$$= -K\, T_x(x + \triangle x, y)\, \triangle y$$

and the amount of heat flowing out of the left edge is

(11-21) $\vec{V_2} \bullet \vec{N_2}\, \Delta s = -K\,(T_x(x, y) + i\, T_x(x, y)) \bullet (-1 + 0\, i)\, \Delta y$

$= K\, T_x(x, y)\, \Delta y$

If we add the contributions in Equations (11-20) and (11-21), the result is

(11-22) $-K\left(\frac{T_x(x+\Delta x, y) - T_x(x, y)}{\Delta x}\right) \Delta x\, \Delta y \approx -K\, T_{xx}(x, y)\, \Delta x\, \Delta y.$

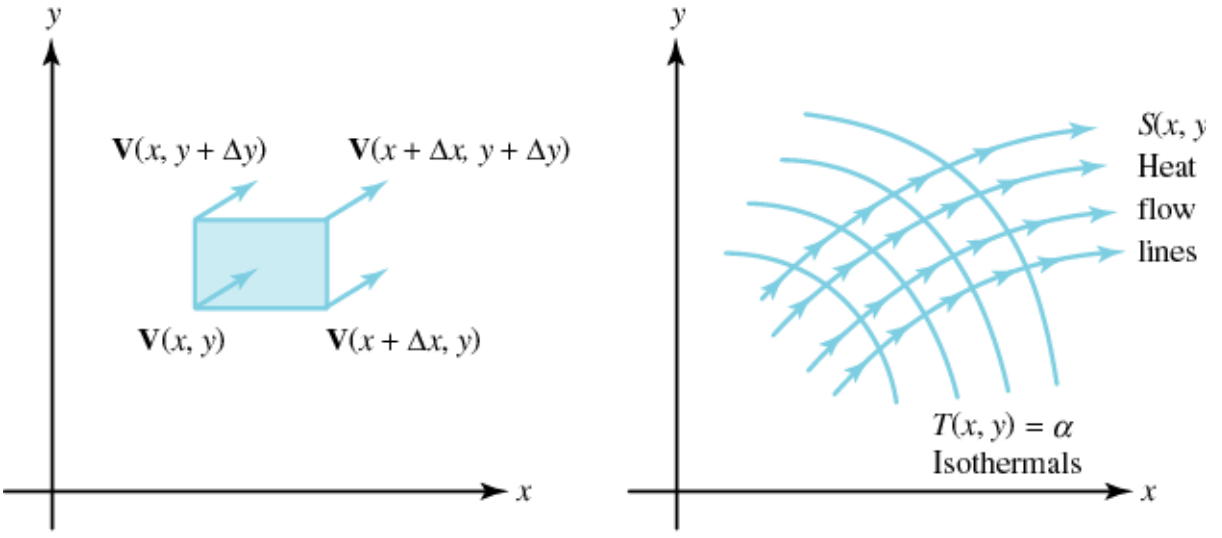


Figure 11.16 Steady state temperatures.

Similarly, the contribution for the amount of heat flowing out of the top and bottom edges is

(11-23) $-K\left(\frac{T_y(x, y+\Delta y) - T_x(x, y)}{\Delta y}\right) \Delta x\, \Delta y \approx -K\, T_{yy}(x, y)\, \Delta x\, \Delta y.$

Adding the quantities in Equations (11-22) and (11-23), we find that the net heat flowing out of the rectangle is approximated by the equation

$$-K\,(T_{xx}(x, y) + T_{yy}(x, y))\, \Delta x\, \Delta y = 0,$$

which implies that

$$T_{xx}(x, y) + T_{yy}(x, y) = 0.$$

Hence $T(x, y)$ satisfies Laplace's equation and is a harmonic function.

If the domain in which $T(x, y)$ is defined is simply connected, then a conjugate harmonic function $S(x, y)$ exists, and

$$F(z) = T(x, y) + i\, S(x, y)$$

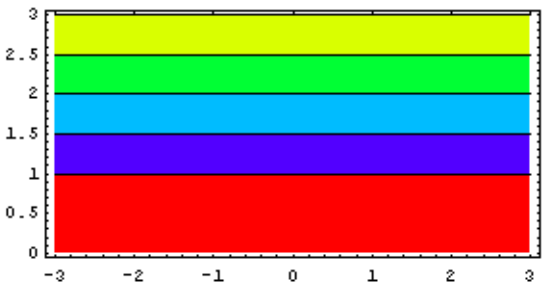
is an analytic function. The curves $T(x, y) = K_1$ are called isothermals and are lines connecting

points of the same temperature. The curves $S(x, y) = K_2$ are called the heat flow lines, and one can visualize the heat flowing along these curves from points of higher temperature to points of lower temperature. The situation is illustrated in Figure 11.16(b).

Boundary value problems for steady state temperatures are realizations of the Dirichlet problem where the value of the harmonic function $T(x, y)$ is interpreted as the temperature at the point (x, y).

Example 11.14. Suppose that two parallel planes are perpendicular to the z plane and pass through the horizontal lines $y = a$ and $y = b$ and that the temperature is held constant at the values $T(x, a) = T_1$ and $T(x, b) = T_2$, respectively, on these planes. Then $T(x, y)$ is given by

$$T(x, y) = T_1 + \frac{T_2 - T_1}{b - a}(y - a).$$



Solution. The two-dimensional solution is constructed at points in the horizontal strip $a < \mathrm{Im}(z) < b$ in the complex plane. A reasonable assumption is that the temperature at all points on the plane passing through the line $y = y_0$ is constant. Hence $T(x, y) = t(y)$, where $t(y)$ is a function of y alone. Laplace's equation implies that $t''(y) = 0$, and an argument similar to that in Example 11.1 will show that the solution $T(x, y)$ has the form given in the preceding equation.

The isothermals $T(x, y) = \alpha$ are easily seen to be horizontal lines. The conjugate harmonic function is

$$S(x, y) = \frac{T_1 - T_2}{b - a} x,$$

and the heat flow lines $S(x, y) = \beta$ are vertical segments between the horizontal lines. If $T_1 > T_2$, then the heat flows along these segments from the plane through $y = a$ to the plane through $y = b$, as illustrated in Figure 11.17.

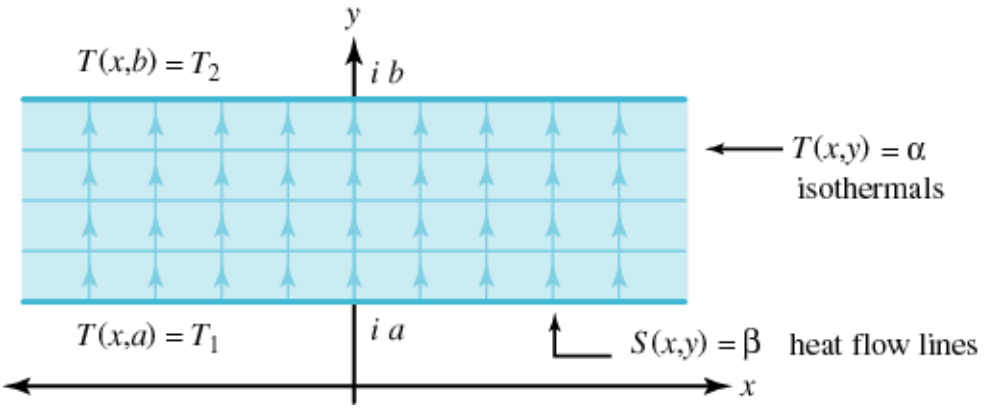


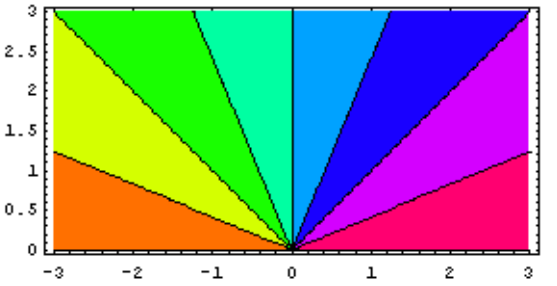
Figure 11.17 The temperature between parallel planes where $T_1 > T_2$.

Explore Solution 11.14.

Example 11.15. Find the temperature $T(x, y)$ at each point in the upper half plane $\mathrm{Im}(z) > 0$, if the temperature at points on the x-axis on the boundary satisfy

$$T(x, 0) = T_1, \quad \text{for} \quad x > 0;$$

$$T(x, 0) = T_2, \quad \text{for} \quad x < 0.$$



Solution. Since $T(x, y)$ is a harmonic function, this problem is an example of a Dirichlet problem. From Example 11.2, it follows that the solution is

$$T(x, y) = T_1 + \frac{T_2 - T_1}{\pi} \mathrm{Arg}(x + i\, y)$$

The isotherms $T(x, y) = \alpha$ are rays emanating from the origin. The conjugate harmonic function is

$$S(x, y) = \frac{T_1 - T_2}{\pi} \ln|x + i\, y|,$$

and the heat flow lines $S(x, y) = \beta$ are semicircles centered at the origin. If $T_1 > T_2$, then the heat flows counterclockwise along the semicircles, as shown in Figure 11.18.

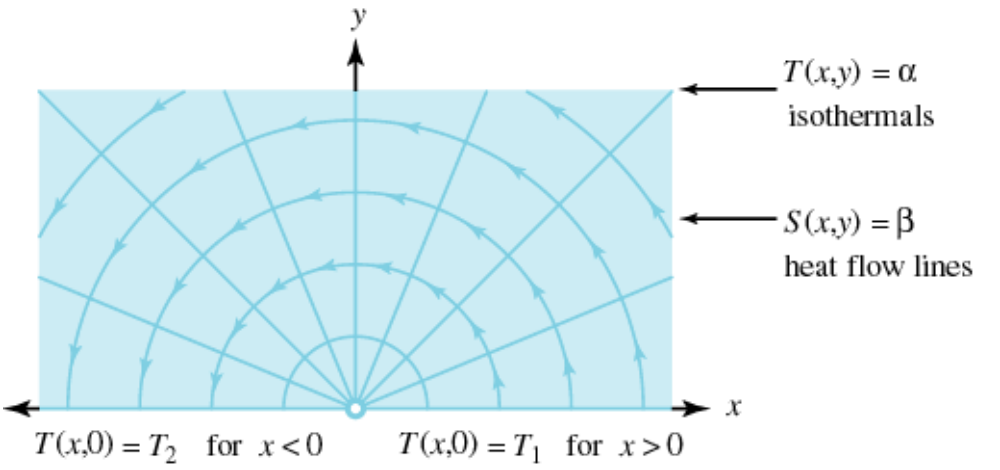


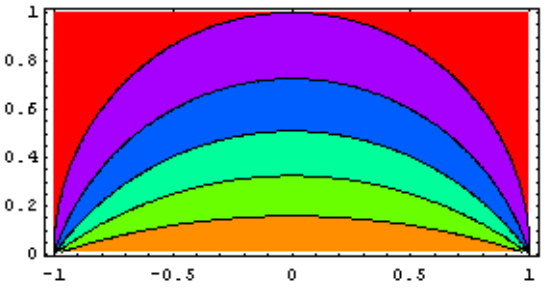
Figure 11.18 The temperature $T(x, y)$ in the upper half-plane where $T_1 > T_2$.

Explore Solution 11.15.

Example 11.16. Find the temperature $T(x, y)$ at each point in the upper half-disk $H: y > 0, \ |z| < 1$, if the temperature at points on the boundary satisfies

$$T(x, y) = 100, \quad \text{for} \quad x + i\,y = e^{i\theta}, \ 0 < \theta < \pi \quad \text{on the upper semi-circle;}$$

$$T(x, 0) = 50, \quad \text{for} \quad -1 < x < 1 \quad \text{on the diameter.}$$



Solution. As discussed in Example 11.9, the transformation

$$w = u + i\,v = \frac{i\,(1-z)}{1+z} = \frac{i\,(1-x-i\,y)}{1+x+i\,y}$$

is a one-to-one conformal mapping of the half-disk H onto the first quadrant $Q: u > 0, v > 0$, and can be written as

(11-24)
$$u + i\,v = \frac{i\,(1-x-i\,y)\,(1+x-i\,y)}{(1+x+i\,y)\,(1+x-i\,y)}$$

$$= \frac{2\,y+i\,\left(1-x^2-y^2\right)}{1+2\,x+x^2+y^2}$$

$$= \frac{2\,y}{(x+1)^2+y^2} + i\,\frac{1-x^2-y^2}{(x+1)^2+y^2}$$

The conformal map given by Equation (11-24) gives rise to a new problem of finding the temperature $T^*(u, v)$ that satisfies the boundary conditions

$$T^*(u, 0) = 100, \quad \text{for} \quad u > 0;$$

$$T^*(0, v) = 50, \quad \text{for} \quad v > 0.$$

If we use Example 11.2, the harmonic function $T^*(u, v)$ is given by

(11-25)
$$T^*(u, v) = 100 + \frac{50-100}{\pi/2}\,\text{Arg}\,(u + i\,v) = 100 - \frac{100}{\pi}\,\text{Arctan}\left(\frac{v}{u}\right).$$

Substituting the expressions for u and v from Equation (11-24) into Equation (11-25) yields the desired solution:

$$T(x, y) = 100 - \frac{100}{\pi}\,\text{Arctan}\left(\frac{1-x^2-y^2}{2\,y}\right).$$

The isothermals $T(x, y) = \text{constant}$ are circles that pass through the points ±1, as shown in Figure 11.19.

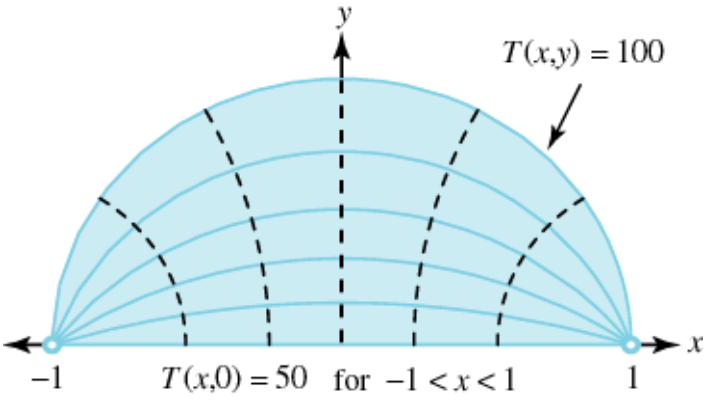


Figure 11.19 The temperature $T(x, y) = 100 - \frac{100}{\pi}\,\text{Arctan}\left(\frac{1-x^2-y^2}{2y}\right)$ in a half-disk.

Explore Solution 11.16.

11.5.1 An Insulated Segment on the Boundary

We now turn to the problem of finding the steady state temperature function $T(x, y)$ inside the simply connected domain D whose boundary consists of three adjacent curves C_1, C_2, and C_3, where $T(x, y) = T_1$ along C_1; $T(x, y) = T_2$ along C_2, and the region is insulated along C_3. Zero heat flowing across C_3 implies that

$$\vec{V}(x, y) \bullet \vec{N}(x, y) = -K\,\vec{N}(x, y) \bullet \text{grad}\,T(x, y) = 0,$$

where $\vec{N}(x, y)$ is perpendicular to C_3. Thus the direction of heat flow must be parallel to this portion of the boundary. In other words, C_3 must be part of a heat flow line $S(x, y) = \text{constant}$ and the isothermals $T(x, y) = \text{constant}$ intersect C_3 orthogonally.

We can solve this problem by finding a conformal mapping

(11-26) $$w = f(z) = u(x, y) + i\,v(x, y)$$

from D onto the semi-infinite strip $G: 0 < u < 1,\ v > 0$ so that the image of the curve C_1 is the ray $u = 0,\ v > 0$; the image of the curve C_2 is the ray given by $u = 1,\ v > 0$; and the thermally insulated curve C_3 is mapped onto the thermally insulated segment $0 < u < 1$ of the u axis, as shown in Figure 11.20.

The new problem in G is to find the steady state temperature function $T^*(u, v)$ so that along the rays, we have the boundary values

(11-27) $$T^*(0, v) = T_1, \quad \text{for} \quad v > 0;$$

$$T^*(1, v) = T_2, \quad \text{for} \quad v > 0.$$

The condition that a segment of the boundary is insulated can be expressed mathematically by saying that the normal derivative of $T^*(u, v)$ is zero. That is,

(11-28) $\frac{\partial T^*}{\partial n} = T_v^*(u, 0) = 0$

where n is a coordinate measured perpendicular to the segment. We can easily verify that the function

$$T^*(u, v) = T_1 + (T_2 - T_1)\, u$$

satisfies the conditions stated in Equations (11-27) and (11-28) for region G. Therefore, using Equation (11-26), we find that the solution in D is

$$T(x, y) = T_1 + (T_2 - T_1)\, u(x, y).$$

The isothermals $T(x, y) = \text{constant}$ and their images under $w = f(z)$ are also illustrated in Figure 11.20.

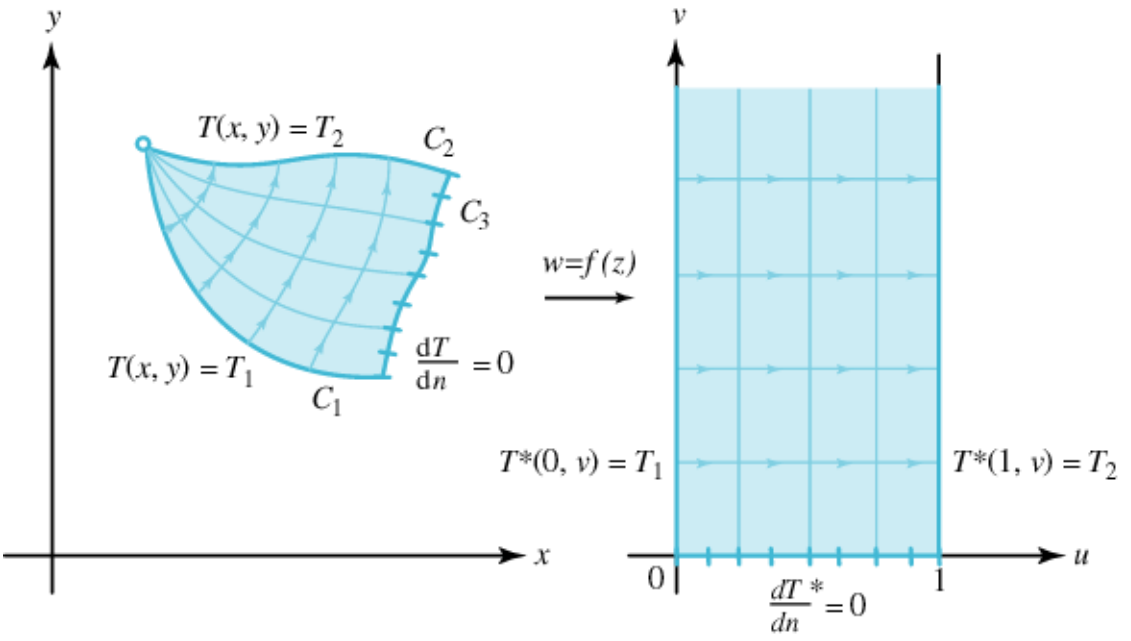


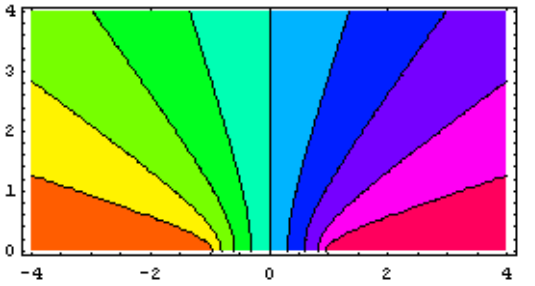
Figure 11.20 Steady state temperatures with one boundary portion insulated.

Example 11.17. Find the temperature $T(x, y)$ for the domain D consisting of the the upper half-plane $\text{Im}(z) > 0$ where the temperature at points on the boundary satisfies

$$T(x, 0) = 1, \quad \text{for} \quad x > 1;$$

$$T(x, 0) = -1, \quad \text{for} \quad x < -1;$$

$$\frac{\partial T}{\partial N} = T_y(x, 0) = 0, \quad \text{for} \quad -1 < x < 1.$$



Solution. The mapping $w = \text{Arcsin}\, z$ conformally maps D onto the semi-infinite strip $v > 0, -\frac{\pi}{2} < u < \frac{\pi}{2}$, where the new problem is to find the steady state temperature $T^*(u, v)$ that has the boundary conditions

$$T^*\left(\frac{\pi}{2}, v\right) = 1, \quad \text{for} \quad v > 0;$$

$$T^*\left(\frac{-\pi}{2}, v\right) = -1, \quad \text{for} \quad v > 0;$$

$$\frac{\partial T^*}{\partial n} = T^*_v(u, 0) = 0, \quad \text{for} \quad -\frac{\pi}{2} < u < \frac{\pi}{2}.$$

Using the result of Example 11.1, we can easily obtain the solution:

$$T^*(u, v) = \frac{2}{\pi} u.$$

Therefore the solution in D is

$$T(x, y) = \frac{2}{\pi} \text{Re}(\text{Arcsin}\, z).$$

If an explicit solution is required, then we can use Formula (10-26) to obtain

$$T(x, y) = \frac{2}{\pi} \text{Arcsin}\left(\frac{\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}}{2}\right)$$

where the function $\text{Arcsin}\, t$ has range values satisfying $-\frac{\pi}{2} < \text{Arcsin}\, t < \frac{\pi}{2}$; see Figure 11.21.

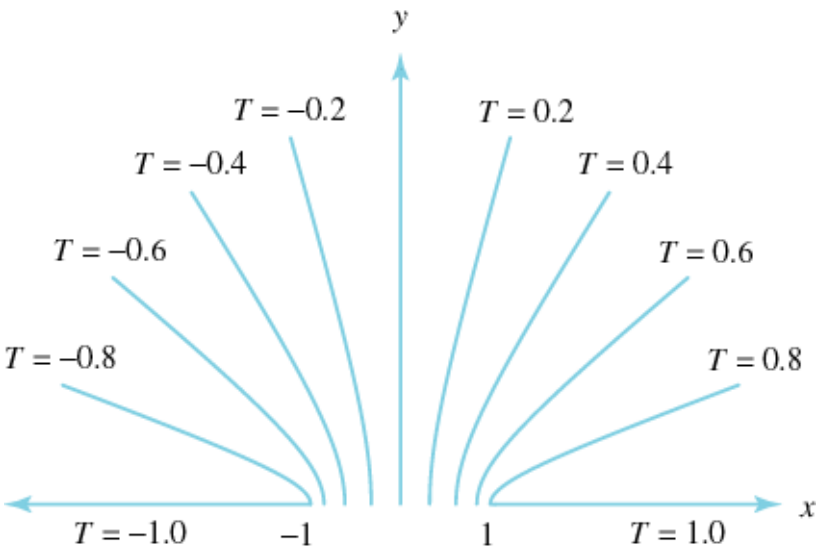


Figure 11.21 The temperature $T(x, y)$ with $T_y(x, 0) = 0$ for $-1 < x < 1$, and boundary values $T(x, 0) = -1$ for $x < -1$, and $T(x, 0) = 1$ for $x > 1$.

Explore Solution 11.17.

Using Previous Techniques

The techniques of the N-value Dirichlet problem in Sections 11.2 and Poisson's integral formula in 11.3 can be used to find steady state temperatures. We recast two previous examples in the context of steady state temperatures.

Revisited Example 11.6. Find the temperature $T(x, y)$ in the upper half-plane $\mathrm{Im}(z) > 0$ where the temperature at points on the boundary satisfies Figure 11.5. That is

$$T(x, 0) = 4, \quad \text{for} \quad x < -1,$$

$$T(x, 0) = 1, \quad \text{for} \quad -1 < x < 0,$$

$$T(x, 0) = 3, \quad \text{for} \quad 0 < x < 1,$$

$$T(x, 0) = 2, \quad \text{for} \quad 1 < x.$$

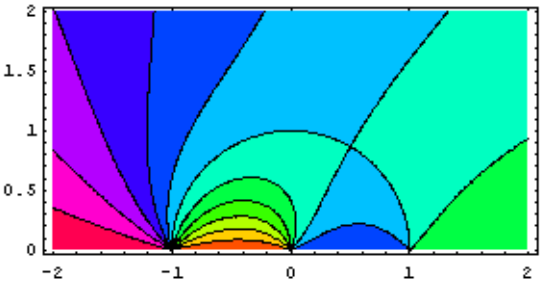


Figure 11.5 The Dirichlet solution for the steady state temperatures.

Solution. This is a four-value Dirichlet problem in the upper half-plane defined by $\mathrm{Im}(z) > 0$. For the z plane, the solution in Equation (11-5) becomes

$$T(x, y) = a_3 + \frac{1}{\pi}\sum_{k=1}^{3}(a_{k-1} - a_k)\,\text{Arg}(z - x_k)$$

$$T(x, y) = a_3 + \frac{a_0 - a_1}{\pi}\text{Arg}(z - x_1) + \frac{a_1 - a_2}{\pi}\text{Arg}(z - x_2) + \frac{a_2 - a_3}{\pi}\text{Arg}(z - x_3)$$

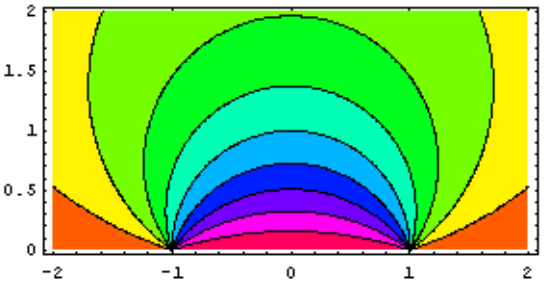
Here we have $a_0 = 4$, $a_1 = 1$, $a_2 = 3$, and $a_3 = 2$ and $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$, which we substitute into equation for $T(x, y)$ to obtain

$$\begin{aligned} T(x, y) &= 2 + \frac{4-1}{\pi}\text{Arg}(z - (-1)) + \frac{1-3}{\pi}\text{Arg}(z - 0) + \frac{3-2}{\pi}\text{Arg}(z - 1) \\ &= 2 + \frac{3}{\pi}\text{Arg}(z + 1) - \frac{2}{\pi}\text{Arg}(z) + \frac{1}{\pi}\text{Arg}(z - 1) \\ &= 2 + \frac{3}{\pi}\text{Arg}(x + 1 + i\,y) - \frac{2}{\pi}\text{Arg}(x + i\,y) + \frac{1}{\pi}\text{Arg}(x - 1 + i\,y) \\ &= 2 + \frac{3}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right) - \frac{2}{\pi}\text{Arctan}\left(\frac{y}{x}\right) + \frac{1}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right) \end{aligned}$$

Explore Revisited Solution 11.6.

Revisited Example 11.11. Find the temperature $T(x, y)$ in the upper half-plane $\text{Im}(z) > 0$, where the temperature at points on the boundary satisfies

$$T(x, 0) = 1, \quad \text{for} \quad -1 \le x \le 1;$$
$$T(x, 0) = 0, \quad \text{for} \quad |x| > 1.$$



Solution. Using Equation (11-12), we obtain

$$T(x, y) = \frac{y}{\pi}\int_{-1}^{1}\frac{1}{(x-t)^2+y^2}\,dt$$

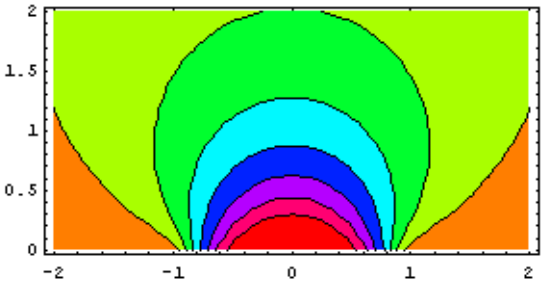
Using techniques from calculus we have the integration formula $\int\frac{y}{(x-t)^2+y^2}\,dt = \text{Arctan}\left(\frac{y}{x-t}\right)$. We obtain the solution as follows

$$\begin{aligned} T(x, y) &= \frac{1}{\pi}\left(\text{Arctan}\frac{y}{x-t}\right)\Bigg|_{t=-1}^{t=1} \\ &= \frac{1}{\pi}\text{Arctan}\left(\frac{y}{x-1}\right) - \frac{1}{\pi}\text{Arctan}\left(\frac{y}{x+1}\right) \end{aligned}$$

Explore Revisited Solution 11.11.

Extra Example 1. Find the temperature $T(x, y)$ in the upper half-plane $\mathrm{Im}(z) > 0$, where the temperature at points on the boundary satisfies

$$T(x, 0) = 1 - x^2, \quad \text{for} \quad -1 \le x \le 1;$$
$$T(x, 0) = 0, \quad \text{for} \quad |x| > 1.$$



Explore Extra Solution 1.

Complex Analysis for Mathematics and Engineering

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11.6 Two-Dimensional Electrostatics

A two-dimensional electrostatic field is produced by a system of charged wires, plates, and cylindrical conductors that are perpendicular to the z plane. The wires, plates, and cylinders are assumed to be so long that the effects at the ends can be neglected, as mentioned in Section 11.4. This assumption results in an electric field $\mathbf{E}(x, y)$ that can be interpreted as the force acting on a unit positive charge placed at the point (x, y). In the study of electrostatics the vector field $\mathbf{E}(x, y)$ is shown to be conservative and is derivable from a function $\phi(x, y)$, called the electrostatic potential, expressed as

$$\mathbf{E}(x, y) = -\operatorname{grad}\phi(x, y) = -\phi_x(x, y) - i\,\phi_y(x, y).$$

If we make the additional assumption that there are no charges within the domain D, then Gauss' Law for electrostatic fields implies that the line integral of the outward normal component of $\mathbf{E}(x, y)$ taken around any small rectangle lying inside D is identically zero. A heuristic argument similar to the one for steady state temperatures with $T(x, y)$ replaced by $\phi(x, y)$ will show that the value of the line integral is

$$-[\phi_{xx}(x, y) + \phi_{yy}(x, y)]\,\Delta x\,\Delta y.$$

This quantity is zero, so we conclude that $\phi(x, y)$ is a harmonic function. If we let $\psi(x, y)$ be the harmonic conjugate, then

$$F(z) = \phi(x, y) + i\,\psi(x, y)$$

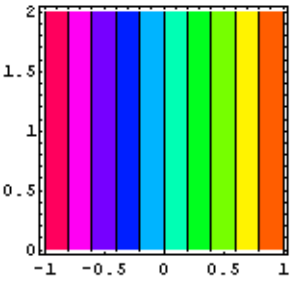
is the complex potential (not to be confused with the electrostatic potential).

The curves $\phi(x, y) = K_1$ are called the equipotential curves, and the curves $\psi(x, y) = K_2$ are called the lines of flux. If a small test charge is allowed to move under the influence of the field

E (x, y), then it will travel along a line of flux. Boundary value problems for the potential function $\phi(x, y)$ are mathematically the same as those for steady state heat flow, and they are realizations of the Dirichlet problem where the harmonic function is $\phi(x, y)$.

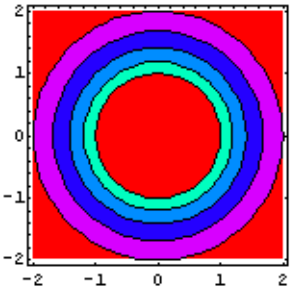
Example 11.18. Consider two parallel conducting planes that pass perpendicular to the z plane through the lines $x = a$ and $x = b$, which are kept at the potentials U_1 and U_2, respectively. Then according to the result of Example 11.1, the electrical potential is

$$\phi(x, y) = U_1 + \frac{U_2 - U_1}{b - a}(x - a).$$



Explore Solution 11.18.

Example 11.19. Find the electrical potential $\phi(x, y)$ in the region between two infinite coaxial cylinders $r = a$ and $r = b$, which are kept at potentials U_1 and U_2, respectively.



Solution. The transformation

$$w = u + i v = \log(z)$$

$$= \log(x + i y)$$

$$= \ln|x + i y| + i \arg(x + i y)$$

maps the annular region between the circles $r = a$ and $r = b$ onto the infinite strip $\ln a < u < \ln b$ in the w plane, as shown in Figure 11.36. The potential $\Phi(u, v)$ in the infinite strip has the boundary values

$$\Phi(\ln a, v) = U_1 \text{ and } \Phi(\ln b, v) = U_2 \text{ for all } v.$$

If we use the result of Example 11.18, the electrical potential $\Phi(u, v)$ is

$$\Phi(u, v) = U_1 + \frac{U_2 - U_1}{\ln b - \ln a}(u - \ln a).$$

Because $u = \ln|x + i y|$, we can use this equation to conclude that the potential $\phi(x, y)$ is

$$\phi(x, y) = U_1 + \frac{U_2 - U_1}{\ln b - \ln a}(\ln|x + i y| - \ln a).$$

The equipotentials $\phi(x, y) = \text{constant}$ are concentric circles centered on the origin, and the lines of flux are portions of rays emanating from the origin. If $U_2 < U_1$, then the situation is as illustrated in Figure 11.36.

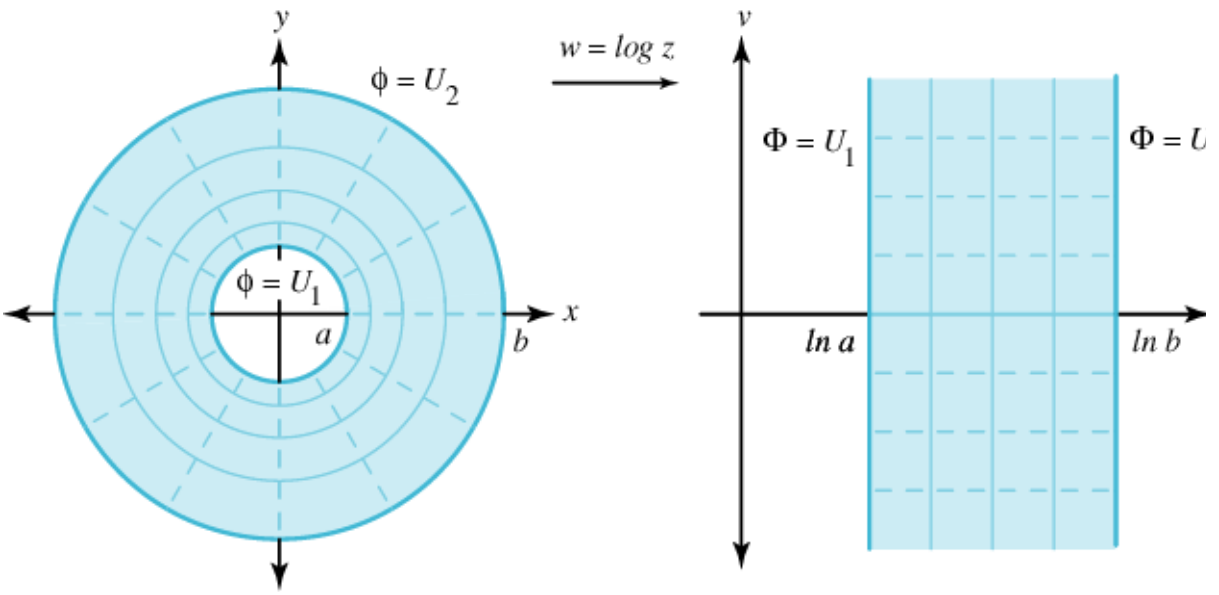
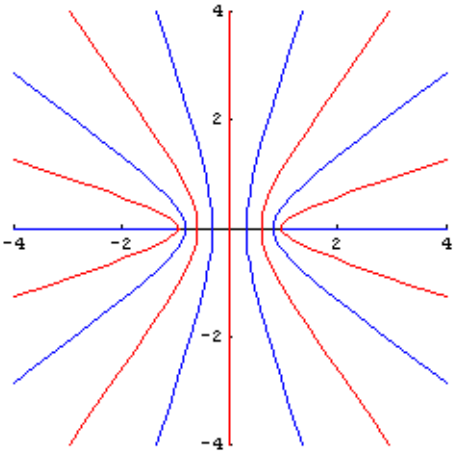


Figure 11.36 The electrical field in a coaxial cylinder, where $U_2 < U_1$.

Explore Solution 11.19.

Example 11.20. Find the electrical potential $\phi(x, y)$ produced by two charged half-planes that are perpendicular to the z plane and pass through the rays $x < -1, y = 0$ and $x > 1, y = 0$ where the planes are kept at the fixed potentials

$$\phi(x, 0) = -300, \quad \text{for} \quad x < -1;$$
$$\phi(x, 0) = 300, \quad \text{for} \quad x > 1.$$



Solution. The result of Example 10.13 shows that the function

$$w = u + i\,v$$

$$= \text{Arcsin}(z)$$

$$= \text{Arcsin}(x + i\,y)$$

is a conformal mapping of the z plane slit along the two rays $x < -1, y = 0$ and $x > 1, y = 0$ onto the vertical strip $-\frac{\pi}{2} < u < \frac{\pi}{2}$. The new problem is to find the potential $\Phi(u, v)$ that satisfies the boundary values

$$\Phi\left(-\frac{\pi}{2}, v\right) = -300, \quad \text{for all } v;$$
$$\Phi\left(\frac{\pi}{2}, v\right) = 300, \quad \text{for all } v.$$

From Example 11.1,

$$\Phi(u, v) = \frac{600}{\pi}\,u.$$

As in the discussion of Example 11.17, the solution in the z plane is

$$\phi(x, y) = \frac{600}{\pi}\,\text{Re}(\text{Arcsin}(x + i\,y)) \quad .$$

$$= \frac{600}{\pi}\,\text{Arcsin}\left(\frac{\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}}{2}\right)$$

Several equipotential curves are shown in Figure 11.37.

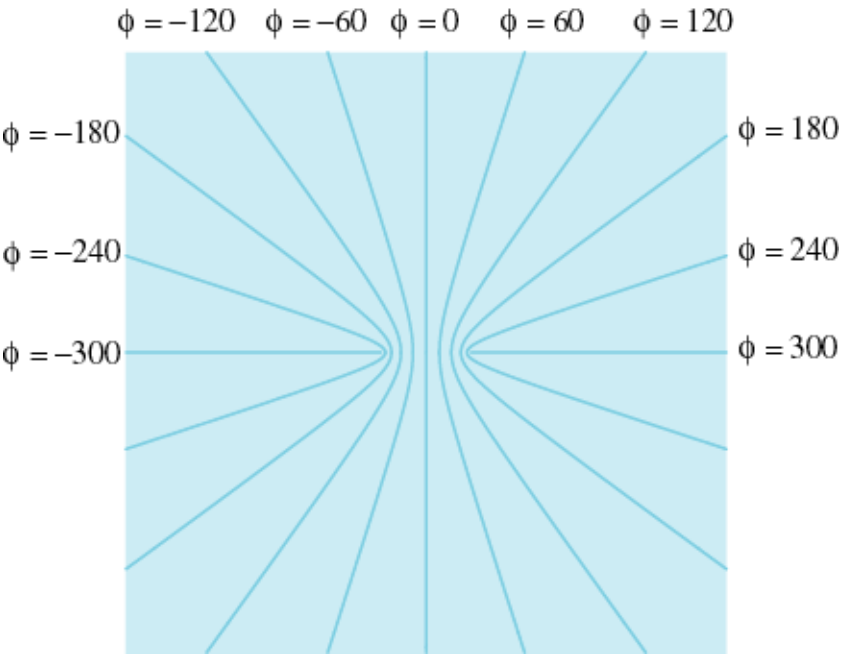
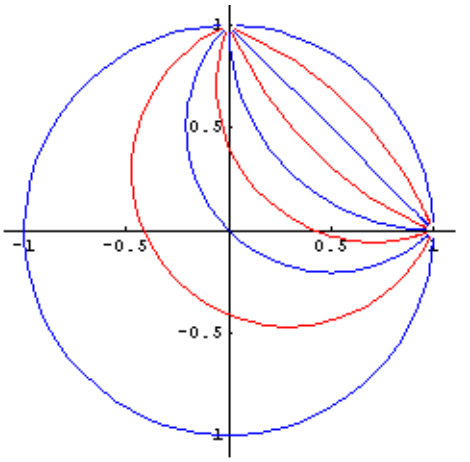


Figure 11.37 The electric field produced by two charged half-planes that are perpendicular to the complex plane.

Explore Solution 11.20.

Example 11.21. Find the electrical potential $\phi(x, y)$ in the disk $D: |z| < 1$ that satisfies the boundary values

$$\phi(x, y) = 80, \quad \text{for } x + i\,y = z \text{ on } C_1 = \left\{z = e^{i\theta} : 0 < \theta < \frac{\pi}{2}\right\};$$
$$\phi(x, y) = 0, \quad \text{for } x + i\,y = z \text{ on } C_2 = \left\{z = e^{i\theta} : \frac{\pi}{2} < \theta < 2\pi\right\}.$$



Solution. The mapping $w = S(z) = \frac{(1-i)(z-i)}{(z-1)}$ is a one-to-one conformal mapping of D onto the upper half-plane $\text{Im}(w) > 0$ with the property that C_1 is mapped onto the negative u axis and C_2 is mapped onto the positive u axis. The potential $\Phi(u, v)$ in the upper half-plane that satisfies the new boundary values

$$\Phi(u, 0) = 80, \quad \text{for } u < 0,$$
$$\Phi(u, 0) = 0, \quad \text{for } u > 0,$$

is given by

(11-29) $$\Phi(u, v) = \frac{80}{\pi}\,\text{Arg}(w) = \frac{80}{\pi}\,\text{Arctan}\left(\frac{v}{u}\right).$$

A straightforward calculation shows that

$$w = u + i\,v = S(z) = \frac{(1-i)(x+i\,y-i)}{(x+i\,y-1)}$$

$$= \frac{(1-i)(x+i(y-1))}{(x-1+i\,y)}$$

$$= \frac{(1-i)(x+i(y-1))(x-1-i\,y)}{(x-1+i\,y)(x-1-i\,y)}$$

$$= \frac{1-2x+x^2-2y+y^2+i\left(1-x^2-y^2\right)}{1-2x+x^2+y^2}$$

$$= \frac{(x-1)^2+(y-1)^2-1+i\left(1-x^2-y^2\right)}{1-2x+x^2+y^2}$$

$$= \frac{(x-1)^2+(y-1)^2-1}{(x-1)^2+y^2} + i\,\frac{1-x^2-y^2}{(x-1)^2+y^2}$$

We substitute the real and imaginary parts, $u = u(x, y) = \frac{(x-1)^2+(y-1)^2-1}{(x-1)^2+y^2}$ and $v = v(x, y) = \frac{1-x^2-y^2}{(x-1)^2+y^2}$ from this equation, into Equation (11-29) to obtain the desired solution:

$$\phi(x, y) = \frac{80}{\pi}\,\text{Arctan}\left(\frac{1-x^2-y^2}{(x-1)^2+(y-1)^2-1}\right).$$

The level curve $\Phi(u, v) = \alpha$ in the upper half-plane is a ray emanating from the origin, and the preimage $\phi(x, y) = \alpha$ in the unit disk is an arc of a circle that passes through the points $z = 1$ and $z = i$. Several level curves are illustrated in Figure 11.38.

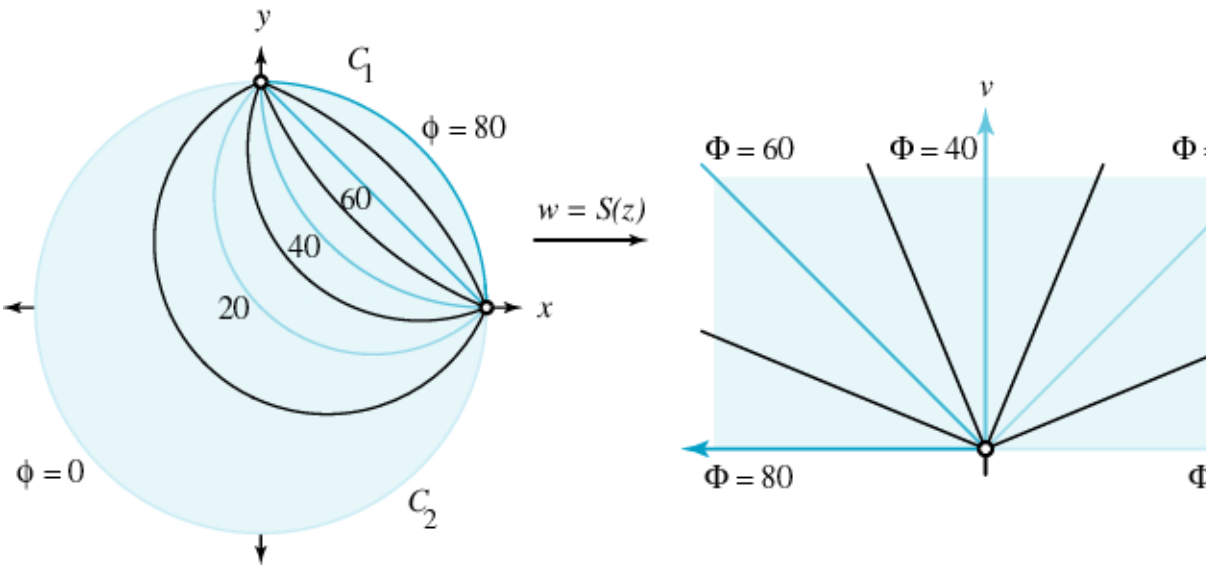


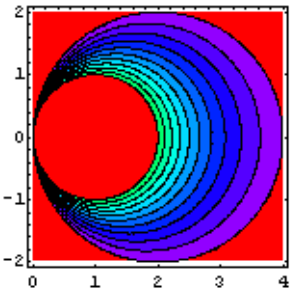
Figure 11.38 The potentials $\phi(x, y)$ and $\Phi(u, v)$.

Explore Solution 11.21.

Extra Example 1. Find the electrical potential $\phi(x, y)$ in the crescent-shaped region that lies inside the disk D: $|z-2| < 2$ and outside the circle $|z-1| = 1$ that satisfies the following

boundary values shown in Figure 11.41:

$$\phi(x, y) = 100, \quad \text{for} \quad |z-2| = 2,\ z \neq 0;$$
$$\phi(x, y) = 50, \quad \text{for} \quad |z-1| = 1,\ z \neq 0.$$



Solution. The result of Example 10.7 shows that the function

$$w = u + i\,v = \frac{i\,(4-z)}{z}$$

$$= \frac{i\,(4-x-i\,y)}{(x+i\,y)}$$

is a conformal mapping of the crescent-shaped region that lies inside the disk $D: |z-2| < 2$ and outside the circle $|z-1| = 1$ onto the horizontal strip $0 < v < 1$.

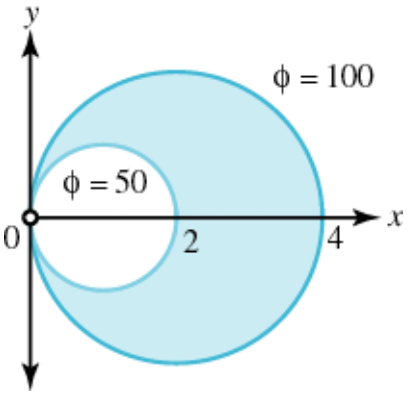


Figure 11.41 The electrical potential $\phi(x, y)$ inside $D: |z-2| < 2$ and outside $|z-1| = 1$.

Revisit and Explore Solution 10.7.

The new problem in the w plane is to find the potential $\Phi(u, v)$ that satisfies the boundary values

$$\Phi(u, 0) = 100, \quad \text{for all } u;$$
$$\Phi(u, 1) = 50, \quad \text{for all } u.$$

In the w plane the solution is

$$\Phi(u, v) = 100 + \frac{50 - 100}{1 - 0}(v - 0)$$

$$= 100 - 50\,v$$

A straightforward calculation shows that

$$w = u + i\,v = \frac{i\,(4 - x - i\,y)}{(x + i\,y)}$$

$$= \frac{i\,(4 - x - i\,y)\,(x - i\,y)}{(x + i\,y)\,(x - i\,y)}$$

$$= \frac{4\,y + i\,(4\,x - x^2 - y^2)}{x^2 + y^2}$$

$$= \frac{4\,y}{x^2 + y^2} + i\,\frac{4\,x - x^2 - y^2}{x^2 + y^2}$$

We substitute the imaginary part, $v = v(x, y) = \frac{4\,x - x^2 - y^2}{x^2 + y^2}$ from this equation, into $\Phi(u, v) = 100 - 50\,v$ to obtain the desired solution:

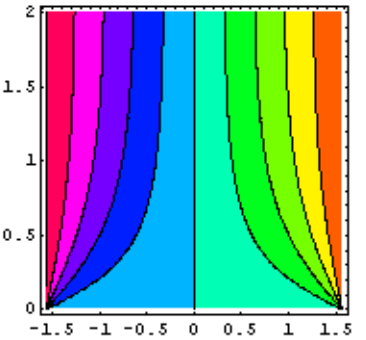
$$\phi(x, y) = 100 - 50\left(\frac{4\,x - x^2 - y^2}{x^2 + y^2}\right)$$

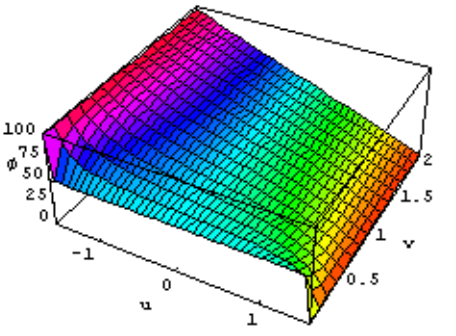
$$= 150 - \frac{200\,x}{x^2 + y^2}$$

Explore Extra Solution 1.

Extra Example 2. Find the electrical potential $\phi(x, y)$ in the semi-infinite strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y > 0$ that has the boundary values shown in Figure 11.42:

$$\phi\left(\frac{\pi}{2}, y\right) = 0, \quad \text{for } y > 0,$$
$$\phi(y, 0) = 50, \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2},$$
$$\phi\left(-\frac{\pi}{2}, y\right) = 100, \quad \text{for } y < 0.$$





Solution. The result of Example 10.13 shows that the function

$$w = u + i\,v = \sin z$$

$$= \sin x \cosh y + i \cos x \sinh y$$

is a conformal mapping of the semi-infinite strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y > 0$ onto the upper half-plane $\text{Im}(w) > 0$.

The new problem in the w plane is to find the electrical potential $\Phi(u, v)$ that satisfies the boundary values

$$\Phi(u, 0) = 0, \quad \text{for } u > 1,$$
$$\Phi(u, 0) = 50, \quad \text{for } -1 < u < 1,$$
$$\Phi(u, 0) = 100, \quad \text{for } u < -1.$$

This is a three-value Dirichlet problem in the upper half-plane defined by $\text{Im}(w) > 0$. For the w plane, the solution in Equation (11-5) becomes

$$\Phi(u, v) = a_2 + \frac{1}{\pi}\sum_{k=1}^{2}(a_{k-1} - a_k)\,\text{Arg}(w - u_k)$$

$$\Phi(u, v) = a_2 + \frac{a_0 - a_1}{\pi}\text{Arg}(w - u_1) + \frac{a_1 - a_2}{\pi}\text{Arg}(w - u_2)$$

Here we have $a_0 = 100$, $a_1 = 50$, and $a_2 = 0$ and $u_1 = -1$, and $u_2 = 1$, which we substitute into the above equation for $\Phi(u, v)$ to obtain

$$\Phi(u, v) = 0 + \frac{100 - 50}{\pi}\text{Arg}(w - (-1)) + \frac{50 - 0}{\pi}\text{Arg}(w - 1)$$

$$= \frac{50}{\pi}\text{Arg}(w + 1) + \frac{50}{\pi}\text{Arg}(w - 1)$$

$$= \frac{50}{\pi}\text{Arg}(u + 1 + i\,v) + \frac{50}{\pi}\text{Arg}(u - 1 + i\,v)$$

$$= \frac{50}{\pi}\text{Arctan}\left(\frac{v}{u+1}\right) + \frac{50}{\pi}\text{Arctan}\left(\frac{v}{u-1}\right)$$

Now make the substitutions $u = u(x, y) = \sin x \cosh y$ and $v = v(x, y) = \cos x \sinh y$ to get the solultion $\phi(x, y)$ in the z plane

$$\phi(x, y) = \frac{50}{\pi}\text{Arg}(1 + \sin(x + i\,y)) + \frac{50}{\pi}\text{Arg}(-1 + \sin(x + i\,y))$$

$$\phi(x, y) = \frac{50}{\pi}\text{Arctan}\left(\frac{\cos x \sinh y}{1 + \sin x \cosh y}\right) + \frac{50}{\pi}\text{Arctan}\left(\frac{\cos x \sinh y}{-1 + \sin x \cosh y}\right)$$

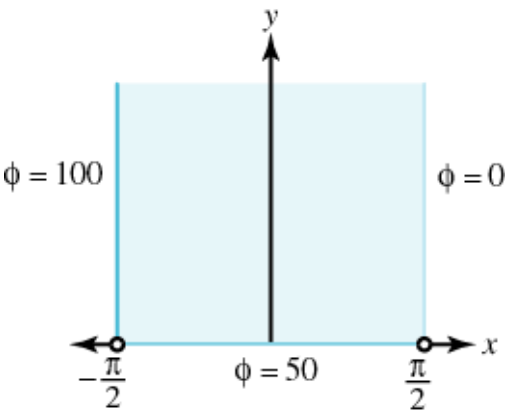


Figure 11.42 The electrical potential $\phi(x, y)$ in the semi-infinite strip $-\frac{\pi}{2} < x < \frac{\pi}{2}, y > 0$.

Explore Extra Solution 2.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

by

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11.7 Two-Dimensional Fluid Flow

Suppose that a fluid flows over the complex plane and that the velocity at the point $z = x + i\,y$ is given by the velocity vector

(11-30) $$\mathbf{V}(x, y) = p(x, y) + i\,q(x, y).$$

We also require that the velocity does not depend on time and that the components $p(x, y)$ and $q(x, y)$ have continuous partial derivatives. The divergence of the vector field is given by

$$\operatorname{div}\mathbf{V}(x, y) = p_x(x, y) + i\,q_y(x, y)$$

and is a measure of the extent to which the velocity field diverges near the point. We ill consider only fluid flows for which the divergence is zero. This is more precisely characterized by requiring that the net flow through any simple closed contour be identically zero.

If we consider the flow out of the small rectangle shown in Figure 11.47, then the rate of outward flow equals the line integral of the exterior normal component of $\mathbf{V}(x, y)$ taken over the sides of the rectangle. The exterior normal component is given by $-q(x, y)$ on the bottom edge, $p(x, y)$ on the right edge, $q(x, y)$ on the top edge, and $-p(x, y)$ on the left edge. Integrating and setting the resulting net flow to zero yield

(11-31) $$\int_y^{y+\Delta y} (p(x + \Delta x, t) - p(x, t))\,dt + \int_x^{x+\Delta x} (q(t, y + \Delta y) - q(t, y))\,dt = 0.$$

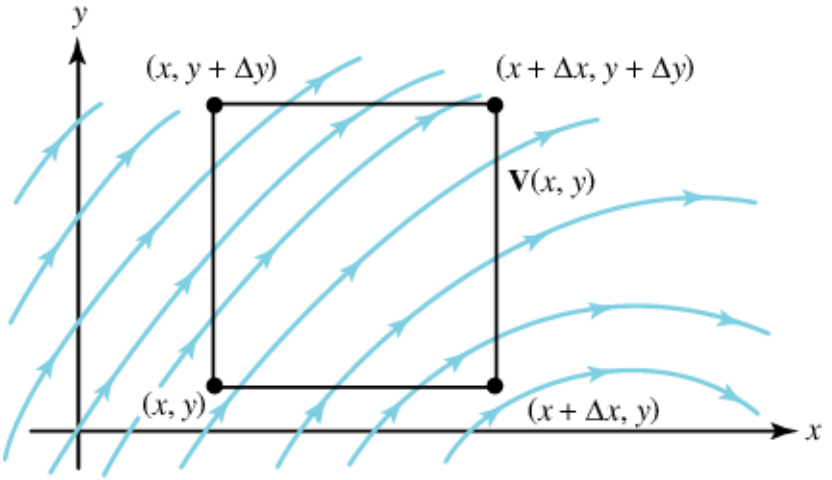


Figure 11.47 A two-dimensional vector field.

Both $p(x, y)$ and $q(x, y)$ are continuously differentiable, so we can use the mean value theorem to show that

(11-32)
$$p(x + \Delta x, t) - p(x, t) = p_x(x_1, y)\,\Delta x, \quad \text{and}$$

$$q(t, y + \Delta y) - q(t, y) = q_y(x, y_2)\,\Delta y,$$

where $x < x_1 < x + \Delta x$ and $y < y_2 < y + \Delta y$. Substitution of the expressions in Equation (11-32) into Equation (11-31) and subsequently dividing through by $\Delta x\,\Delta y$ results in

$$\frac{1}{\Delta y}\int_y^{y+\Delta y} p_x(x_1, y)\,dt + \frac{1}{\Delta x}\int_x^{x+\Delta x} q_y(x, y_2)\,dt = 0.$$

We can use the mean value theorem for integrals with this equation to show that

$$p_x(x_1, y_1) + q_y(x_2, y_2) = 0,$$

where $y < y_1 < y + \Delta y$ and $x < x_2 < x + \Delta x$. Letting $\Delta x \to 0$ and $\Delta y \to 0$ in this equation yields

(11-33)
$$p_x(x, y) + q_y(x, y) = 0,$$

which is called the equation of continuity.

The curl of the vector field in Equation (11-30) has magnitude

$$|\,\mathrm{curl}\,\mathbf{V}(x, y)\,| = q_x(x, y) - p_y(x, y)$$

and is an indication of how the field swirls in the vicinity of a point. Imagine that a "fluid element" at the point (x, y) is suddenly frozen and then moves freely in the fluid. The fluid element will rotate with an angular velocity given by

$$\frac{1}{2}q_x(x, y) - \frac{1}{2}p_y(x, y) = \frac{1}{2}\left|\,\mathrm{curl}\,\mathbf{V}(x, y)\,\right|.$$

We consider only fluid flows for which the curl is zero. Such fluid flows are called irrotational. This condition is more precisely characterized by requiring that the line integral of the tangen-

tial component of $\mathbf{V}(x, y)$ along any simply closed contour be identically zero. If we consider the rectangle in Figure 11.47, then the tangential component is given by $p(x, y)$ on the bottom edge, $q(x, y)$ on the right edge, $-p(x, y)$ on the top edge, and $-q(x, y)$ on the left edge. Integrating and equating the resulting circulation integral to zero yields

$$\int_y^{y+\Delta y} (q(x+\Delta x, t) - q(x, t))\, dt - \int_x^{x+\Delta x} (p(t, y+\Delta y) - p(t, y))\, dt = 0.$$

As before, we apply the mean value theorem and divide through by $\Delta x\, \Delta y$, and obtain the equation

$$\frac{1}{\Delta y} \int_y^{y+\Delta y} q_x(x_1, y)\, dt - \frac{1}{\Delta x} \int_x^{x+\Delta x} p_y(x, y_2)\, dt = 0.$$

We can use the mean value for integrals with this equation to deduce that

$$q_x(x_1, y_1) - p_y(x_2, y_2) = 0.$$

Letting $\Delta x \to 0$ and $\Delta y \to 0$ yields

$$q_x(x, y) - p_y(x, y) = 0.$$

Equations (11-33) and this equation show that the complex function

$$f(z) = p(x, y) - i\, q(x, y)$$

satisfies the Cauchy-Riemann equations and hence it is an analytic function. If we let $F(z)$ denote the antiderivative of $f(z)$, then

(11-34) $$F(z) = \phi(x, y) + i\, \psi(x, y),$$

which is the complex potential of the flow and has the property

$$\begin{aligned} \overline{F'(z)} &= \phi_x(x, y) - i\, \psi_x(x, y) \\ &= p(x, y) + i\, q(x, y) \\ &= \mathbf{V}(x, y) \end{aligned}$$

Since $\phi_x(x, y) = p(x, y)$ and $\phi_y(x, y) = q(x, y)$, we also have

$$\text{grad}\, \phi(x, y) = p(x, y) + i\, q(x, y) = \mathbf{V}(x, y),$$

so $\phi(x, y)$ is the velocity potential for the flow, and the curves

$$\phi(x, y) = K_1$$

are called equipotentials. The function $\psi(x, y)$ is called the stream function. The curves

$$\psi(x, y) = K_2$$

are called streamlines, and describe the paths of the fluid particles.

To demonstrate this result, we implicitly differentiate $\psi(x, y) = K_2$ and find that the slope of a vector tangent is given by

$$\frac{dy}{dx} = \frac{-\psi_x(x,y)}{\psi_y(x,y)}.$$

Using the fact that $\psi_y(x, y) = \phi_x(x, y)$ and this equation, we find that the tangent vector to the curve is

$$\vec{T} = \phi_x(x, y) - i\,\psi_x(x, y) = p(x, y) + i\,q(x, y) = \mathbf{V}(x, y).$$

The main idea of the preceding discussion is the conclusion that, if

(11-35) $$F(z) = \phi(x, y) + i\,\psi(x, y),$$

is an analytic function, then the family of curves

$$\{\psi(x, y) = K_2\}$$

represents the streamlines of a fluid flow.

The boundary condition for an ideal fluid flow is that $\mathbf{V}(x, y)$ should be parallel to the boundary curve containing the fluid (the fluid flows parallel to the walls of a containing vessel). In other words, if Equation (11-35) is the complex potential for the flow, then the boundary curve must be given by $\psi(x, y) = K$ for some constant K; that is, the boundary curve must be a streamline.

Theorem 11.5 (Invariance of Flow). Let

$$F_1(w) = \Phi(u, v) + i\,\Psi(u, v)$$

denote the complex potential for a fluid flow in a domain G in the w plane where the velocity is

$$\mathbf{V}_1(u, v) = \overline{F_1'(w)}.$$

If the function $w = S(z) = u(x, y) + i\,v(x, y)$ is a one-to-one conformal mapping from a domain D in the z plane onto G, then the composite function

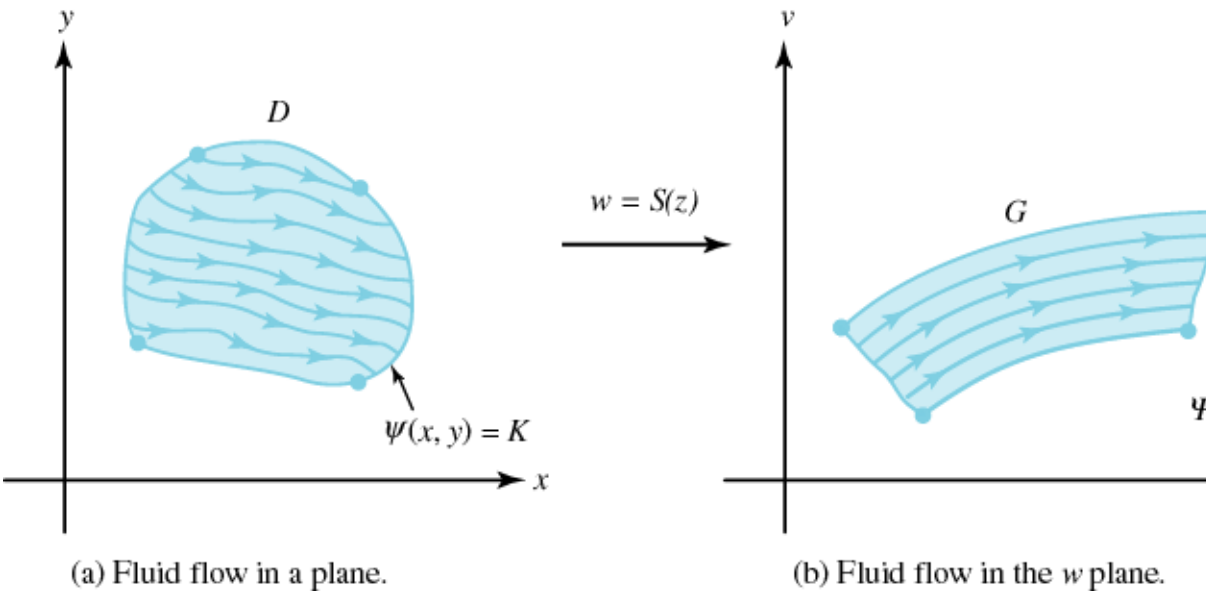
$$F_2(z) = F_1(S(z))$$

$$= \Phi(u(x, y), v(x, y)) + i\,\Psi(u(x, y), v(x, y))$$

is the complex potential for a fluid flow in D where the velocity is

$$\mathbf{V}_2(x, y) = \overline{F_2'(z)}.$$

The situation is shown in Figure 11.48.



(a) Fluid flow in a plane. (b) Fluid flow in the w plane.

Figure 11.48 The image of a fluid flow under conformal mapping.

Proof.

We note that the functions

$$\phi(x, y) = \Phi(u(x, y), v(x, y)), \text{ and}$$

$$\psi(x, y) = \Psi(u(x, y), v(x, y)),$$

are the new velocity potential and stream function, respectively, for the flow in D. A streamline or natural boundary curve

$$\psi(x, y) = K_2$$

in the z plane is mapped onto a streamline or natural boundary curve

$$\Psi(u, v) = K_2$$

in the w plane by the transformation $w = S(z)$. One method for finding a flow inside a domain D in the z plane is to conformally map D onto a domain G in the w plane in which the flow is known.

For an ideal fluid with uniform density , the fluid pressure $P(x, y)$ and speed $|\mathbf{V}(x, y)|$ are related by the following special case of Bernoulli's equation:

$$\frac{P(x,y)}{\rho} + \frac{1}{2}\left|\mathbf{V}(x, y)\right| = \text{constant}.$$

Note that the pressure is greatest when the speed is least.

Example 11.22. The complex potential $F(z) = (a + i\,b)\,z$ has the velocity potential and stream function of

$$\phi(x, y) = a x - b y \text{ and } \psi(x, y) = b x + a y,$$

respectively, and gives rise to the fluid flow defined in the entire complex plane that has a uniform parallel velocity of

$$\mathbf{V}(x, y) = \overline{F'(z)} = \overline{a + i b} = a - i b.$$

The streamlines are parallel lines given by the equation $bx + ay = \text{constant}$ and are inclined at an angle $\alpha = -\text{Arctan}\left(\frac{b}{a}\right)$, as indicated in Figure 11.49.

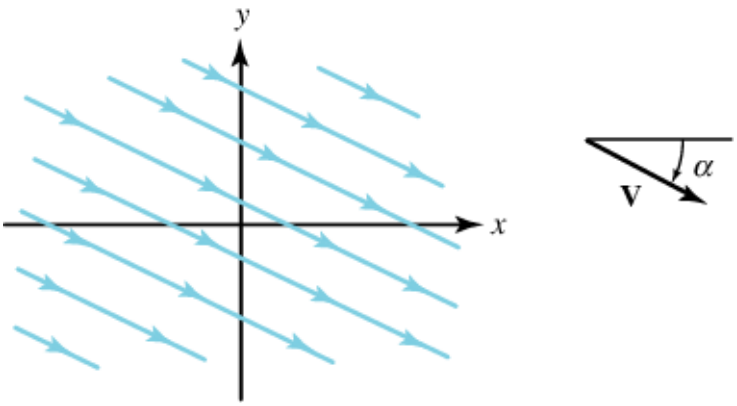


Figure 11.49 A uniform parallel flow.

Solution 11.22.

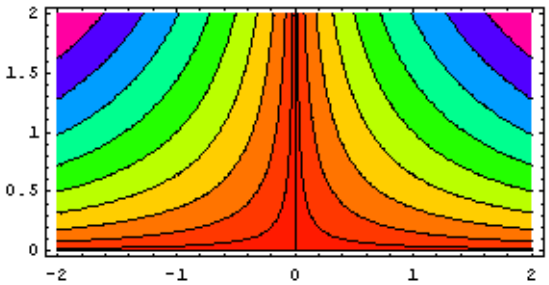
Example 11.23. Consider the complex potential $F(z) = \frac{A}{2} z^2$ where A is a positive real number. The velocity potential and stream function are given by

$$\phi(x, y) = \frac{A}{2}(x^2 - y^2),$$

and

$$\psi(x, y) = A x y,$$

respectively.



The streamlines $\psi(x, y) = \text{constant}$ form a family of hyperbolas with asymptotes along the coordinate axes. The velocity vector $\mathbf{V}(x, y) = \overline{F'(z)} = \overline{A z} = A \overline{z}$ indicates that in the upper half-plane $\text{Im}(z) > 0$, the fluid flows down along the streamlines and spreads out along the x axis, as against a wall, as depicted in Figure 11.50.

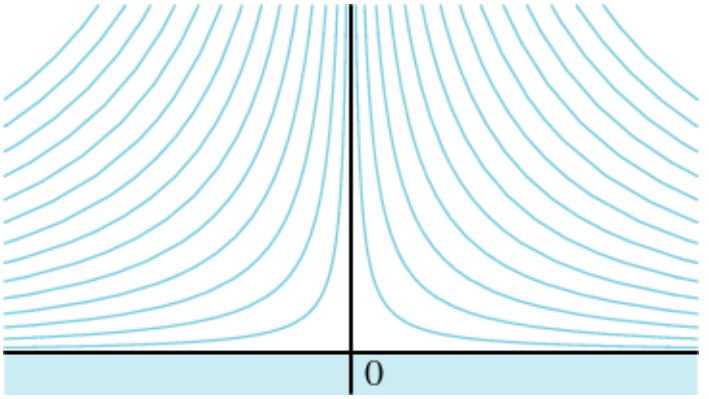


Figure 11.50 The fluid flow with complex potential $F(z) = \frac{A}{2} z^2$.

Explore Solution 11.23.

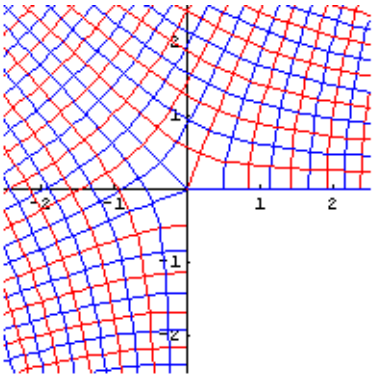
Extra Example 1. Consider the complex potential $F(z) = A z^{4/3}$ where A is a positive real number. The velocity potential and stream function are given by

$$\phi(x, y) = A \operatorname{Re}\left[(x + i y)^{4/3}\right],$$

and

$$\psi(x, y) = A \operatorname{Im}\left[(x + i y)^{4/3}\right],$$

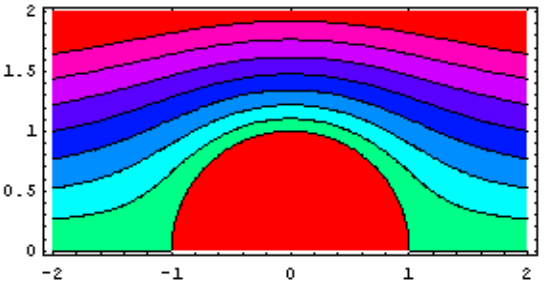
respectively.



The streamlines $\psi(x, y) = \text{constant}$ form a family of hyperbolas with asymptotes along the coordinate axes. The velocity vector $\mathbf{V}(x, y) = \overline{F'(z)} = \frac{4}{3} A \overline{(x + i y)^{1/3}}$ indicates that in the upper half-plane $\operatorname{Im}(z) > 0$, the fluid flows down along the streamlines and spreads out along the positive x axis and negative y axis, as shown in Figure 11.55.

Explore Extra Solution 1.

Example 11.24. Find the complex potential for an ideal fluid flowing from left to right across the complex plane and around the unit circle $|z| = 1$.



Solution. We use the fact that the conformal mapping $w = S(z) = z + \frac{1}{z}$ maps the domain $D = \{z : |z| < 1\}$ one-to-one and onto the w plane slit along the segment $-2 \le u \le 2, v = 0$. The complex potential for a uniform horizontal flow parallel to this slit in the w plane is

$$F_1(w) = A w,$$

where A is a positive real number. The stream function for the flow in the w plane is $\Psi(u, v) = A v$ so that the slit lies along the streamline $\Psi(x, y) = 0$.

The composite function $F_2(z) = F_1(S(z))$ determines the fluid flow in the domain D, where the complex potential is

$$F_2(z) = F_1(S(z)) = A\left(z + \frac{1}{z}\right),$$

where $A > 0$. We can use polar coordinates to express $F_2(z)$ as

$$F_2(z) = F_2(r e^{i\theta}) = A\left(r e^{i\theta} + \frac{1}{r e^{i\theta}}\right)$$

$$= A\left(r\cos\theta + i r\sin\theta + \frac{1}{r\cos\theta + i r\sin\theta}\right)$$

$$= A\left(r\cos\theta + i r\sin\theta + \frac{r\cos\theta - i r\sin\theta}{(r\cos\theta + i r\,\mathrm{Sin}\,\theta)(r\cos\theta - i r\sin\theta)}\right)$$

$$= A\left(r\cos\theta + i r\sin\theta + \frac{r(\cos\theta - i r\sin\theta)}{r^2}\right)$$

$$= A\left(r + \frac{1}{r}\right)\cos\theta + i A\left(r - \frac{1}{r}\right)\sin\theta$$

The streamline $\psi(r, \theta) = A\left(r - \frac{1}{r}\right)\sin\theta = 0$ consists of the rays

$$r > 1, \theta = 0 \quad \text{and} \quad r > 1, \theta = \pi$$

along the x axis and the curve $r - \frac{1}{r} = 0$, which is the unit circle $|z| = r = 1$. Thus the unit circle can be considered as a boundary curve for the fluid flow.

The approximation $F_2(z) = A\left(z + \frac{1}{z}\right) \approx A z$ is valid for large values of z, so we can approximate the flow with a uniform horizontal flow having speed $|\mathbf{V}(x, y)| = A$ at points that are distant from the origin. The streamlines $\psi(x, y) = \text{constant}$ and their images $\Psi(u, v) = \text{constant}$ under the mapping $w = S(z) = z + \frac{1}{z}$ are illustrated in Figure 11.51.

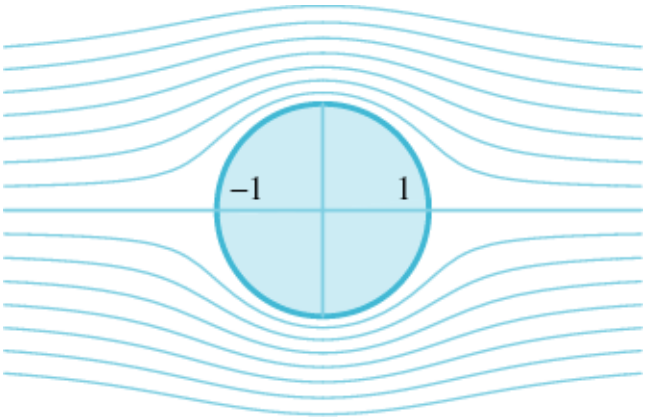
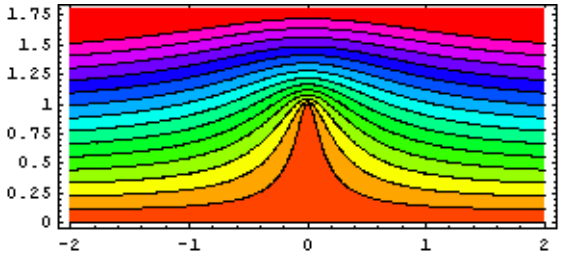


Figure 11.51 Fluid flow around a circle.

Explore Solution 11.24.

Example 11.25. Find the complex potential for an ideal fluid flowing from left to right across the complex plane and around the segment from $z = -i$ to $z = -i$.



Solution. We use the conformal mapping

$$w = S(z) = (z^2+1)^{\frac{1}{2}} = (z+i)^{\frac{1}{2}}(z-i)^{\frac{1}{2}}$$

where the branch of the square root of $Z = z \pm i$ in each factor is $Z^{\frac{1}{2}} = R^{\frac{1}{2}} e^{i\frac{\Theta}{2}}$, where $R = |Z|$, and $\Theta = \arg_{-\pi/2}(Z)$, where $-\frac{\pi}{2} < \Theta \le \frac{3\pi}{2}$. The transformation $w = S(z) = (z^2+1)^{\frac{1}{2}}$ is a one-to-one conformal mapping of the domain D consisting of the z plane slit along the segment $x = 0, -1 \le y \le 1$ onto the domain G consisting of the w plane slit along the segment $-1 \le u \le 1, v = 0$. The complex potential for a uniform horizontal flow parallel to the slit in the w plane is given by $F_1(w) = A w$, where for convenience we choose $A = 1$ and where the slit lies along the streamline $\Psi(u, v) = c = 0$.

The composite function

$$F_2(z) = F_1(S(z)) = A(z^2+1)^{\frac{1}{2}}$$

is the complex potential for a fluid flow in the domain D. The streamlines given by $\psi(x, y) = c$ for the flow in D are obtained by finding the preimage of the streamline $\Psi(u, v) = c$ in G given by the parametric equations

$$v = c \quad \text{and} \quad u = t, \quad \text{for} \quad -\infty < t < \infty.$$

The corresponding streamline in D is found by solving the equation

$$t + i\,c = (z^2+1)^{\frac{1}{2}} = \left((x+i\,y)^2+1\right)^{\frac{1}{2}}$$

for x and y in terms of t. Squaring both sides of this equation yields

$$(t + i\,c)^2 = (x+i\,y)^2 + 1$$

$$t^2 - c^2 + 2\,i\,c\,t = 1 + x^2 - y^2 + 2\,i\,x\,y$$

Equating the real and imaginary parts leads to the system of equations

$$t^2 - c^2 = 1 + x^2 - y^2 \quad \text{and} \quad 2\,c\,t = 2\,x\,y.$$

To eliminate the parameter t in the last two equations, first use the equations

$$c^2\,t^2 - c^4 = c^2 + c^2\,x^2 - c^2\,y^2 \quad \text{and} \quad c^2\,t^2 = x^2\,y^2,$$

then get $x^2\,y^2 - c^4 = c^2 + c^2\,x^2 - c^2\,y^2$ which can be rewritten as

$$c^2\,y^2 + x^2\,y^2 = c^2 + c^4 + c^2\,x^2$$

$$y^2\,(c^2+x^2) = c^2\,(1+c^2+x^2)$$

$$y\,\sqrt{c^2+x^2} = c\,\sqrt{1+c^2+x^2}$$

and we can solve for y in terms of x to obtain

$$y = c\,\sqrt{\frac{1+c^2+x^2}{c^2+x^2}}$$

for streamlines in D. For large values of x, this streamline approaches the asymptote $y = c$ and approximates a horizontal flow, as shown in Figure 11.52.

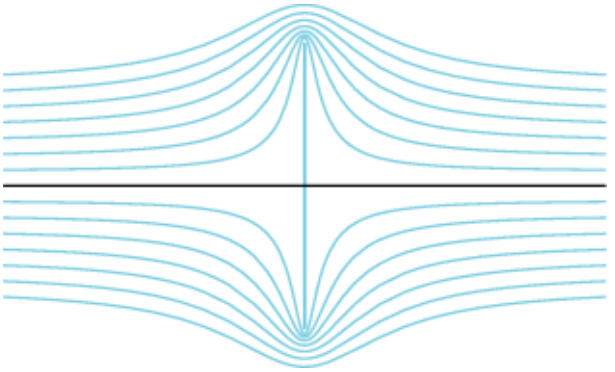


Figure 11.52 Flow around a segment.

Explore Solution 11.25.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering
by
John H. Mathews and Russell W. Howell

Jones and Bartlett Learning

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11.8 The Joukowski Airfoil

The Russian scientist Nikolai Egorovich Joukowsky studied the function

$$J(z) = z + \frac{1}{z}.$$

He showed that the image of a circle passing through $z_1 = 1$ and containing the point $z_2 = -1$ is mapped onto a curve shaped like the cross section of an airplane wing. We call this curve the Joukowski airfoil. If the streamlines for a flow around the circle are known, then their images under the mapping $w = J(z)$ will be streamlines for a flow around the Joukowski airfoil, as shown in Figure 11.60.

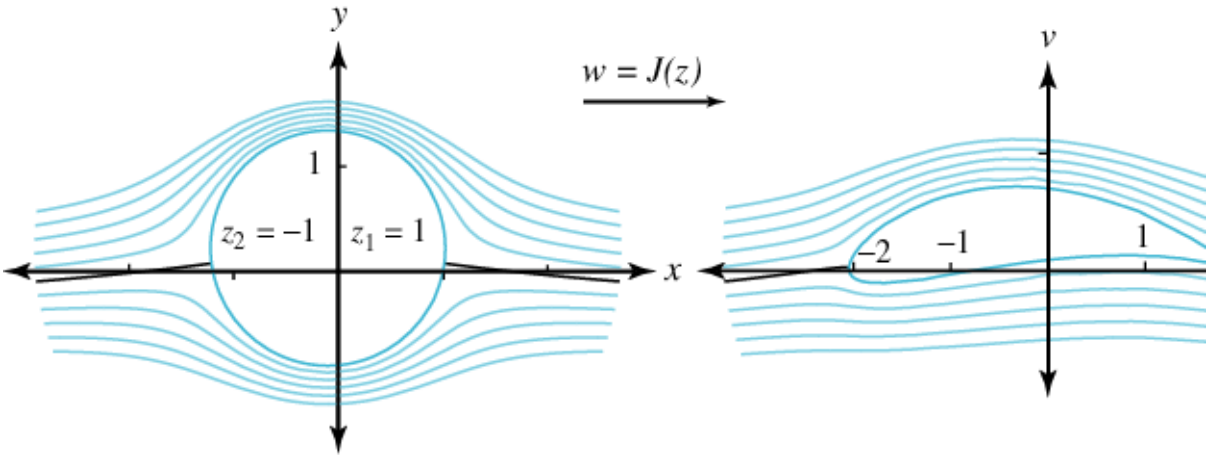


Figure 11.60 Image of a fluid flow under $w = J(z) = z + \frac{1}{z}$.

The mapping $w = J(z)$ is two-to-one, because $J(z) = J\left(\frac{1}{z}\right)$, for $z \neq 0$. The region $|z| > 1$ is mapped one-to-one onto the w plane slit along the segment of the real axis $-2 \leq u \leq 2, v = 0$. To visualize this mapping, we investigate the implicit form, which we obtain by

using the substitutions

$$w - 2 = z - 2 + \frac{1}{z} = \frac{z^2 - 2z + 1}{z} = \frac{(z-1)^2}{z}, \text{ and}$$

$$w + 2 = z + 2 + \frac{1}{z} = \frac{z^2 + 2z + 1}{z} = \frac{(z+1)^2}{z}.$$

Forming the quotient of these two quantities results in the relationship

$$\frac{w-2}{w+2} = \frac{(z-1)^2}{z} \Big/ \frac{(z+1)^2}{z} = \frac{(z-1)^2}{(z+1)^2}.$$

The inverse of $z = T(w) = \frac{w-2}{w+2}$ is $w = S_3(z) = \frac{2+2z}{1-z}$. If we use the notation $S_1(z) = \frac{z-1}{z+1}$ and $S_2(z) = z^2$, then we can express $J(z)$ as the composition of $S_1(z)$, $S_2(z)$, and $S_3(z)$, that is

(11-36) $$w = J(z) = S_3(S_2(S_1(z))).$$

Which is verified by the calculation

$$S_3(S_2(S_1(z))) = S_3\left(S_2\left(\frac{z-1}{z+1}\right)\right)$$

$$= S_3\left(\frac{(z-1)^2}{(z+1)^2}\right)$$

$$= \frac{2 + 2\frac{(z-1)^2}{(z+1)^2}}{1 - \frac{(z-1)^2}{(z+1)^2}}$$

$$= z + \frac{1}{z}$$

We can easily show that $w = J(z) = z + \frac{1}{z}$ maps the four points $z_1 = -i$, $z_2 = 1$, $z_3 = i$, and $z_4 = -1$ onto $w_1 = 0$, $w_2 = 2$, $w_3 = 0$, and $w_4 = -2$, respectively. However, the composition functions in Equation (11-36) must be considered in order to visualize the geometry involved. First, the bilinear transformation $Z = S_1(z) = \frac{z-1}{z+1}$ maps the region $|z| > 1$ onto the right half-plane $\mathrm{Re}(Z) > 0$, and the points $z_1 = -i$, $z_2 = 1$, $z_3 = i$, and $z_4 = -1$ are mapped onto $Z_1 = -i$, $Z_2 = 0$, $Z_3 = i$, and $Z_4 = i\infty$, respectively. Second, the function $W = S_2(Z) = Z^2$ maps the right half plane onto the W plane slit along its negative real axis, and the points $Z_1 = -i$, $Z_2 = 0$, $Z_3 = i$, and $Z_4 = i\infty$, are mapped onto $W_1 = -1$, $W_3 = 0$, $W_3 = -1$, and $W_4 = -\infty$, respectively. Then the bilinear transformation $w = S_3(W) = \frac{2+2W}{1-W}$ maps the latter region onto the W plane slit along the portion of the real axis $U \le -2$, $V = 0$, and the points $W_1 = -1$, $W_3 = 0$, $W_3 = -1$, and $W_4 = -\infty$ are mapped onto $w_1 = 0$, $w_2 = 2$, $w_3 = 0$, and $w_4 = -2$, respectively. These three compositions are shown in Figure 11.61.

Exploration

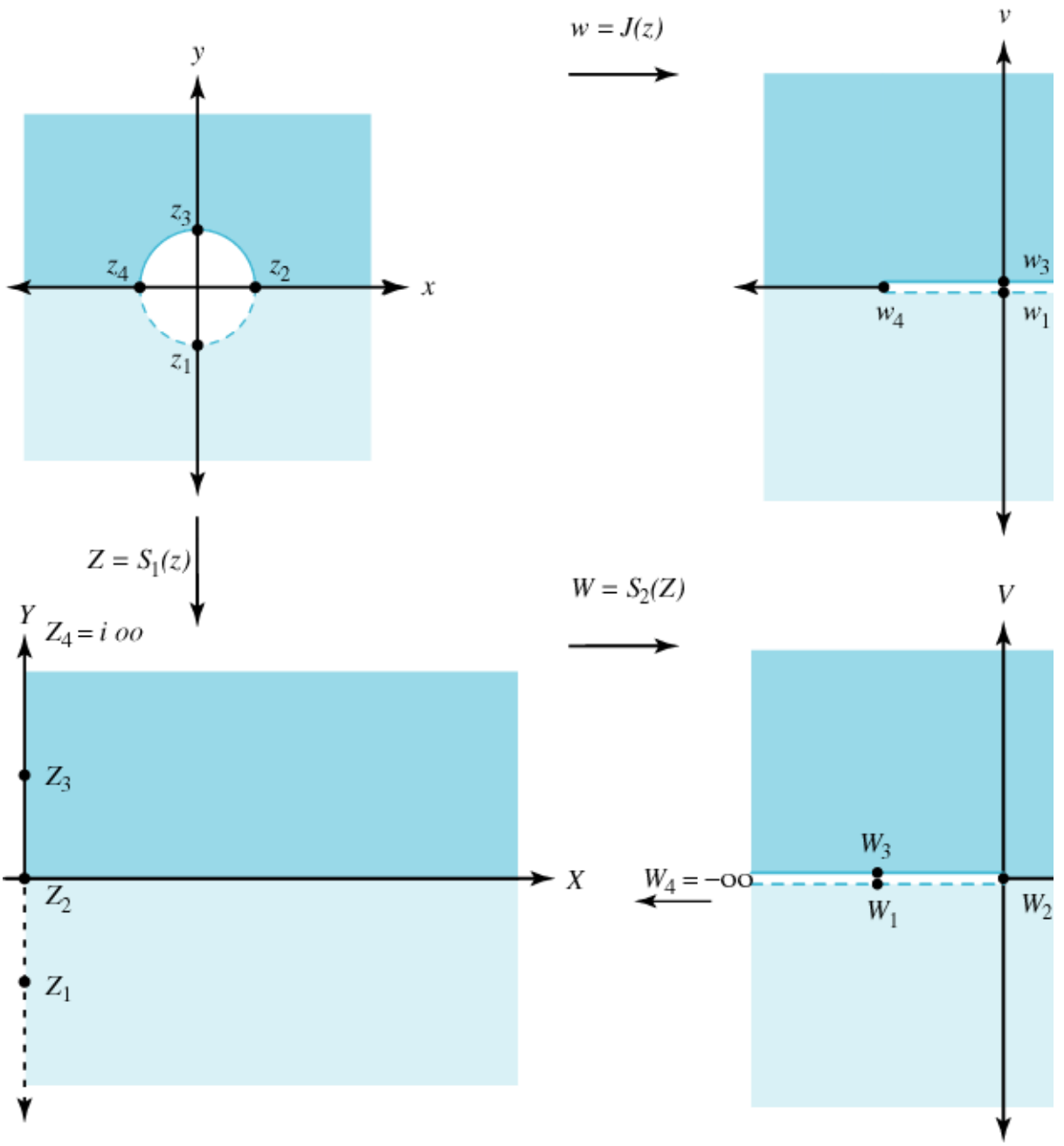


Figure 11.61 The composition mappings for $w = J(z) = S_3(S_2(S_1(z)))$.

The circle C_0 with center $c_0 = i\,a$ on the imaginary axis passes through the points $z_2 = 1$, and $z_4 = -1$ and has radius $r_0 = \sqrt{1 + a^2}$. With the restriction that $0 < a < 1$, then this circle intersects the x axis at the point $z_2 = 1$ with angle $\alpha_0 = \frac{\pi}{2} - \text{Arctan}\,a$, with $\frac{\pi}{2} < \alpha_0 < \frac{\pi}{2}$. We want to track the image of C_0 in the Z, W, and w planes. First, the image of this circle C_0 under $Z = S_1(z) = \frac{z-1}{z+1}$ is the line L_0 that passes through the origin and is inclined at the angle α_0. Second, the function $W = S_2(Z) = Z^2$ maps the line L_0 onto the ray R_0 inclined at the angle $2\,\alpha_0$. Finally, the transformation given by $w = S_3(W) = \frac{2+2W}{1-W}$ maps the ray R_0 onto the arc of the circle A_0 that passes through the points $w_2 = 2$, and $w_4 = -2$ and intersects the u axis at $w_2 = 2$ with angle $2\,\alpha_0$, where $\pi < 2\,\alpha_0 < \pi$. The restriction on the angle α_0, and hence $2\,\alpha_0$, is necessary in order for the arc A_0 to have a low profile. The arc A_0 lies in the center of the Joukowski airfoil and is

shown in Figure 11.62.

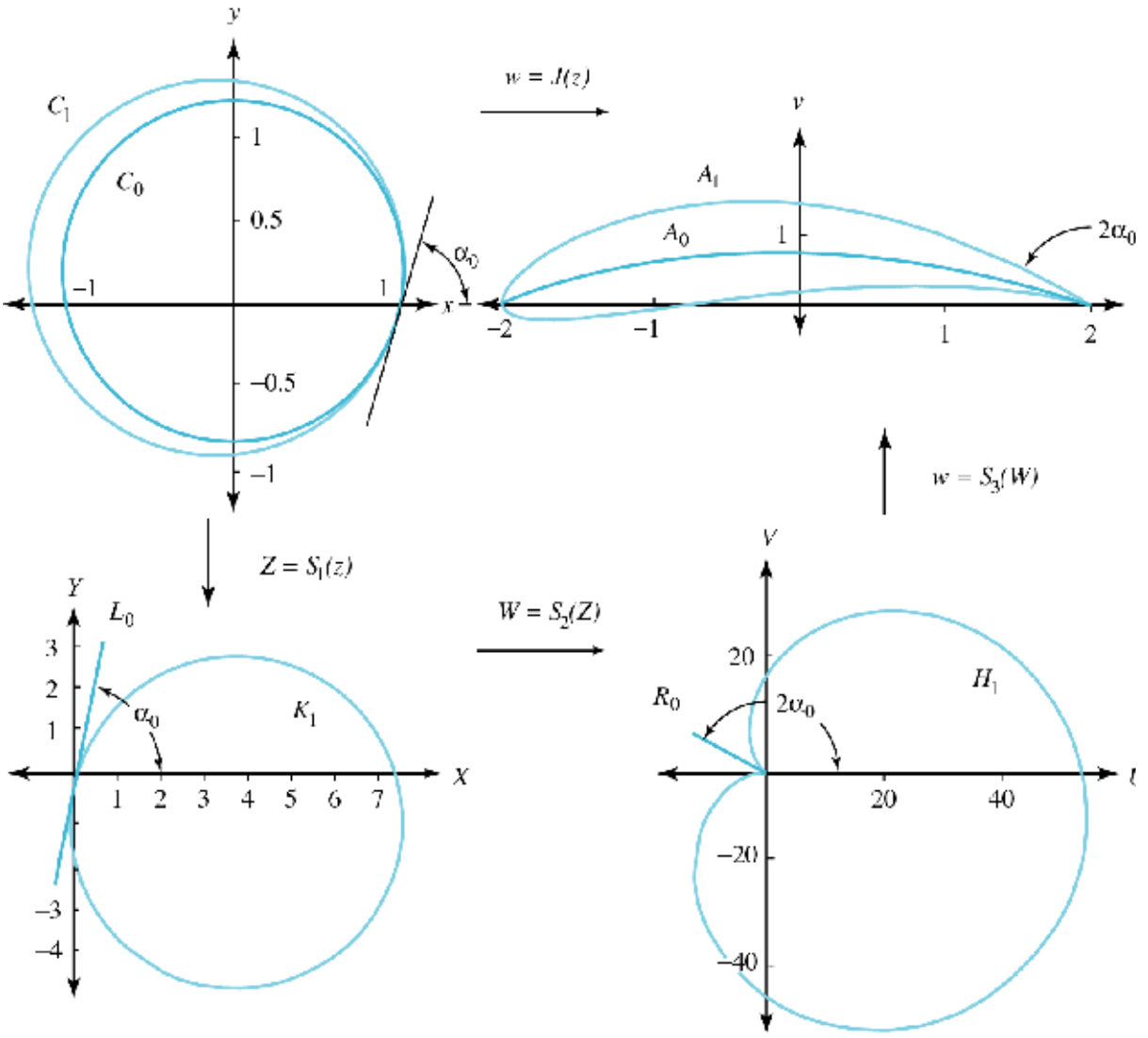


Figure 11.62 The images of the circles C_0 and C_1 under the composition mappings for $w = J(z) = S_3(S_2(S_1(z)))$.

If we let b be fixed, $0 < b < 1$, then the larger circle C_1 with center given by $c_1 = -h + i(1+h)b$ (just a bit to the left of the imaginary axis) will pass through the points $z_2 = 1$, and $z_4^* = -1 - 2h$ and has radius $r_1 = (1+h)\sqrt{1+b^2}$. The circle C_1 also intersects the x axis at the point $z_2 = 1$ at the angle α_0. The image of circle C_1 under $Z = S_1(z) = \frac{z-1}{z+1}$ is the circle K_1, which is tangent to L_0 at the origin in the Z-plane. The function $W = S_2(Z) = Z^2$ maps the circle K_1 onto the cardioid H_1 in the W-plane. Finally, $w = S_3(W) = \frac{2+2W}{1-W}$ maps the cardioid H_1 onto the Joukowski airfoil A_1 that passes through the point $w_2 = 2$ and surrounds the point $w_4 = -2$, as shown in Figure 11.62. An observer traversing C_1 counterclockwise will traverse the image curves K_1 and H_1 clockwise but will traverse A_1 counterclockwise. Thus the points z_4, Z_4, W_4, and w_4 will always be to the observer's left.

Now we are ready to visualize the flow around the Joukowski airfoil. We start with the fluid flow around a circle (see Figure 11.51). This flow is adjusted with a linear transformation $z^* = az + b$ so that it flows horizontally around the circle C_1, as shown in Figure 11.63. Then the

mapping $w = J(z) = z^* + \frac{1}{z^*}$ creates a flow around the Joukowski airfoil, as illustrated in Figure 11.64.

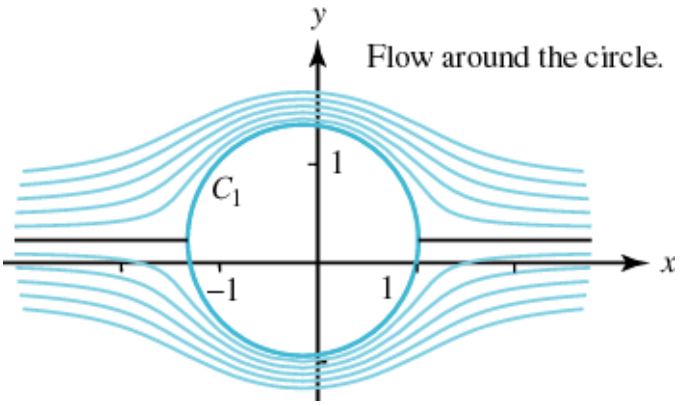


Figure 11.63 The horizontal flow around the circle C_1.

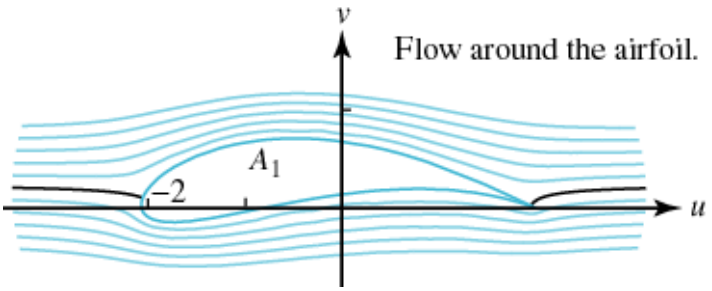


Figure 11.64 The horizontal flow around the Joukowski airfoil A_1.

11.8.1 Flow with Circulation

The function $F(z) = s z + \frac{s}{z} + \frac{k}{2\pi i} \operatorname{Log} z$, where $s > 0$ and k is real, is the complex potential for a uniform horizontal flow past the unit circle $|z| = 1$, with circulation strength k and velocity at infinity $V_\infty = s$.

For illustrative purposes, we let $s = 1$ and use the substitution $a = \frac{-k}{2\pi}$. Now the complex potential has the form

(11-37) $$F(z) = z + \frac{1}{z} + a i \operatorname{Log} z$$

and the corresponding velocity function is

$$V(x, y) = \overline{F'(z)} = 1 - (\overline{z})^{-2} - a i (\overline{z})^{-1}.$$

We can express the complex potential in $F(x + i y) = \phi(x, y) + i \psi(x, y)$ form:

$$F(r e^{i\theta}) = r e^{i\theta} + \frac{1}{r e^{i\theta}} + a\,i\,\text{Log}(r e^{i\theta})$$

$$= r e^{i\theta} + \frac{1}{r} e^{-i\theta} + a\,i\,(\ln r + i\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta - a\,\theta + i\left(r - \frac{1}{r}\right)\sin\theta + a \ln r$$

and we have the formulas for the velocity potential $\phi(x, y)$ stream function $\psi(x, y)$

$$\phi(r\cos\theta, r\sin\theta) = \left(r + \frac{1}{r}\right)\cos\theta - a\,\theta, \text{ and}$$

$$\psi(r\cos\theta, r\sin\theta) = \left(r - \frac{1}{r}\right)\sin\theta + a \ln r.$$

For the flow given by $\psi(x, y) = c$, where c is a constant, we have

$$\psi(r\cos\theta, r\sin\theta) = \left(r - \frac{1}{r}\right)\sin\theta + a \ln r = c. \quad \text{(Streamlines.)}$$

Setting $r = 1$ in this equation, we get $\psi(\cos\theta, \sin\theta) = \left(1 - \frac{1}{1}\right)\sin\theta + a \ln 1 = 0$ for all θ, so the unit circle is a natural boundary curve for the flow.

Points at which the flow has zero velocity are called stagnation points. To find them we solve $F'(z) = 0$; for the function in Equation (11-37) we have

$$F'(z) = 1 - (z)^{-2} + a\,i\,(z)^{-1} = 0$$

$$F'(z) = 1 - \frac{1}{z^2} + a\,i\,\frac{1}{z} = 0$$

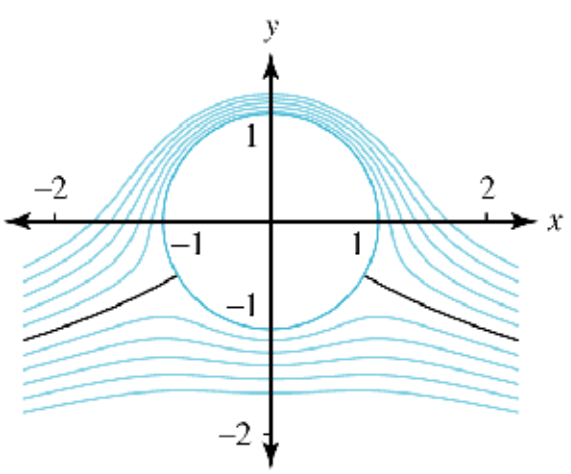
Multiplying through by z^2 and rearranging terms gives

$$z^2 + a\,i\,z - 1 = 0$$

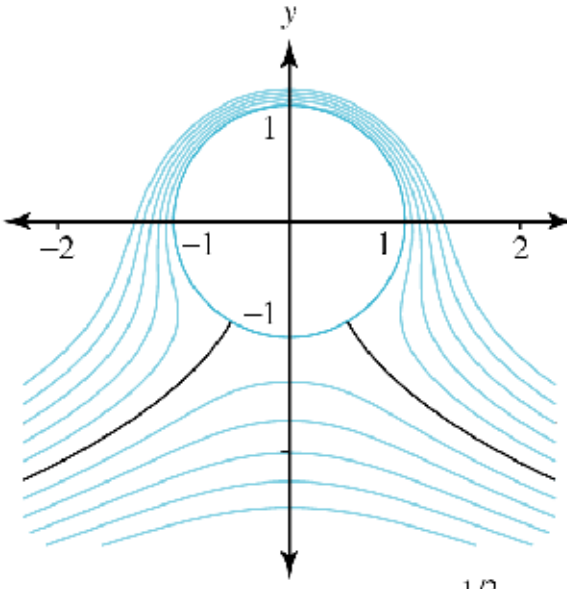
Now we invoke the quadratic equation to obtain

$$z = \frac{\pm\sqrt{4 - a^2} - a\,i}{2} \quad \text{(stagnation point(s).)}$$

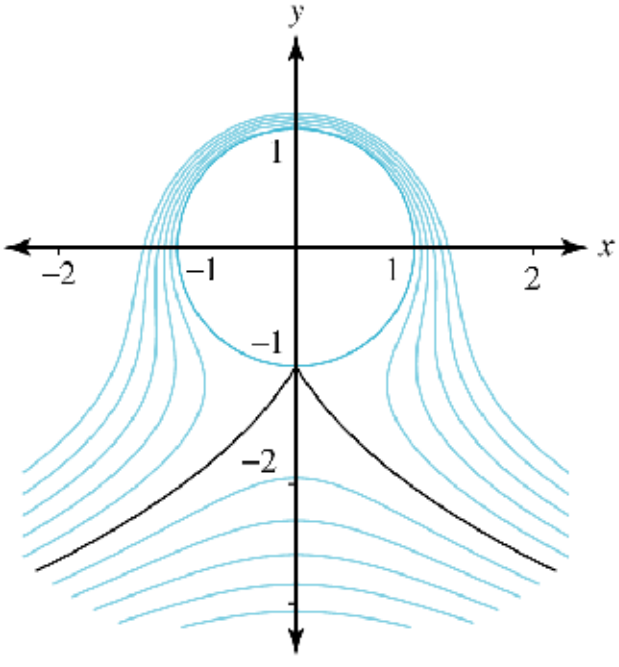
If $0 \le |a| < 2$, then there are two stagnation points on the unit circle $|z| = 1$. If $a = 2$, then there is one stagnation point on the unit circle. If $a > 2$, then the stagnation point lies outside the unit circle. We are mostly interested in the case with two stagnation points. When $a = 0$, the two stagnation points are $z = \pm 1$, which is the flow discussed in Example 11.25. The cases $a = 1$, $a = \sqrt{3}$, $a = 2$, and $a = 2.2$ are shown in Figure 11.65.



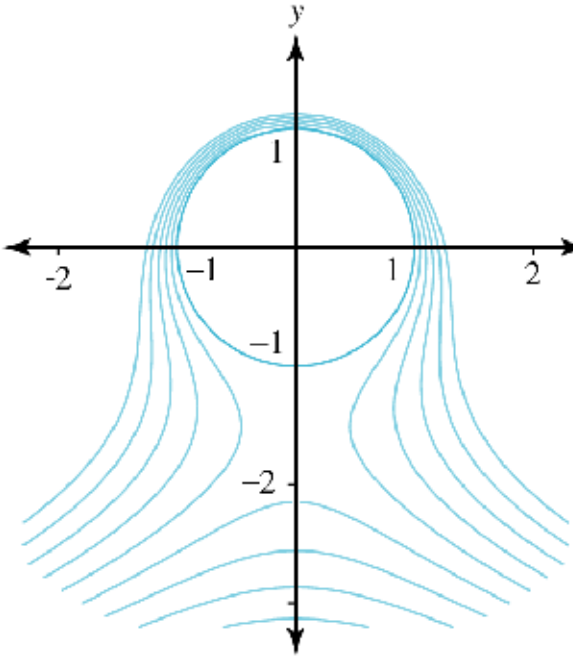
Flow with circulation $a = 1$



Flow with circulation $a = 3^{1/2}$



Flow with circulation $a = 2$



Flow with circulation $a = 2.2$

Figure 11.65 Flows past the unit circle with circulation $a = 1$, $a = \sqrt{3}$, $a = 2$, and $a = 2.2$.

We are now ready to combine the preceding ideas. For illustrative purposes, we consider a C_1 circle with center $c_0 = -0.15 + 0.23\,i$ that passes through the points $z_2 = 1$, and $z_4^* = -1.3$ and has radius $r_0 = 0.23\sqrt{\frac{13}{2}}$. We use the linear transformation $Z = S(z) = -0.15 + 0.23\,i + r_0\,z$ to map the flow with circulation $k = -0.52\,\pi$ (or $a = \frac{-k}{2\pi} = \frac{-(-0.52\,\pi)}{2\,\pi} = 0.26$) around $|z| = 1$ onto the flow around the circle C_1, as shown in Figure 11.66.

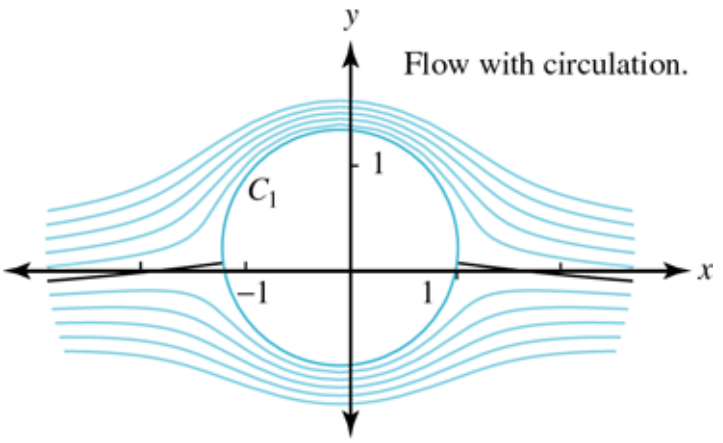


Figure 11.66 Flow with circulation around C_1.

Then we use the mapping $J(Z) = Z + \frac{1}{Z}$ to map this flow around the Joukowski airfoil, as shown in Figure 11.67 and compare it to the flows shown in Figures 11.63 and 11.64.

If the second transformation in the composition given by $w = J(z) = S_3(S_2(S_1(z)))$ is modified to be $S_2(z) = z^{1.925}$, then the image of the flow shown in Figure 11.66 will be the flow around the modified airfoil shown in Figure 11.68. The advantage of this latter airfoil is that the sides of its tailing edge form an angle of $0.15\,\pi$ radians, or 27°, which is more realistic than the angle of 0° of the traditional Joukowski airfoil.

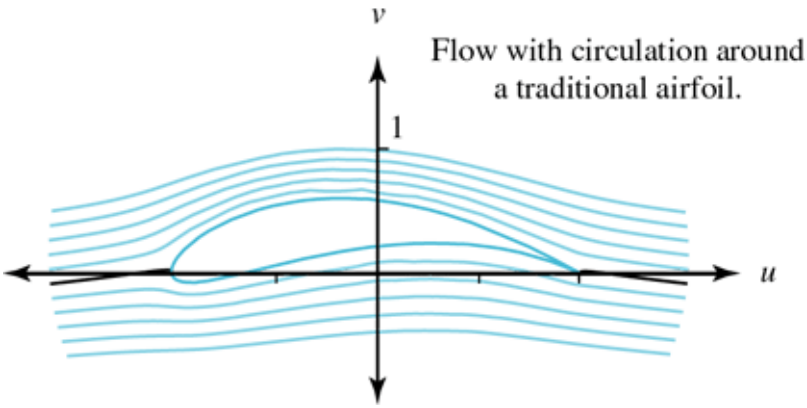


Figure 11.67 Flow with circulation around a traditional Joukowski airfoil.

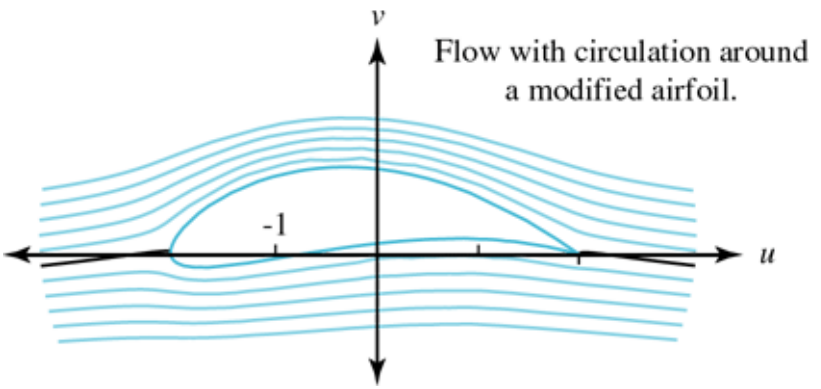
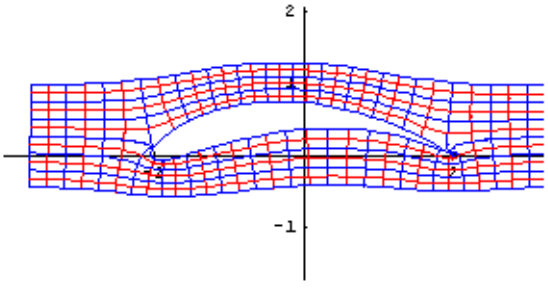


Figure 11.68 Flow with circulation around a modified Joukowski airfoil.

The following *Mathematica* subroutine will form the functions that are needed to graph a Joukowski airfoil.

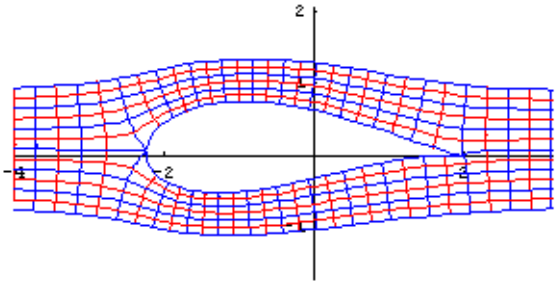
```
Z = 0; Remove[Z];
Clear[F, F1, F2, F3, F4, dx, dy, r0, w, z, Z];
JoukowskiFun[dx_, dy_] := Module[{},
   r0 = Sqrt[(1 - dx)^2 + (0 - dy)^2];
   Z1 = (z + Sqrt[z^2 - 4])/2; Z2 = (z - Sqrt[z^2 - 4])/2;
   Z3 = (-z - Sqrt[z^2 - 4])/2; Z4 = (-z + Sqrt[z^2 - 4])/2;
   w[z_] = r0 z + dx + i dy;
   F[z_] = z + 1/z;
   F1[z_] = F[w[Z1]]; F2[z_] = F[w[Z2]];
   F3[z_] = F[w[Z3]]; F4[z_] = F[w[Z4]]; ];
```

Example 1. For a fixed value `dx`, increasing the parameter `dy` will bend the airfoil.



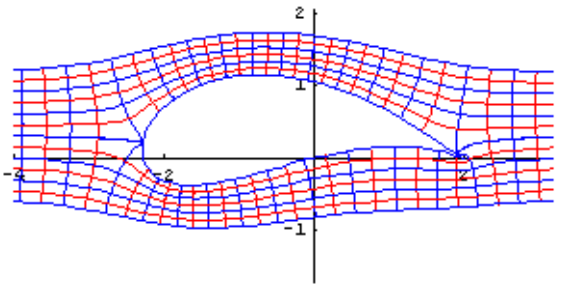
Explore Solution 1.

Example 2. For a fixed value `dy`, increasing the parameter `dx` will fatten out the airfoil.



Explore Solution 2.

Example 3. Increasing both parameters `dx` and `dy` will bend and fatten out the airfoil.



Explore Solution 3.

Example 4. Consider the modified Joukowski airfoil when `W = S2 (z) = z^1.925` is used to map the `Z` plane onto the `W` plane. Refer to Figure 11.69 and discuss why the angle of the trailing edge of the modified Joukowski airfoil A_1 forms an angle of `0.15` radians.

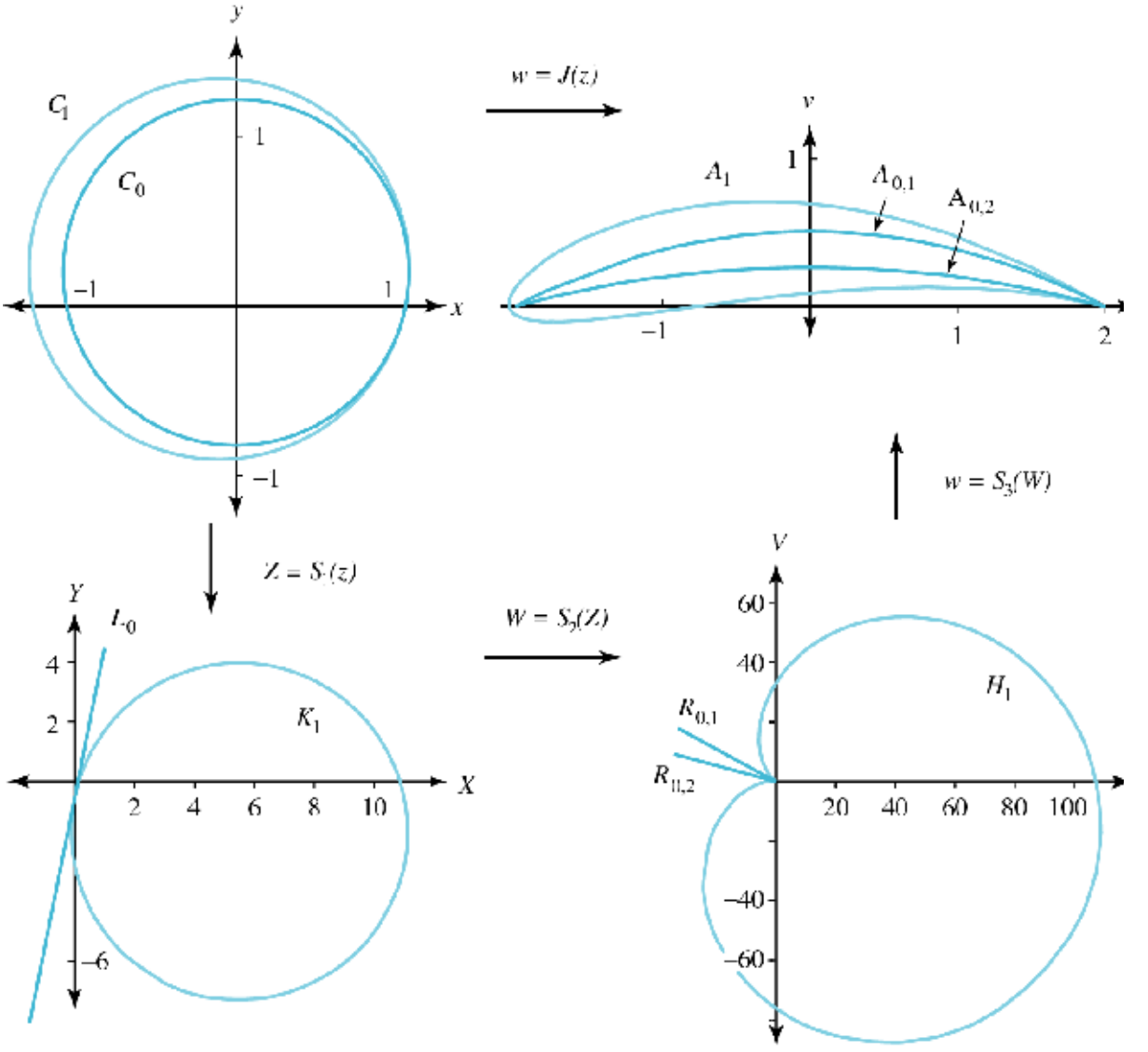


Figure 11.69 The images of the circles C_0 and C_1 under the modified Joukowski airfoil.

Solution 4.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

by

John H. Mathews and Russell W. Howell

Jones and Bartlett Learning

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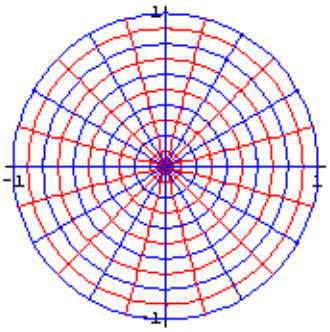
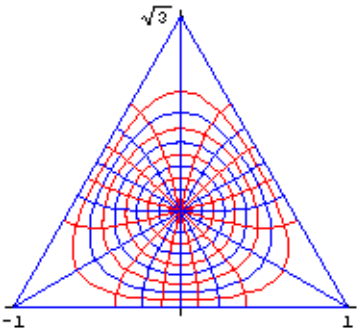
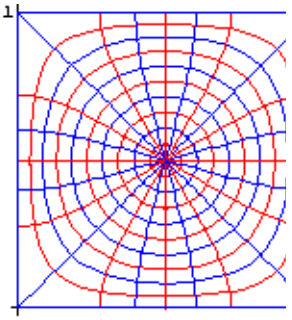
11.9 The Schwarz-Christoffel Transformation

In Chapter 10 we mentioned the Riemann mapping theorem: If D is any simply connected domain in the plane (other than the entire plane itself), then there exists a one-to-one conformal mapping $w = f(z)$ that maps D onto the unit disk $|w| < 1$. The Mobius transformation $w = S(z) = \frac{i(1+z)}{1-z}$ is a one-to-one conformal mapping of the unit disk $|z| < 1$ onto the upper half-plane $\mathrm{Im}(w) > 0$, and the inverse is $z = S^{-1}(w) = \frac{w-i}{w+i}$. Hence the Riemann mapping theorem could have been stated: If D is any simply connected domain in the w-plane, then there exists a one-to-one conformal mapping $w = f(z)$ that maps the upper half-plane $\mathrm{Im}(z) > 0$ onto D. In this section we will introduce the Schwarz-Christoffel formula for constructing a conformal mapping from the upper half plane onto a region G bounded by a polygonal curve. At the end of this section we will introduce the remarkable mappings of the unit disk onto an equilateral triangle and a square.







To proceed further, we must review the rotational effect of a conformal mapping $w = f(z)$ at a point z_0. If the contour C has the parameterization $z(t) = x(t) + i\,y(t)$, then a vector $\mathcal{T}$ tangent to C at the point $z_0 = z(t_0)$ is

$$\mathcal{T} = z'(t_0) = x'(t_0) + i\,y'(t_0).$$

The image of C is a contour K given by $w(t) = u(x(t), y(t)) + i\,v(x(t), y(t))$, and a vector T tangent to K at the point $w_0 = f(z_0)$ is

$$T = w'(t_0) = f'(z_0)\, z'(t_0).$$

If the angle of inclination of $\mathcal{T}$ is $\beta = \text{Arg}\, z'(t_0)$, then the angle of inclination of T is

$$\text{Arg}\, T = \text{Arg}\,(f'(z_0)\, z'(t_0)) = \text{Arg}\, f'(z_0) + \beta.$$

Hence the angle of inclination of the tangent $\mathcal{T}$ to C at z_0 is rotated through the angle $\text{Arg}\, f'(z_0)$ to obtain the angle of inclination of the tangent T to K at the point w_0.

Many applications involving conformal mappings require the construction of a one-to-one conformal mapping from the upper half plane $\text{Im}(z) > 0$ onto a domain G in the w-plane where the boundary consists of straight line segments. Let's consider the case where G is the interior of a polygon P with vertices $w_1, w_2, \ldots, w_n$ specified in the positive sense (counterclockwise). We want to find a function $w = f(z)$ with the property

(11-38) $$w_k = f(x_n) \text{ for } k = 1, 2, \ldots, n-1, \text{ and } w_n = f(\infty)$$

where

$$x_1 < x_2 < \ldots < x_{n-1} < \infty.$$

Two German mathematicians Herman Amandus Schwarz (1843-1921) and Elwin Bruno Christoffel (1829-1900) independently discovered a method for finding f, and that is our next theorem.

Theorem 11.6 (Schwarz-Christoffel Formula). Let P be a polygon in the w plane with vertices $w_1, w_2, \ldots, w_n$ and exterior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$, where $-\pi < \alpha_k < \pi$. There exists a one-to-one conformal mapping $w = f(z)$ from the upper half plane $\text{Im}(z) > 0$ onto G that satisfies the boundary conditions

$$w_k = f(x_k) \text{ for } k = 1, 2, \ldots, n-1 \text{ and } w_n = f(\infty)$$
$$\text{where } x_1 < x_2 < \ldots < x_{n-1} < \infty.$$

The derivative $f'(z)$ is

(11-39) $$f'(z) = A\,(z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi} \ldots (z - x_{n-1})^{-\alpha_{n-1}/\pi}$$

and the function $f(z)$ can be expressed as an indefinite integral,

(11-40) $$f(z) = B + A \int (z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi} \cdots (z - x_{n-1})^{-\alpha_{n-1}/\pi}\, dz$$

where A and B are suitably chosen constants. Two of the points $\{x_k\}$ may be chosen arbitrarily, and the constants A and B determine the size and position of P.

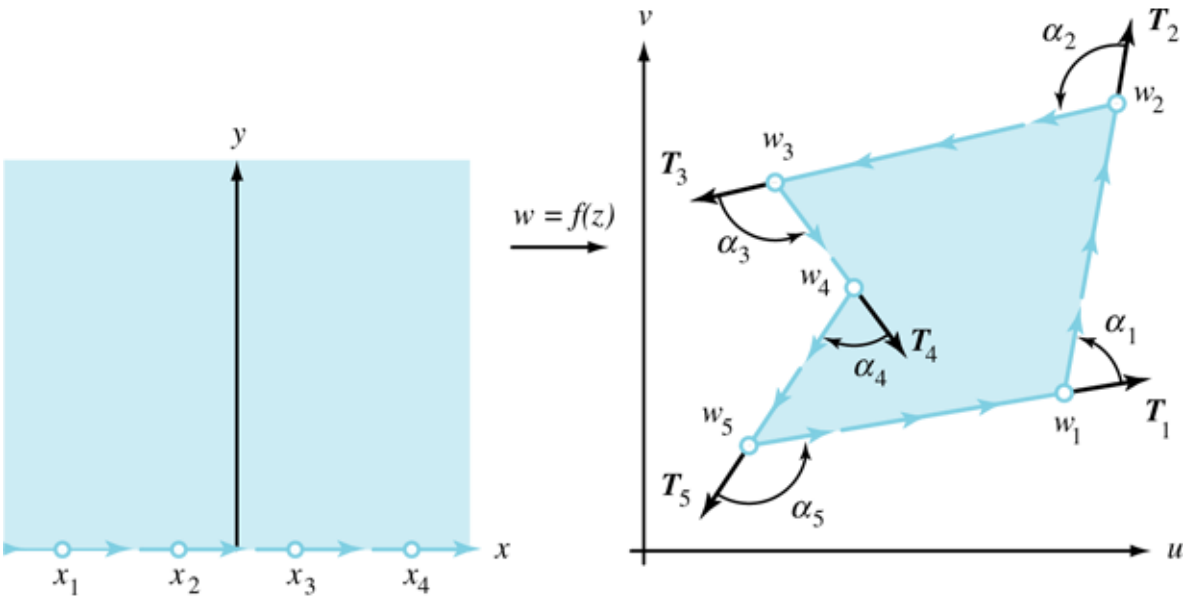


Figure 11.70 A Schwarz-Christoffel mapping with n = 5 and $\alpha_1 + \alpha_2 + \alpha_2 + \alpha_4 > \pi$.

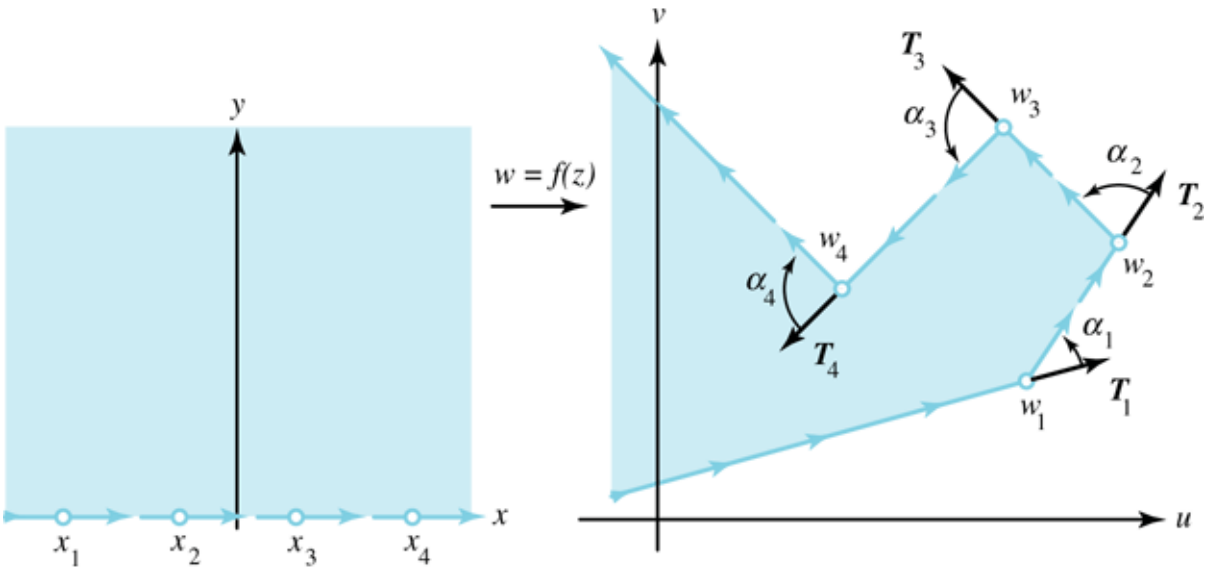


Figure 11.71 A Schwarz-Christoffel mapping with n = 5 and $\alpha_1 + \alpha_2 + \alpha_2 + \alpha_4 \leq \pi$

Proof.

Proof of Theorem 11.6 is in the book.

Equation (11-40) gives a representation for f in terms of an indefinite integral. Note that these integrals do not represent elementary functions unless the image is an infinite region. Also, the integral will involve a multivalued function, and we must select a specific branch to fit the boundary values specified in the problem. Table 11.2 is useful for our purposes.

The following table of integrals is useful for hand computations and are compatible with the versions used in the textbook. Occasionally they have been adjusted by a constant of integration which could be changed for indefinite integrands. Also, depending on which quadrant you use it might be necessary to use various combinations of the branches of the multivalued terms.

$$\int \frac{1}{(1-z^2)^{\frac{1}{2}}}\,dz = \text{Arcsin}\,z$$

$$\int \frac{1}{(z^2-1)^{\frac{1}{2}}}\,dz = \text{Log}\left(z+(z^2-1)^{\frac{1}{2}}\right) - \frac{i\pi}{2}$$

$$\int \frac{1}{(z^2-1)^{\frac{1}{2}}}\,dz = i\,\text{Arcsin}\,z$$

$$\int \frac{1}{z^2+1}\,dz = \text{Arctan}\,z$$

$$\int \frac{1}{z^2+1}\,dz = \frac{i}{2}\text{Log}\left(\frac{i+z}{i-z}\right)$$

$$\int \frac{1}{z\,(z^2-1)^{\frac{1}{2}}}\,dz = -\text{Arcsin}\left(\frac{1}{z}\right)$$

$$\int \frac{1}{z\,(z^2-1)^{\frac{1}{2}}}\,dz = i\,\text{Log}\left(\frac{1}{z}+\left(\frac{1}{z^2}-1\right)^{\frac{1}{2}}\right)$$

$$\int \frac{1}{z\,(z+1)^{\frac{1}{2}}}\,dz = -2\,\text{Arctanh}\left((z+1)^{\frac{1}{2}}\right)$$

$$\int \frac{1}{z\,(z+1)^{\frac{1}{2}}}\,dz = \text{Log}\left(\frac{1-(z+1)^{\frac{1}{2}}}{1+(z+1)^{\frac{1}{2}}}\right)$$

$$\int (1-z^2)^{\frac{1}{2}}\,dz = \frac{1}{2}z\,(1-z^2)^{\frac{1}{2}} + \frac{1}{2}\text{Arcsin}\,z$$

$$\int (1-z^2)^{\frac{1}{2}}\,dz = \frac{i}{2}z\,(z^2-1)^{\frac{1}{2}} - \frac{i}{2}\text{Log}\left(z+(z^2-1)^{\frac{1}{2}}\right)$$

Table 11.2 Indefinite integrals.

Exploration

Example 11.26. Use the Schwarz-Christoffel formula to verify that $w = f(z) = \text{Arcsin}\,z$ maps the upper half plane $\text{Im}(z) > 0$ onto the semi infinite strip $\frac{-\pi}{2} < u < \frac{\pi}{2}$, $v > 0$ shown in Figure 11.72.

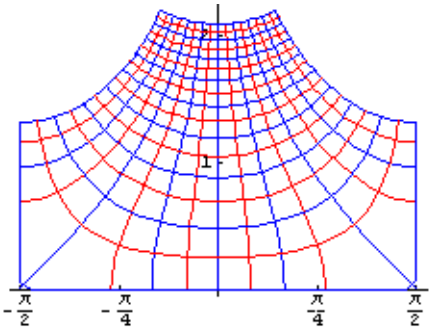


Figure 11.72 The region with $w_1 = -\frac{\pi}{2}$, $w_2 = \frac{\pi}{2}$ and $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \frac{\pi}{2}$.

Solution. If we choose $x_1 = -1$, $x_2 = 1$, and $w_1 = -\frac{\pi}{2}$, $w_2 = \frac{\pi}{2}$, then the exterior angles are $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \frac{\pi}{2}$, and Equation (11-39) for $f'(z)$ becomes

$$f'(z) = A(z-x_1)^{-\alpha_1/\pi}(z-x_2)^{-\alpha_2/\pi}$$

$$= A(z-(-1))^{-\left(\frac{\pi}{2}\right)/\pi}(z-1)^{-\left(\frac{\pi}{2}\right)/\pi}$$

$$= A(z+1)^{-\frac{1}{2}}(z-1)^{-\frac{1}{2}}$$

$$= A(z^2-1)^{-\frac{1}{2}}$$

$$= A\frac{1}{(z^2-1)^{\frac{1}{2}}}$$

Then, using Table 11.2, the indefinite integral becomes

$$f(z) = \int \frac{1}{(z^2-1)^{\frac{1}{2}}}\,dz + B$$

$$= A\times i\,\text{Arcsin}\,z + B$$

Using the image values $f(-1) = -\frac{\pi}{2}$ and $f(1) = \frac{\pi}{2}$, we obtain the system

$$-\frac{\pi}{2} = A\times i\,\text{Arcsin}\left(\frac{-\pi}{2}\right) + B$$

$$\frac{\pi}{2} = A\times i\,\text{Arcsin}\left(\frac{\pi}{2}\right) + B$$

and simplifies to be

$$-\frac{\pi}{2} = -\frac{\pi}{2}A + B$$

$$\frac{\pi}{2} = \frac{\pi}{2}iA + B$$

which we can solve to obtain $B = 0$ and $A = -i$. Hence the required function is

$$f(z) = \text{Arcsin}\,z$$

Explore Solution 11.26.

Example 11.27. Use the Schwarz Christoffel formula to verify that $w = f(z) = (z^2-1)^{\frac{1}{2}}$ maps the upper half plane $\text{Im}(z) > 0$ onto the upper half plane $\text{Im}(w) > 0$ slit along the line segment from $w = 0$ to $w_2 = i$. (Use the principal square root throughout.)

Solution. If we choose $x_1 = -1, x_2 = 0, x_3 = 1$ and $w_1 = -d, w_2 = i, w_3 = d$, then the formula

$$g'(z) = A(z - x_1)^{-\alpha_1/\pi}(z - x_2)^{-\alpha_2/\pi}(z - x_3)^{-\alpha_3/\pi}$$

$$= A(z - (-1))^{-\alpha_1/\pi}(z - 0)^{-\alpha_2/\pi}(z - 1)^{-\alpha_3/\pi}$$

$$= A(z + 1)^{-\alpha_1/\pi}(z)^{-\alpha_2/\pi}(z - 1)^{-\alpha_3/\pi}$$

will determine a mapping $w = g(z)$ from the upper half-plane $\mathrm{Im}(z) > 0$ onto the portion of the upper half-plane $\mathrm{Im}(w) > 0$ that lies outside the triangle with vertices $w_1 = -d, w_2 = i, w_3 = d$ as indicated in Figure 11.73(a). If we let $d \to 0$, then $w_1 \to 0$ and $w_3 \to 0$, and $\alpha_1 \to \frac{\pi}{2}, \alpha_2 \to -\pi, \alpha_3 \to \frac{\pi}{2}$.

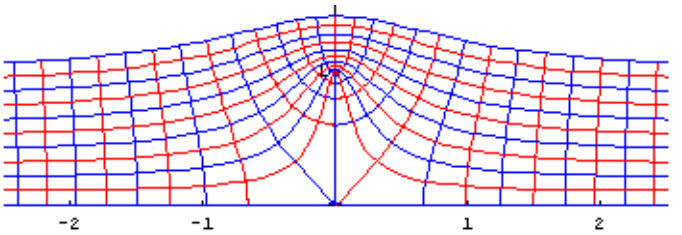


Figure 11.73 The region with $w_1 \to 0, w_2 = i, w_3 \to 0$ and $\alpha_1 \to \frac{\pi}{2}, \alpha_2 \to -\pi, \alpha_3 \to \frac{\pi}{2}$.

The limiting formula for the derivative $g'(z)$ becomes

$$f'(z) = A(z + 1)^{-\alpha_1/\pi}(z)^{-\alpha_2/\pi}(z - 1)^{-\alpha_3/\pi}$$

$$= A(z + 1)^{-\frac{\pi}{2}/\pi}(z)^{-(-\pi)/\pi}(z - 1)^{-\frac{\pi}{2}/\pi}$$

$$= A(z + 1)^{-\frac{1}{2}}(z)^{1}(z - 1)^{-\frac{1}{2}}$$

$$= A\,z\,(z^2 - 1)^{-\frac{1}{2}}$$

$$= A\,\frac{z}{(z^2-1)^{\frac{1}{2}}}$$

which will determine a mapping $w = f(z)$ from the upper half-plane $\mathrm{Im}(z) > 0$ onto the upper half plane $\mathrm{Im}(w) > 0$ slit along the line segment from $w = 0$ to $w_2 = i$ as indicated in Figure 11.73(b). An easy integration reveals that $f(z)$ is given by

$$f(z) = A\int \frac{z}{(z^2-1)^{\frac{1}{2}}}\,dz + B$$

$$= A(z^2 - 1)^{\frac{1}{2}} + B$$

and the boundary values $f(\pm 1) = 0$ and $f(0) = i$ lead to system of equations

$$A\left((\pm 1)^2 - 1\right)^{\frac{1}{2}} + B = 0 \quad \text{and} \quad A(0 - 1)^{\frac{1}{2}} + B = i$$

the solution is easily found to be $A = 1$ and $B = 0$ and the desired function is

$$f(z) = (z^2 - 1)^{\frac{1}{2}}.$$

Explore Solution 11.27.

Example 11.28. Show that $w = f(z) = \frac{1}{\pi}\,\text{Arcsin}\, z + \frac{i}{\pi}\,\text{Arcsin}\,\frac{1}{z} + \frac{1+i}{2}$ maps the upper half plane $\text{Im}(z) > 0$ onto the right angle channel in the first quadrant, which is bounded by the coordinate axes and the rays $x \geq 0$, $y = 1$ and $y \geq 1$, $x = 1$, as depicted in Figure 11.74(b).

Solution. If we choose $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ and $w_1 = 0$, $w_2 = d$, $w_3 = 1 + i$, then the formula

$$\begin{aligned} g'(z) &= A_1 (z - x_1)^{-\alpha_1/\pi} (z - x_2)^{-\alpha_2/\pi} (z - x_3)^{-\alpha_3/\pi} \\ &= A_1 (z - (-1))^{-\alpha_1/\pi} (z - 0)^{-\alpha_2/\pi} (z - 1)^{-\alpha_3/\pi} \\ &= A_1 (z + 1)^{-\alpha_1/\pi} (z)^{-\alpha_2/\pi} (z - 1)^{-\alpha_3/\pi} \end{aligned}$$

will determine a mapping $w = g(z)$ of the upper half-plane onto the domain indicated in Figure 11.74(a). With $\alpha_1 = \frac{\pi}{2}$, we let $d \to \infty$, then $\alpha_2 \to \pi$ and $\alpha_3 \to -\frac{\pi}{2}$.

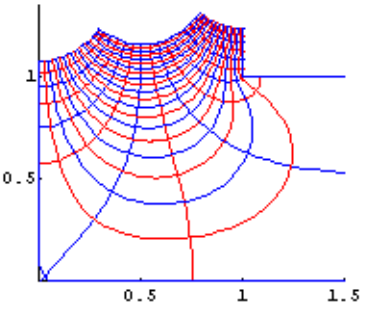


Figure 11.74 The region with $w_1 = 0$, $w_2 = \to \infty$, $w_3 = 1 + i$ and $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 \to \pi$, $\alpha_3 \to -\frac{\pi}{2}$.

The limiting formula for the derivative $g'(z)$ becomes

$$f'(z) = A_1 (z+1)^{-\alpha_1/\pi} (z)^{-\alpha_2/\pi} (z-1)^{-\alpha_3/\pi}$$

$$= A_1 (z+1)^{-\frac{\pi}{2}/\pi} (z)^{-\pi/\pi} (z-1)^{-\left(-\frac{\pi}{2}\right)/\pi}$$

$$= A_1 (z+1)^{-\frac{1}{2}} (z)^{-1} (z-1)^{\frac{1}{2}}$$

$$= A_1 \frac{1}{z} \frac{(z-1)^{\frac{1}{2}}}{(z+1)^{\frac{1}{2}}}$$

$$= A_1 \frac{(z-1)^{\frac{1}{2}} (z-1)^{\frac{1}{2}}}{z (z+1)^{\frac{1}{2}} (z-1)^{\frac{1}{2}}}$$

$$= A_1 \frac{z-1}{z \left(z^2-1\right)^{\frac{1}{2}}}$$

$$= A \frac{z-1}{z \left(1-z^2\right)^{\frac{1}{2}}}$$

where $A = -i A_1$, which will determine a mapping $w = f(z)$ from the upper half plane onto the channel as indicated in Figure 11.74(b).

Using integrals in Table 11.2, we obtain

$$f(z) = A \int \frac{z-1}{z \left(1-z^2\right)^{\frac{1}{2}}} dz + B$$

$$= A \int \frac{z}{z \left(1-z^2\right)^{\frac{1}{2}}} dz - A \int \frac{1}{z \left(1-z^2\right)^{\frac{1}{2}}} dz + B$$

$$= A \int \frac{1}{\left(1-z^2\right)^{\frac{1}{2}}} dz - A i \int \frac{1}{z \left(z^2-1\right)^{\frac{1}{2}}} dz + B$$

$$= A \left(\text{Arcsin}\, z + i \,\text{Arcsin}\left(\frac{1}{z}\right)\right) + B$$

If we use the principal branch of the inverse sine function, then the boundary values $f(-1) = 0$ and $f(1) = 1 + i$ lead to the system

$$A \left(-\frac{\pi}{2} - \frac{i\pi}{2}\right) + B = 0 \quad \text{and} \quad A \left(\frac{\pi}{2} + \frac{i\pi}{2}\right) + B = 1 + i,$$

which we can solve to obtain $A = \frac{1}{\pi}$ and $B = \frac{1+i}{2}$. Hence the required solution is

$$w = f(z) = \frac{1}{\pi} \text{Arcsin}\, z + \frac{i}{\pi} \text{Arcsin}\, \frac{1}{z} + \frac{1+i}{2}$$

Explore Solution 11.28.

Extra Example 1. Show that $w = f(z) = \frac{1}{\pi} (z^2-1)^{\frac{1}{2}} + \frac{1}{\pi} \text{Log}\left(z + (z^2-1)^{\frac{1}{2}}\right) - i$ maps the upper half-plane onto the domain indicated in Figure 11.77.

Hint: Set $x_1 = -1, x_2 = 1$ and $w_1 = 0, w_2 = -i$.

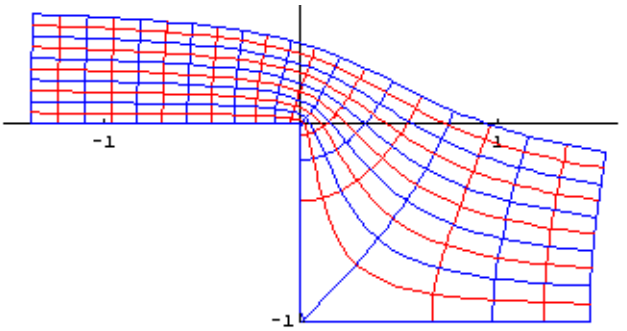


Figure 11.77 The region with $w_1 = 0$, $w_2 = -i$ and $\alpha_1 = -\frac{\pi}{2}$, $\alpha_2 = \frac{\pi}{2}$.

Solution. The exterior angles are $\alpha_1 = -\frac{\pi}{2}, \alpha_2 = \frac{\pi}{2}$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A\,(z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi}$$

$$= A\,(z - (-1))^{-\left(-\frac{\pi}{2}\right)/\pi}\,(z - 1)^{-\left(\frac{\pi}{2}\right)/\pi}$$

$$= A\,(z+1)^{\frac{1}{2}}\,(z-1)^{-\frac{1}{2}}$$

$$= A\,\frac{(z+1)^{\frac{1}{2}}}{(z-1)^{\frac{1}{2}}} = A\,\frac{(z+1)^{\frac{1}{2}}\,(z+1)^{\frac{1}{2}}}{(z+1)^{\frac{1}{2}}\,(z-1)^{\frac{1}{2}}}$$

$$= A\,\frac{z+1}{(z+1)^{\frac{1}{2}}\,(z-1)^{\frac{1}{2}}}$$

$$= A\left(\frac{z}{\left(z^2-1\right)^{\frac{1}{2}}} + \frac{1}{\left(z^2-1\right)^{\frac{1}{2}}}\right)$$

integrate and get

$$f(z) = A\,(z^2-1)^{\frac{1}{2}} + A\,\text{Log}\left(z + (z^2-1)^{\frac{1}{2}}\right)$$

use the conditions $f(-1) = 0$ and $f(1) = -i$ and solve the resulting system for A and B. The desired result is

$$f(z) = \frac{1}{\pi}\,(z^2-1)^{\frac{1}{2}} + \frac{1}{\pi}\,\text{Log}\left(z + (z^2-1)^{\frac{1}{2}}\right) - i.$$

Explore Extra Solution 1.

Extra Example 2. Show that $w = f(z) = \frac{2}{\pi}\,(z^2-1)^{\frac{1}{2}} + \frac{2}{\pi}\,\text{Arcsin}\left(\frac{1}{z}\right)$ maps the upper half-plane onto the domain indicated in Figure 11.78.

Hint: Set $x_1 = -1, x_2 = 0, x_3 = 1$ and $w_1 = -1, w_2 = -i\,d, w_3 = 1$ and let $d \to \infty$

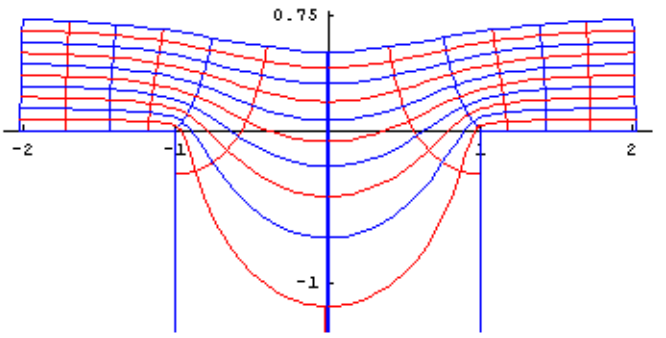


Figure 11.78 The region with $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ and $\alpha_1 = -\frac{\pi}{2}$, $\alpha_2 = \pi$, $\alpha_3 = -\frac{\pi}{2}$.

Solution. The exterior angles are $\alpha_1 = -\frac{\pi}{2}$, $\alpha_2 = \pi$, $\alpha_3 = -\frac{\pi}{2}$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A\,(z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi}\,(z - x_3)^{-\alpha_3/\pi}$$

$$= A\,(z - (-1))^{-\left(-\frac{\pi}{2}\right)/\pi}\,(z - 0)^{-(\pi)/\pi}\,(z - 1)^{-\left(-\frac{\pi}{2}\right)/\pi}$$

$$= A\,(z+1)^{\frac{1}{2}}\,(z)^{-1}\,(z-1)^{\frac{1}{2}} = A\,z^{-1}\,(z^2-1)^{\frac{1}{2}}$$

$$= A\,\frac{(z^2-1)^{\frac{1}{2}}}{z} = A\,\frac{(z^2-1)^{\frac{1}{2}}\,(z^2-1)^{\frac{1}{2}}}{z\,(z^2-1)^{\frac{1}{2}}}$$

$$= A\,\frac{(z^2-1)}{z\,(z^2-1)^{\frac{1}{2}}} = A\,\frac{z^3-z}{z^2\,(z^2-1)^{\frac{1}{2}}}$$

$$= A\,\frac{z}{(z^2-1)^{\frac{1}{2}}} - A\,\frac{1}{\left(1-\frac{1}{z^2}\right)^{\frac{1}{2}}\,z^2}$$

integrate and get

$$f(z) = A\,(z^2-1)^{\frac{1}{2}} + A \times \text{Arcsin}\left(\frac{1}{z}\right)$$

use the conditions $f(-1) = -1$ and $f(1) = 1$ and solve the resulting system for A and B. The desired result is

$$f(z) = \frac{2}{\pi}\,(z^2-1)^{\frac{1}{2}} + \frac{2}{\pi}\,\text{Arcsin}\left(\frac{1}{z}\right).$$

Explore Extra Solution 2.

Extra Example 3. Show that $w = f(z) = 4\,(z+1)^{1/4} + \log\frac{(z+1)^{1/4}-1}{(z+1)^{1/4}+1} + i\,\log\frac{i-(z+1)^{1/4}}{i+(z+1)^{1/4}}$ maps the upper half-plane onto the domain indicated in Figure 11.84.

Hint: Set $x_1 = -1$, $x_2 = 0$ and $w_1 = i\,\pi$, $w_2 = -d$ and let $d \to \infty$. Use the change of variable $z + 1 = s^4$ in the resulting integral.

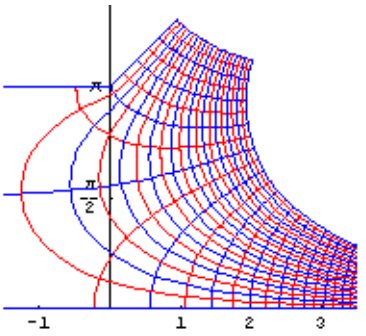


Figure 11.84 The region with $x_1 = -1$, $x_2 = 0$ and $\alpha_1 = -\frac{\pi}{4}$, $\alpha_2 = \pi$.

Solution. The exterior angles are $\alpha_1 = -\frac{\pi}{4}$, $\alpha_2 = \pi$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A\,(z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi}$$

$$= A\,(z - (-1))^{-\left(-\frac{\pi}{4}\right)/\pi}\,(z - 0)^{-(\pi)/\pi}$$

$$= A\,(z+1)^{\frac{1}{4}}\,(z)^{-1}$$

$$= A\,\frac{(z+1)^{\frac{1}{4}}}{z}$$

integrate

$$f(z) = B + A\int \frac{(z+1)^{\frac{1}{4}}}{z}\,dz$$

use the substitutions

$$z + 1 = s^4$$
$$z = s^4 - 1$$
$$dz = 4\,s^3\,ds$$

and get

$$\int \frac{(z+1)^{\frac{1}{4}}}{z}\,dz = \int \frac{\left(s^4\right)^{1/4}}{s^4-1}\,4\,s^3\,ds$$

$$= \int \frac{4\,s^4}{s^4-1}\,ds$$

$$= 4\,s - 2\,\text{Arctan}\,s + \text{Log}\,(s-1) - \text{Log}\,(s+1)$$

Use the identity $-2\,\text{ArcTan}\,s = -2\,\frac{i}{2}\,\text{Log}\left(\frac{i+s}{i-s}\right) = i\,\text{Log}\left(\frac{i-s}{i+s}\right)$ and write the integral as

$$\int \frac{(z+1)^{\frac{1}{4}}}{z}\,dz = 4\,s + \text{Log}\left(\frac{s-1}{s+1}\right) + i\,\text{Log}\left(\frac{i-s}{i+s}\right)$$

Now use the substitution $s \to (z+1)^{\frac{1}{4}}$ and get

$$f(z) = B + A \int \frac{(z+1)^{\frac{1}{4}}}{z}\, dz$$
$$= B + A\left(4(z+1)^{1/4} + \text{Log}\,\frac{(z+1)^{1/4}-1}{(z+1)^{1/4}+1} + i\,\text{Log}\,\frac{i-(z+1)^{1/4}}{i+(z+1)^{1/4}}\right)$$

use the conditions $f(0) = 0$, $f(1) = i\,\pi$ and solve the resulting system for A and B. The desired result is

$$f(z) = 4(1+z)^{1/4} + \text{Log}\,\frac{(z+1)^{1/4}-1}{(z+1)^{1/4}+1} + i\,\text{Log}\,\frac{i-(z+1)^{1/4}}{i+(z+1)^{1/4}}.$$

Explore Extra Solution 3.

Extra Example 4. Show that $w = f(z) = \frac{-i}{2} z^{\frac{1}{2}} (z-3)$ maps the upper half-plane onto the domain indicated in Figure 11.85.
Hint: Set $x_1 = 0, x_2 = 1$ and $w_1 = -d, w_2 = i$ and let $d \to 0$.

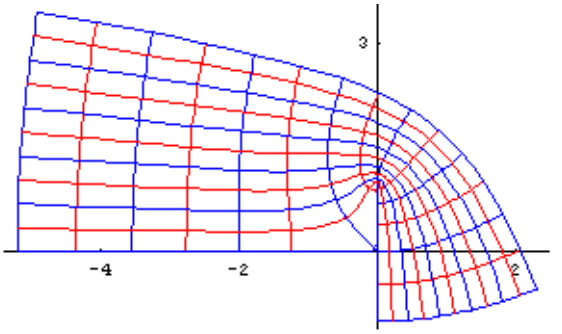


Figure 11.85 The region with $x_1 = 0$, $x_2 = 1$ and $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = -\pi$.

Solution. The exterior angles are $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = -\pi$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A(z-x_1)^{-\alpha_1/\pi}(z-x_2)^{-\alpha_2/\pi}$$
$$= A(z-0)^{-\frac{\pi}{2}/\pi}(z-1)^{-(-\pi)/\pi}$$
$$= A\,z^{-\frac{1}{2}}(z-1)^1$$
$$= A\,z^{-\frac{1}{2}}(z-1)$$
$$= A\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)$$

integrate and get

$$f(z) = B + A\left(\frac{2}{3}z^{3/2} - 2\,z^{\frac{1}{2}}\right)$$

use the conditions $f(0) = 0$, $f(1) = i$ and solve the resulting system for A and B. The desired result is

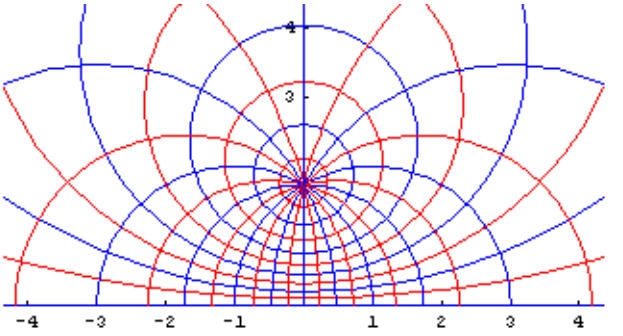
$$f(z) = \frac{-i}{2} z^{\frac{1}{2}} (z-3).$$

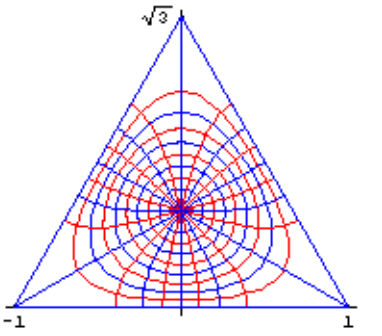
Explore Extra Solution 4.

Extra Example 5. Use the Schwarz Christoffel formula to show that

$w = f(z) = B + A \int \frac{1}{(z+1)^{\frac{2}{3}} (z-1)^{\frac{2}{3}}} dz$ will map the upper-half-plane onto an equilateral triangle.

Hint: For convenience we have chosen $x_1 = -1, x_2 = 1$.





Solution. The exterior angles of an equilateral triangle are $\alpha_1 = \frac{2\pi}{3}, \alpha_2 = \frac{2\pi}{3}$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A (z - x_1)^{-\left(\frac{2\pi}{3}\right)/\pi} (z - x_2)^{-\left(\frac{2\pi}{3}\right)/\pi}$$

$$= A (z - (-1))^{-\frac{2}{3}} (z - 1)^{-\frac{2}{3}}$$

$$= A (z + 1)^{-\frac{2}{3}} (z - 1)^{-\frac{2}{3}}$$

$$= A \frac{1}{(z+1)^{\frac{2}{3}} (z-1)^{\frac{2}{3}}}$$

integrate and get

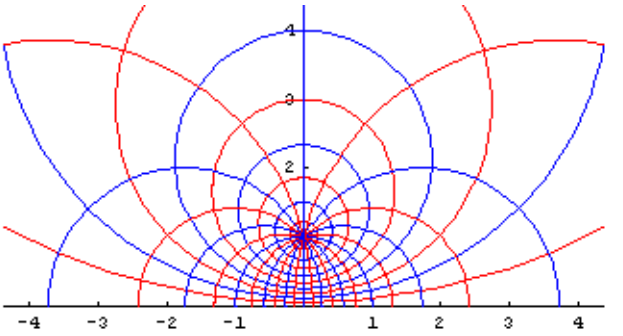
$$f(z) = B + A \int \frac{1}{(z+1)^{\frac{2}{3}} (z-1)^{\frac{2}{3}}} dz$$

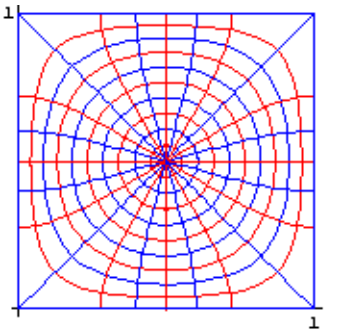
Explore Extra Solution 5.

Extra Example 6. Use the Schwarz Christoffel formula to show that

$w = f(z) = B + A \int \frac{1}{(z+1)^{\frac{1}{2}} z^{\frac{1}{2}} (z-1)^{\frac{1}{2}}} dz$ will map the upper-half-plane onto a square.

Hint: For convenience we have chosen $x_1 = -1, x_2 = 0, x_3 = 1$.





Solution. The exterior angles of a square are $\alpha_1 = \frac{\pi}{2}, \alpha_2 = \frac{\pi}{2}, \alpha_2 = \frac{\pi}{2},$ and the formula for the derivative f'(z) is

$$f'(z) = A\,(z - x_1)^{-\frac{\pi}{2}/\pi}\,(z - x_2)^{-\frac{\pi}{2}/\pi}\,(z - x_3)^{-\frac{\pi}{2}/\pi}$$

$$= A\,(z - (-1))^{-\frac{1}{2}}\,(z - 0)^{-\frac{1}{2}}\,(z - 1)^{-\frac{1}{2}}$$

$$= A\,(z + 1)^{-\frac{1}{2}}\,z^{-\frac{1}{2}}\,(z - 1)^{-\frac{1}{2}}$$

$$= A\,\frac{1}{(z+1)^{\frac{1}{2}}\,z^{\frac{1}{2}}\,(z-1)^{\frac{1}{2}}}$$

integrate and get

$$f(z) = B + A\int \frac{1}{(z+1)^{\frac{1}{2}}\,z^{\frac{1}{2}}\,(z-1)^{\frac{1}{2}}}\,dz$$

Explore Extra Solution 6.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

by

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Jones and Bartlett Learning

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11.10 Image of a Fluid Flow

We have already examined several two-dimensional fluid flows and have discovered that the image of a flow under a conformal transformation is a flow. The conformal mapping $w = u(x, y) + i\, v(x, y)$, which is obtained by using the Schwarz-Christoffel formula, allows us to find the streamlines for flows in domains in the w plane that are bounded by straight-line segments.

The first technique is finding the image of a fluid flowing horizontally from left to right across the upper half plane $\mathrm{Im}(z) > 0$. The image of the streamline $-\infty < t < \infty,\ y = c$ is a streamline given by the parametric equations

$$u = u(t, c) \quad \text{and},$$

$$v = v(t, c) \quad \text{for} \quad -\infty < t < \infty.$$

and is oriented in the positive sense (counterclockwise). The streamline $u = u(t, 0),\ v = v(t, 0)$ for $-\infty < t < \infty$ is considered to be the boundary wall for a containing vessel for the fluid flow.

Example 11.29. Show that the mapping $w = \frac{1}{\pi}\left((z^2 - 1)^{\frac{1}{2}} + \mathrm{Log}\left(z + (z^2 - 1)^{\frac{1}{2}}\right)\right)$ maps the upper half plane $\mathrm{Im}(z) > 0$ onto the domain in the w-plane that lies above the boundary curve consisting of the rays $u \le 0, v = 1$ and $u \ge 0, v = 0$ and the segment $u = 0, -1 \le v \le 0$ (see Figure 11.87).

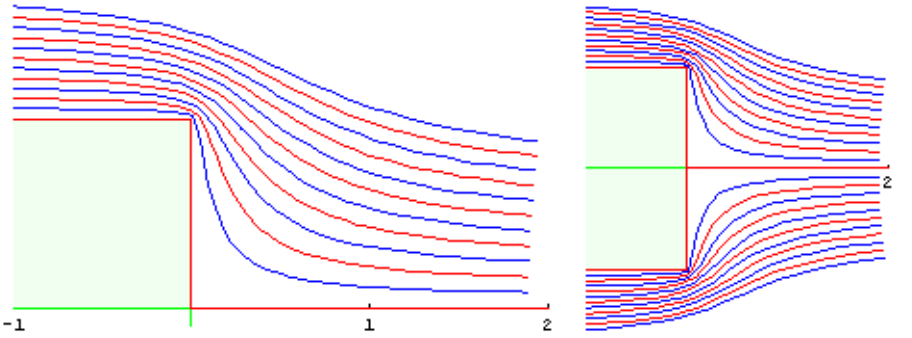

Figure 11.87 (a) Flow over a step. (b) Flow around a blunt object.

Hint: Set $x_1 = -1$, $x_2 = 1$ and $w_1 = 0, w_2 = -i$. In the w-plane $w_1 = i$, $w_2 = 0$, and the exterior angles are $\alpha_1 = -\frac{\pi}{2}, \alpha_2 = \frac{\pi}{2}$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A\,(z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi}$$

$$= A\,(z - (-1))^{-\left(-\frac{\pi}{2}\right)/\pi}\,(z - 1)^{-\left(\frac{\pi}{2}\right)/\pi}$$

$$= A\,(z + 1)^{\frac{1}{2}}\,(z - 1)^{-\frac{1}{2}}$$

$$= A\,\frac{(z+1)^{\frac{1}{2}}}{(z-1)^{\frac{1}{2}}}$$

Integrate and get

$$f(z) = B + A\int \frac{(z+1)^{\frac{1}{2}}}{(z-1)^{\frac{1}{2}}}\,dz$$

$$= B + A\left((z^2 - 1)^{\frac{1}{2}} + \text{Log}\left(z + (z^2 - 1)^{\frac{1}{2}}\right)\right)$$

Solve for the coefficients A and B and obtain

$$w = f(z) = \frac{1}{\pi}\left((z^2 - 1)^{\frac{1}{2}} + \text{Log}\left(z + (z^2 - 1)^{\frac{1}{2}}\right)\right)$$

It maps the upper half-plane $\text{Im}(z) > 0$ onto the domain in the w plane that lies above the boundary curve consisting of the rays $u \le 0, v = 1$ and $u \ge 0, v = 0$ and the segment $u = 0, -1 \le v \le 0$. Furthermore, the image of horizontal streamlines in the z plane are curves in the w plane given by the parametric equation

$$w = u + i\,v = f(t + i\,c)$$

$$= \frac{1}{\pi}\left(\left((t + i\,c)^2 - 1\right)^{\frac{1}{2}} + \text{Log}\left(t + i\,c + \left((t + i\,c)^2 - 1\right)^{\frac{1}{2}}\right)\right)$$

$$= \frac{1}{\pi}\left(\left(t^2 - c^2 - 1 + i\,2\,c\,t\right)^{\frac{1}{2}} + \text{Log}\left(t + i\,c + \left(t^2 - c^2 - 1 + i\,2\,c\,t\right)^{\frac{1}{2}}\right)\right)$$

for $-\infty < t < \infty$. The new flow is that of a step in the bed of a deep stream and is illustrated in Figure 11.87(a). The function $w = f(z)$ is also defined for values of z in the lower half-plane, and the images of horizontal streamlines that lie above or below the x axis are mapped onto streamlines that flow past a long rectangular obstacle, which is illustrated in Figure 11.87(b).

Explore Solution 11.29.

Extra Example 1. Use Figure 11.88 to find the flow over the vertical segment from 0 to i.

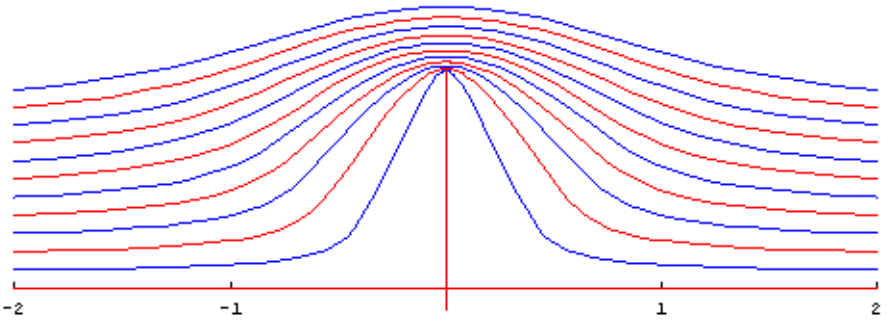

Figure 11.88 Flow over the vertical segment from 0 to i.

Solution. Set $x_1 = -1$, $x_2 = 0$, $x_3 = 1$. In the w-plane $w_1 = 0$, $w_2 = i$, $w_3 = 0$, and the exterior angles are $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = -\pi$, $\alpha_3 = \frac{\pi}{2}$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A\,(z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi}\,(z - x_3)^{-\alpha_3/\pi}$$

$$= A\,(z - (-1))^{-\alpha_1/\pi}\,(z - 0)^{-\alpha_2/\pi}\,(z - 1)^{-\alpha_3/\pi}$$

$$= A\,(z + 1)^{-\alpha_1/\pi}\,(z)^{-\alpha_2/\pi}\,(z - 1)^{-\alpha_3/\pi}$$

$$= A\,(z + 1)^{-\frac{\pi}{2}/\pi}\,(z)^{-(-\pi)/\pi}\,(z - 1)^{-\frac{\pi}{2}/\pi}$$

$$= A\,(z + 1)^{-\frac{1}{2}}\,(z)^{1}\,(z - 1)^{-\frac{1}{2}}$$

$$= A\,z\,(z^2 - 1)^{-\frac{1}{2}}$$

$$= A\,\frac{z}{(z^2 - 1)^{\frac{1}{2}}}$$

Integrate and get

$$f(z) = B + A\int \frac{z}{(z^2-1)^{\frac{1}{2}}}\,dz$$

$$= B + (z^2-1)^{\frac{1}{2}}$$

Solve for the coefficients A and B and obtain

$$f(z) = (z^2-1)^{\frac{1}{2}}$$

Explore Extra Solution 1.

Extra Example 2. Use Figure 11.89 to find the flow around an infinitely long rectangular barrier.

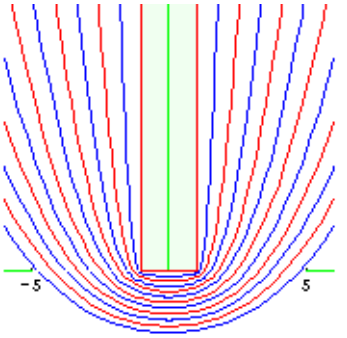

Figure 11.89 Flow around an infinitely long rectangular barrier.

Solution. Set $x_1 = -1$, $x_2 = 1$. In the w-plane $w_1 = 1$, $w_2 = -1$, and the exterior angles are $\alpha_1 = -\frac{\pi}{2}$, $\alpha_2 = -\frac{\pi}{2}$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A(z-x_1)^{-\alpha_1/\pi}(z-x_2)^{-\alpha_2/\pi}$$

$$= A(z-(-1))^{-\alpha_1/\pi}(z-1)^{-\alpha_2/\pi}$$

$$= A(z+1)^{-\alpha_1/\pi}(z-1)^{-\alpha_2/\pi}$$

$$= A(z+1)^{-\left(-\frac{\pi}{2}\right)/\pi}(z-1)^{-\left(-\frac{\pi}{2}\right)/\pi}$$

$$= A(z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}}$$

$$= A(z^2-1)^{\frac{1}{2}}$$

Integrate and get

$$f(z) = B + A\int (z^2-1)^{\frac{1}{2}}\,dz$$

$$= B_1 + A_1\int (1-z^2)^{\frac{1}{2}}\,dz$$

$$= B_1 + A_1\left(\frac{1}{2}z(1-z^2)^{\frac{1}{2}} + \frac{1}{2}\operatorname{Arcsin} z\right)$$

Solve for the coefficients A_1 and B_1 and obtain

$$f(z) = -\frac{2}{\pi} z (1 - z^2)^{\frac{1}{2}} - \frac{2}{\pi} \text{Arcsin } z$$

Explore Extra Solution 2.

Extra Example 3. Use Figure 11.90 to find the flow around

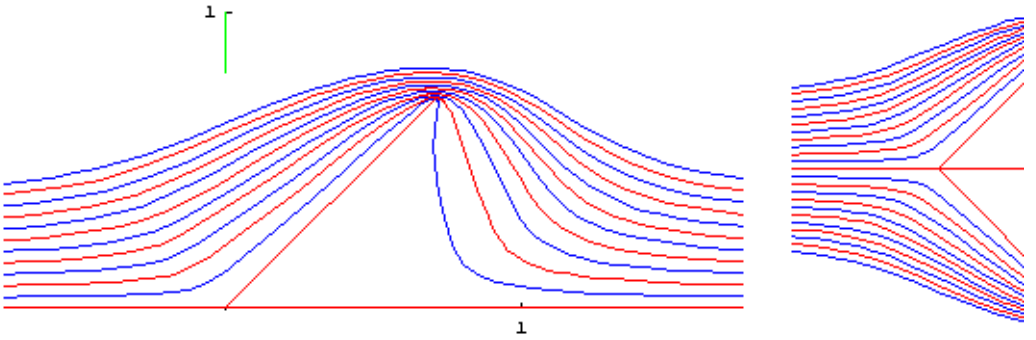

Figure 11.90 (a) Flow around one inclined segment in the upper half-plane. (b) Flow around two inclined segments forming a "V" in the plane.

Solution. Set $x_1 = -\frac{1-\alpha}{\alpha}$, $x_2 = 0$, $x_3 = 1$. In the w-plane $w_1 = 0$, $w_2 = e^{i\alpha\pi}$, $w_3 = 0$, and the exterior angles are $\alpha_1 = \alpha\pi$, $\alpha_2 = -\pi$, $\alpha_3 = (1-\alpha)\pi$, and the formula for the derivative $f'(z)$ is

$$\begin{aligned} f'(z) &= A (z - x_1)^{-\alpha_1/\pi} (z - x_2)^{-\alpha_2/\pi} (z - x_3)^{-\alpha_3/\pi} \\ &= A \left(z - \left(-\frac{1-\alpha}{\alpha}\right)\right)^{-\alpha_1/\pi} (z)^{-\alpha_2/\pi} (z-1)^{-\alpha_3/\pi} \\ &= A \left(z + \frac{1-\alpha}{\alpha}\right)^{-\alpha_1/\pi} (z)^{-\alpha_2/\pi} (z-1)^{-\alpha_3/\pi} \\ &= A \left(z + \frac{1-\alpha}{\alpha}\right)^{-(\alpha\pi)/\pi} (z)^{-(-\pi)/\pi} (z-1)^{-((1-\alpha)\pi)/\pi} \\ &= A \left(z + \frac{1-\alpha}{\alpha}\right)^{-\alpha} (z) (z-1)^{\alpha-1} \end{aligned}$$

Integrate and get

$$\begin{aligned} f(z) &= B + A \int \left(z + \frac{1-\alpha}{\alpha}\right)^{-\alpha} (z) (z-1)^{\alpha-1} \, dz \\ &= B + A (-1+z)^{\alpha} \left(z + \frac{1-\alpha}{\alpha}\right)^{-\alpha} \left(z - \frac{-1+\alpha}{\alpha}\right) \end{aligned}$$

Solve for the coefficients A and B and obtain

$$f(z) = (z-1)^{\alpha} \left(1 + \frac{\alpha z}{(1-\alpha)}\right)^{1-\alpha}$$

Explore Extra Solution 3.

Extra Example 4. Use Figure 11.91 to find the flow over a dam.

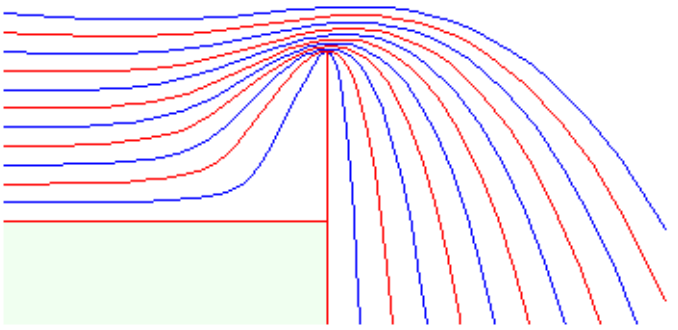

Figure 11.91 Flow over a dam.

Solution. Set $x_1 = 0, x_2 = 1$. In the w-plane $w_1 = 0, w_2 = i$, and the exterior angles are $\alpha_1 = \frac{\pi}{2}, \alpha_2 = -\pi$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A(z - x_1)^{-\alpha_1/\pi}(z - x_2)^{-\alpha_2/\pi}$$

$$= A(z - 0)^{-\frac{\pi}{2}/\pi}(z - 1)^{-(-\pi)/\pi}$$

$$= A z^{-\frac{1}{2}}(z - 1)^1$$

$$= A z^{-\frac{1}{2}}(z - 1)$$

$$= A\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)$$

Integrate and get

$$f(z) = B + A\int\left(z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right)dz$$

$$= B + A\left(-2\sqrt{z} + \frac{2z^{3/2}}{3}\right)$$

Solve for the coefficients A and B and obtain

$$f(z) = \frac{-i}{2}z^{\frac{1}{2}}(z - 3)$$

Explore Extra Solution 4.

Extra Example 5. Use Figure 11.92 to find the flow around

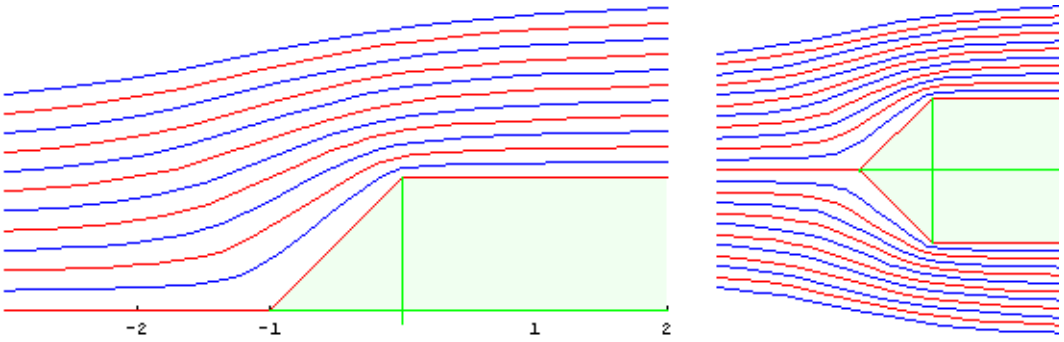

Figure 11.92 (a) Flow up an inclined step
(b) Flow around a pointed object.

Solution. Set $x_1 = 0, x_2 = 1$. In the w-plane $w_1 = -1, w_2 = i$, and the exterior angles are $\alpha_1 = \frac{\pi}{4}, \alpha_2 = -\frac{\pi}{4}$, and the formula for the derivative $f'(z)$ is

$$f'(z) = A\,(z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi}$$

$$= A\,(z - 0)^{-\alpha_1/\pi}\,(z - 1)^{-\alpha_2/\pi}$$

$$= A\,(z)^{-\left(\frac{\pi}{4}\right)/\pi}\,(z - 1)^{-\left(-\frac{\pi}{4}\right)/\pi}$$

$$= A\,(z)^{-\frac{1}{4}}\,(z - 1)^{\frac{1}{4}}$$

$$= A\,\frac{(z-1)^{\frac{1}{4}}}{z^{\frac{1}{4}}}$$

Integrate and get

$$f(z) = B + A\int \frac{(z-1)^{\frac{1}{4}}}{z^{\frac{1}{4}}}\,dz$$

$$= B + A\left((z-1)^{\frac{1}{4}}\,z^{\frac{3}{4}} - \frac{1}{2}\,\text{Arctan}\left(\left(1 - \frac{1}{z}\right)^{\frac{1}{4}}\right) + \frac{1}{4}\,\text{Log}\,\frac{1-\left(1-\frac{1}{z}\right)^{\frac{1}{4}}}{1+\left(1-\frac{1}{z}\right)^{\frac{1}{4}}}\right)$$

Solve for the coefficients A and B and obtain

$$f(z) = i + \frac{1}{\pi}\left(4\,(z-1)^{\frac{1}{4}}\,z^{\frac{3}{4}} - 2\,\text{Arctan}\left(\left(1 - \frac{1}{z}\right)^{\frac{1}{4}}\right) + \text{Log}\,\frac{1-\left(1-\frac{1}{z}\right)^{\frac{1}{4}}}{1+\left(1-\frac{1}{z}\right)^{\frac{1}{4}}}\right)$$

Explore Extra Solution 5.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

by

John H. Mathews and Russell W. Howell

Jones and Bartlett Learning

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11.11 Sources and Sinks

If the two-dimensional motion of an ideal fluid consists of an outward radial flow from a point and is symmetrical in all directions, then the point is called a simple source. A source at the origin can be considered as a line perpendicular to the z plane along which fluid is being emitted. If the rate of emission of volume of fluid per unit length $2\pi m$, then the origin is said to be a source of strength m, the complex potential for the flow is

$$F(z) = m \log(z),$$

and the velocity $\mathbf{V}$ at the point (x, y) is given by

$$\mathbf{V}(x, y) = \overline{F'(z)} = \frac{m}{\bar{z}}.$$

For fluid flows, a sink is a negative source and is a point of inward radial flow at which the fluid is considered to be absorbed or annihilated. Sources and sinks for flows are illustrated in Figure 11.93.

Figure 11.93 Sources and sinks for an ideal fluid.

11.11.1 Source: A Charged Line

In the case of electrostatics a source will correspond to a uniformly charged line perpendicular to the z-plane at the point z_0. We will show that if the line L is located at $z_0 = 0$ and carries a charge density of $\frac{q}{2}$ coulombs per unit length, then the magnitude of the electric field is $|\mathbf{E}(x, y)| = \frac{q}{\sqrt{x^2+y^2}}$. Hence $\mathbf{E}$ is given by

(11-41) $$\mathbf{E}(x, y) = \frac{q\, z}{|z^2|} = \frac{q}{\bar{z}},$$

and the complex potential is

$$F(z) = -q \log(z), \text{ and}$$

$$\mathbf{E}(x, y) = -\overline{F'(z)}.$$

A sink for electrostatics is a negatively charged line perpendicular to the z-plane. The electrostatic field for electrostatic problems corresponds to the velocity field for fluid flow problems, except that their corresponding potentials differ by a sign change.

To establish Equation (11-41), we start with Coulomb's law, which states that two particles with charges q and Q exert a force on one another with magnitude $\frac{C\, q\, Q}{r^2}$, where r is the distance between particles and C is a constant that depends on the scientific units. For simplicity, we assume that $C = 1$ and the test particle at the point z has charge $Q = 1$.

The contribution $\triangle\mathbf{E}_1$ induced by the element of charge $\frac{q\,\triangle h}{2}$ along the segment of length $\triangle h$ situated at a height h above the plane has magnitude $|\triangle\mathbf{E}_1|$ given by

$$|\triangle\mathbf{E}_1| = \frac{(q/2)\,\triangle h}{r^2+h^2}.$$

It has the same magnitude as $\triangle\mathbf{E}_2$ induced by the element $\frac{q\,\triangle h}{2}$ located a distance $-h$ below the plane. From the vertical symmetry involved their sum, $\triangle\mathbf{E}_1 + \triangle\mathbf{E}_2$, lies parallel to the plane along the ray from the origin, as shown in Figure 11.94.

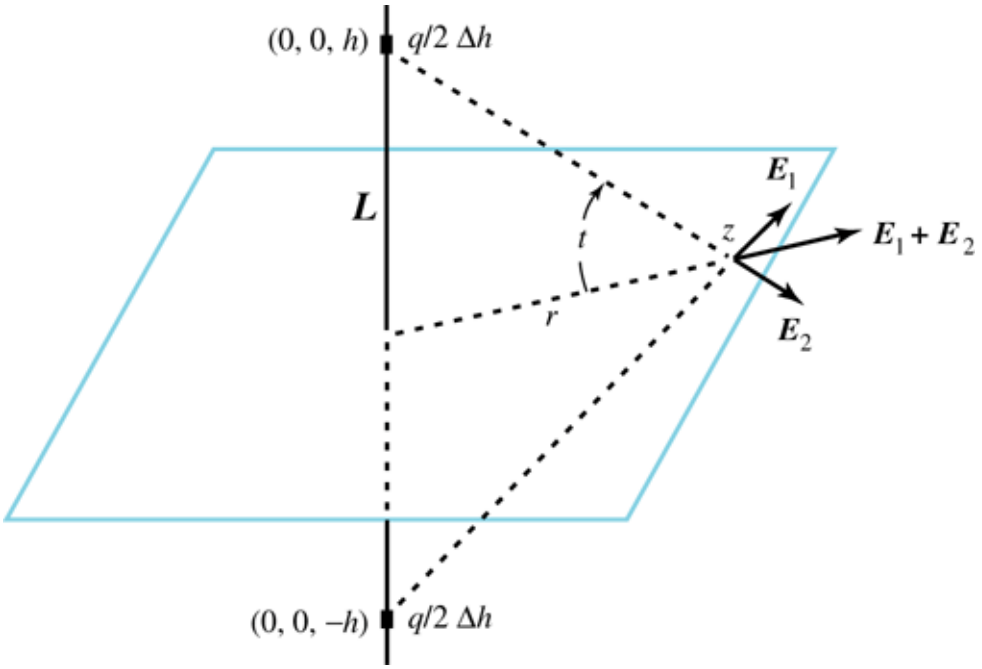

Figure 11.94 Contributions to **E** from the elements of charge $\frac{q\,\triangle h}{2}$ situated at $(0, 0, \pm h)$, above and below the z plane.

By the principal of superposition, we add all contributions from the elements of charge along L to obtain $\mathbf{E} = \sum \triangle \mathbf{E}_k$. By vertical symmetry, **E** lies parallel to the complex plane along the ray from the origin through the point z. Hence the magnitude of **E** is the sum of all components $|\triangle \mathbf{E}| \cos t$ that are parallel to the complex plane, where t is the angle between **E** and the plane. Letting $\triangle h \to 0$ in this summation process produces the definite integral

$$|\mathbf{E}(x, y)| = \int_{-\infty}^{\infty} |\triangle \mathbf{E}| \cos t \, dh = \int_{-\infty}^{\infty} \frac{(q/2)\,\triangle h}{r^2+h^2} \, dh.$$

Next, we use the change of variable $h = r \tan t$ and $dh = r \sec^2 t \, dt$ and the trigonometric identity $\sec^2 t = \frac{r^2+h^2}{r^2}$ to obtain the equivalent integral:

$$|\mathbf{E}(x, y)| = \int_{-\pi/2}^{\pi/2} \frac{(q/2)\cos t}{r^2+h^2} \frac{r^2+h^2}{r} \, dt = \frac{q}{2r} \int_{-\pi/2}^{\pi/2} \cos t \, dt = \frac{q}{r}.$$

Multiplying this magnitude $\frac{q}{r}$ by the unit vector $\frac{z}{|z|}$ establishes Formula (11-41). If $q > 0$, then the field is directed away from $z_0 = 0$ and, if $q < 0$, then it is directed toward $z_0 = 0$. An electrical field located at $z_0 \neq 0$ is given by

$$\mathbf{E}(x, y) = \frac{q\,(z - z_0)}{|(z - z_0)^2|} = \frac{q}{\overline{z} - \overline{z_0}},$$

and the corresponding complex potential is

$$F(z) = -q \log(z - z_0).$$

Example 11.30. (Source and sink of equal strength) Let a source and sink of unit strength be located at the points $z_1 = +1$ and $z_2 = -1$, respectively. The complex potential for a fluid flowing from the source at $z_1 = +1$ to the sink at $z_2 = -1$ is

$$F(z) = \log(z-1) - \log(z+1) = \log\left(\frac{z-1}{z+1}\right).$$

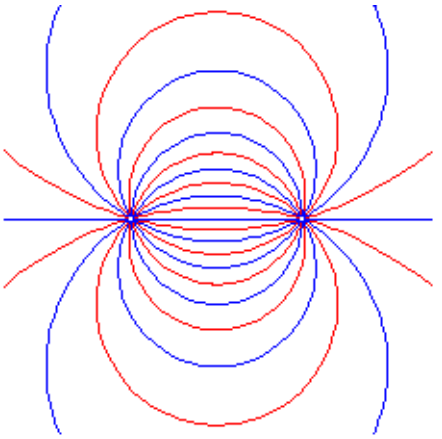

Figure 11.95 (a) Source and sink of unit strength.

The velocity potential and stream function are

$$\phi(x, y) = \ln\left|\frac{z-1}{z+1}\right|, \text{ and}$$

$$\psi(x, y) = \arg\left(\frac{z-1}{z+1}\right),$$

respectively. Solving for the streamline $\psi(x, y) = c$, we start with

$$c = \arg\left(\frac{z-1}{z+1}\right) = \arg\left(\frac{(x+i\,y-1)(x-i\,y+1)}{(x+i\,y+1)(x-i\,y+1)}\right)$$

$$= \arg\left(\frac{x^2+y^2-1+i\,2\,y}{x^2+2\,x+1+y^2}\right) = \arg\left(\frac{x^2+y^2-1+i\,2\,y}{(x+1)^2+y^2}\right)$$

$$= \arctan\left(\frac{2\,y}{x^2+y^2-1}\right)$$

and obtain the equation

$$(\tan c)(x^2 + y^2 - 1) = 2\,y.$$

A straightforward calculation shows that points on the streamline must satisfy the equation

$$x^2 + (y - \cot c)^2 = 1 + \cot^2 c,$$

which is the equation of a circle with center at $(0, \cot c)$ that passes through the points $(\pm 1, 0)$. Several streamlines are indicated in Figure 11.95(a).

Explore Solution 11.30.

Example 11.31. (Two sources of equal strength) Let two sources of unit strength be located at the points $z_1 = 1$ and $z_2 = -1$. The resulting complex potential for a fluid flow is

$$F(z) = \log(z-1) + \log(z+1) = \log(z^2-1).$$

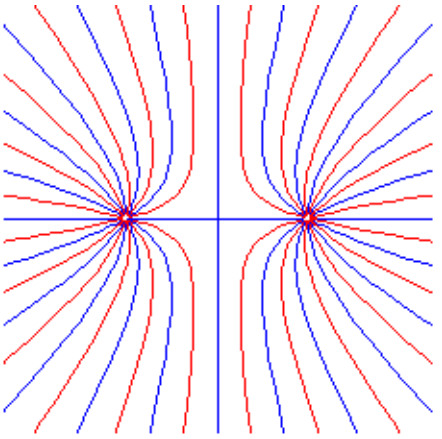

Figure 11.95 (b) Two sources of unit strength

The velocity potential and stream function are

$$\phi(x, y) = \ln|z^2-1|, \text{ and}$$

$$\psi(x, y) = \arg(z^2-1),$$

respectively. Solving for the streamline $\psi(x, y) = c$, we start with

$$c = \arg(z^2-1) = \arg(x^2-y^2-1+i\,2xy)$$

$$= \arctan\left(\frac{2xy}{x^2-y^2-1}\right)$$

and obtain the equation

$$x^2 + 2xy\cot c - y^2 = 1.$$

If we express this equation in the form $\left(x - y\tan\frac{c}{2}\right)\left(x + y\cot\frac{c}{2}\right) = 1$ then we can get

$$\left(x\cos\frac{c}{2} - y\sin\frac{c}{2}\right)\left(x\sin\frac{c}{2} + y\cos\frac{c}{2}\right) = \sin\frac{c}{2}\cos\frac{c}{2}, \text{ or}$$

$$\left(x\cos\frac{c}{2} - y\sin\frac{c}{2}\right)\left(x\sin\frac{c}{2} + y\cos\frac{c}{2}\right) = \frac{\sin c}{2}.$$

Now use the rotation of axes

$$x^* = x\cos\frac{-c}{2} + y\sin\frac{-c}{2} \text{ and}$$

$$y^* = -x\sin\frac{-c}{2} + y\cos\frac{-c}{2},$$

then the streamlines must satisfy the equation

$$x^* y^* = \frac{\sin c}{2}$$

and are rectangular hyperbolas with centers at the origin that pass through the points $z = \pm 1$. Several streamlines are indicated in Figure 11.95(b).

Explore Solution 11.31.

Ideal Fluid Flow

Let an ideal fluid flow in a domain in the z plane be effected by a source located at the point z_0. Then the flow at points z, which lie in a small neighborhood of the point z_0, is approximated by that of a source with the complex potential

$$F(z) = \log(z - z_0) + \text{constant}.$$

If $w = S(z)$ is a conformal mapping and $w_0 = S(z_0)$, then $S(z)$ has a nonzero derivative at z_0 and

$$w - w_0 = (z - z_0)(S'(z_0) + \eta(z))$$

where $\eta(z) \to 0$ as $z \to z_0$. Taking logarithms yields

$$\log(w - w_0) = \log(z - z_0) + \text{Log}(S'(z_0) + \eta(z)).$$

Because $S'(z_0) \neq 0$, the term $\text{Log}(S'(z_0) + \eta(z))$ approaches the constant value $\text{Log}(S'(z_0))$ as $z \to z_0$. As $\log(z - z_0)$ is the complex potential for a source located at the point z_0, the image of a source under a conformal mapping is a source.

We can use the technique of conformal mapping to determine the fluid flow in a domain D in the z plane that is produced by sources and sinks. If we can construct a conformal mapping $w = S(z)$ so that the image of sources, sinks, and boundary curves for the flow in D are mapped onto sources, sinks, and boundary curves in a domain G where the complex potential is known to be $F_1(w)$, then the complex potential in D is given by $F_2(z) = F_1(S(z))$.

Example 11.32. Suppose that the lines $x = \pm\frac{\pi}{2}$ are considered as walls of a containing vessel for a fluid flow produced by a single source of unit strength located at the origin. The conformal mapping $w = S(z) = \sin z$ maps the infinite strip bounded by the lines $x = \pm\frac{\pi}{2}$ onto the w plane slit along the boundary rays $u \leq -1, v = 0$ and $u \geq 1, v = 0$, and the image of the source at $z_0 = 0$ is a source located at $w_0 = 0$. The complex potential

$$F_1(w) = \log w$$

determines a fluid flow in the w plane past the boundary curves $u \leq -1, v = 0$ and $u \geq 1, v = 0$, which lie along streamlines of the flow. Therefore the complex potential for the fluid flow in the infinite strip in the z plane is

$$F_2(z) = F_1(S(z)) = \log(\sin z).$$

Several streamlines for the flow are illustrated in Figure 11.96.

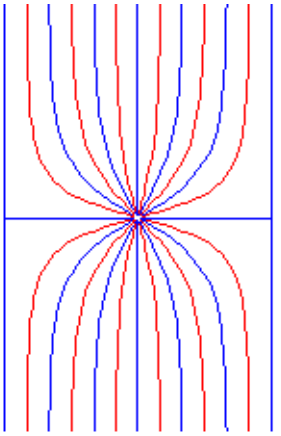

Figure 11.96 A source in the center of a strip.

Explore Solution 11.32.

Example 11.33. Suppose that the lines $x = \pm\frac{\pi}{2}$ are considered as walls of a containing vessel for the fluid flow produced by a single source of unit strength located at the point $z_1 = \frac{\pi}{2}$ and a sink of unit strength located at the point $z_2 = -\frac{\pi}{2}$. The conformal mapping $w = S(z) = \sin z$ maps the infinite strip bounded by the lines $x = \pm\frac{\pi}{2}$ onto the w plane slit along the boundary rays $K_1 : u \le -1,\ v = 0$ and $K_2 : u \ge 1,\ v = 0$. The image of the source at $z_1 = \frac{\pi}{2}$ is a source at $w_1 = 1$, and the image of the sink at $z_2 = \frac{-\pi}{2}$ is a sink at $w_2 = -1$. The potential

$$F_1(w) = \log\frac{w-1}{w+1}$$

determines a fluid flow in the w plane past the boundary curves K_1 and K_2, which lie along streamlines of the flow. Therefore the complex potential for the fluid flow in the infinite strip in the z plane is

$$F_2(z) = F_1(S(z)) = \log\left(\frac{\sin z - 1}{\sin z + 1}\right).$$

Several streamlines for the flow are illustrated in Figure 11.97.

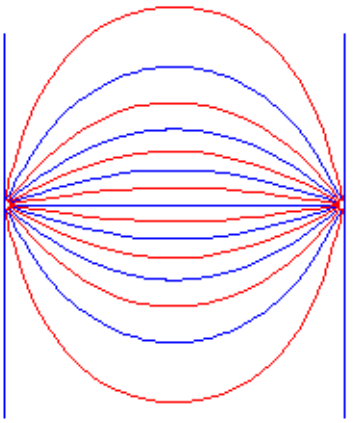

Figure 11.97 A source and a sink on the edges of a strip.

Explore Solution 11.33.

We can use the technique of transformation of a source to determine the effluence from a channel extending from infinity. In this case, we construct a conformal mapping $w = S(z)$ from the upper half-plane $\mathrm{Im}(z) > 0$ so that the single source located at $z_0 = 0$ is mapped to the point $w_0 = -i\infty$ at infinity that lies along the channel. The streamlines emanating from $z_0 = 0$ in the upper half-plane are mapped onto streamlines issuing from the channel.

Example 11.34. Consider the conformal mapping

$$w = S(z) = \frac{2}{\pi}\sqrt{(z^2 - 1)} + \frac{2}{\pi}\,\mathrm{Arcsin}\left(\frac{1}{z}\right),$$

which maps the upper half-plane $\mathrm{Im}(z) > 0$ onto the domain consisting of the upper half-plane $\mathrm{Im}(w) > 0$ joined to the channel $-1 \le u \le 1,\ v \le 0$. The point $z_0 = 0$ is mapped onto the point $w_0 = S(z_0) = S(0) = -i\infty$ and $z_1 = 1$ is mapped onto $w_1 = S(z_1) = S(1) = 1$ along the channel. Images of the rays $r > 0,\ \theta = \alpha$, where α is a constant, are streamlines issuing from the channel as indicated in Figure 11.98.

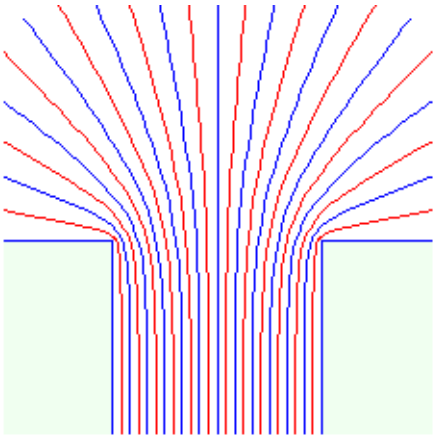

Figure 11.98 Effluence from a channel into a half-plane.

Explore Solution 11.34.

Extra Example 1. Use a Schwarz-Christoffel transformation to show that the conformal mapping

$$w = S(z) = i\pi + 2(z+1)^{\frac{1}{2}} + \mathrm{Log}\left(\frac{1-(z+1)^{\frac{1}{2}}}{1+(z+1)^{\frac{1}{2}}}\right)$$

will map the flow in the upper half-plane from a source at $z_0 = 0$ onto the flow from a channel with $w_0 = S(z_0) = -\infty$ into the first quadrant, as shown in Figure 11.105.

Hint: Set $x_1 = -1$, $x_2 = 0$ and $w_1 = i\pi$, $w_2 = -d$ and let $d \to \infty$.

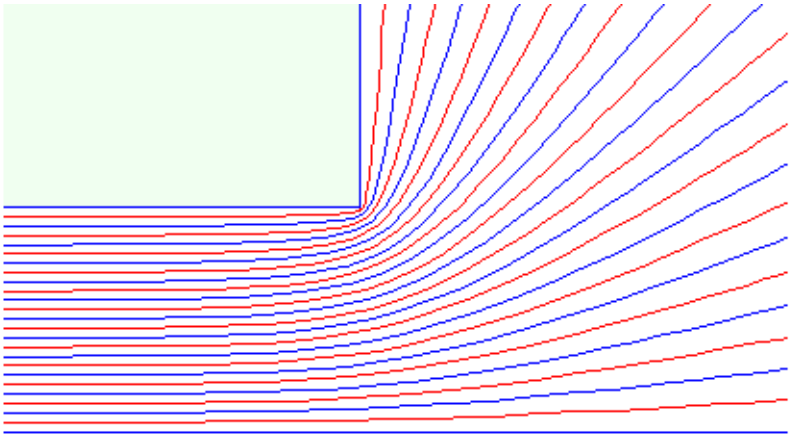

Figure 11.105 Flow from a channel into a quadrant.

Solution. Using the Schwarz-Christoffel transformation method, the exterior angles are $\alpha_1 = -\frac{\pi}{2}$, $\alpha_2 = \pi$, and the formula for the derivative S'(z) is

$$S'(z) = A\,(z - x_1)^{-\alpha_1/\pi}\,(z - x_2)^{-\alpha_2/\pi}$$

$$= A\,(z - (-1))^{-\left(-\frac{\pi}{2}\right)/\pi}\,(z - 0)^{-(\pi)/\pi}$$

$$= A\,(z+1)^{\frac{1}{2}}\,(z)^{-1}$$

$$= A\,\frac{(z+1)^{\frac{1}{2}}}{z}$$

integrate

$$S(z) = B + A\int \frac{(z+1)^{\frac{1}{2}}}{z}\,dz$$

$$S(z) = B + A\left(2\,(z+1)^{\frac{1}{2}} - \text{Log}\left(\frac{1+(z+1)^{\frac{1}{2}}}{1-(z+1)^{\frac{1}{2}}}\right)\right)$$

$$S(z) = B + A\left(2\,(z+1)^{\frac{1}{2}} + \text{Log}\left(\frac{1-(z+1)^{\frac{1}{2}}}{1+(z+1)^{\frac{1}{2}}}\right)\right)$$

use the conditions $S(0) = 0$, $S(1) = i\,\pi$ and solve the resulting system for A and B. The desired result is

$$S(z) = i\,\pi + 2\,(z+1)^{\frac{1}{2}} + \text{Log}\left(\frac{1-(z+1)^{\frac{1}{2}}}{1+(z+1)^{\frac{1}{2}}}\right).$$

Explore Extra Solution 1.

Extra Example 2. Use a Schwarz-Christoffel transformation to show that the conformal mapping

$$w = S(z) = 4\,(1+z)^{1/4} + \text{Log}\,\frac{(z+1)^{1/4}-1}{(z+1)^{1/4}+1} + i\,\text{Log}\,\frac{i-(z+1)^{1/4}}{i+(z+1)^{1/4}}$$

will map the flow in the upper half-plane from a source at $z_0 = 0$ onto the flow from a channel with $w_0 = S(z_0) = -\infty$ into a sector, as shown in Figure 11.106.
Hint: Set $x_1 = -1$, $x_2 = 0$ and $w_1 = i$, $w_2 = -d$ and let $d \to \infty$. Use the change of variable $z + 1 = s^4$ in the resulting integral.

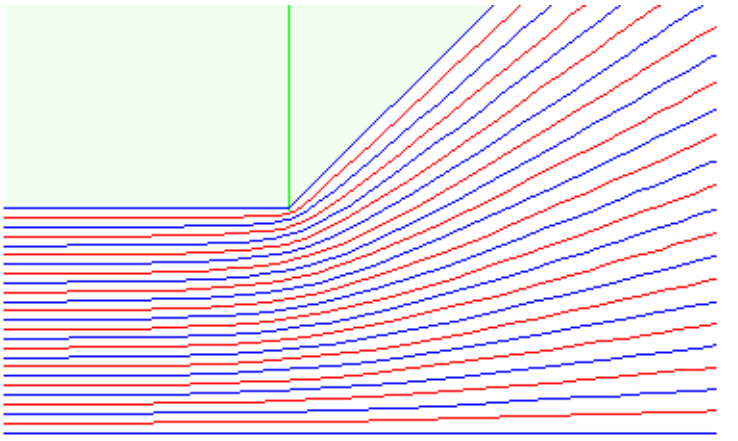

Figure 11.106 Flow from a channel into a sector in the first quadrant.

Solution. Using the Schwarz-Christoffel transformation method, the exterior angles are $\alpha_1 = -\frac{\pi}{4}$, $\alpha_2 = \pi$, and the formula for the derivative $S'(z)$ is

$$S'(z) = A(z - x_1)^{-\alpha_1/\pi}(z - x_2)^{-\alpha_2/\pi}$$

$$= A(z - (-1))^{-\left(-\frac{\pi}{4}\right)/\pi}(z - 0)^{-(\pi)/\pi}$$

$$= A(z + 1)^{\frac{1}{4}}(z)^{-1}$$

$$= A\frac{(z+1)^{\frac{1}{4}}}{z}$$

integrate

$$S(z) = B + A\int \frac{(z+1)^{\frac{1}{4}}}{z}\,dz$$

use the substitutions

$$z + 1 = s^4$$
$$z = s^4 - 1$$
$$dz = 4s^3\,ds$$

and get

$$\int \frac{(z+1)^{\frac{1}{4}}}{z}\,dz = \int \frac{(s^4)^{1/4}}{s^4-1}\,4\,s^3\,ds$$

$$= \int \frac{4\,s^4}{s^4-1}\,ds$$

$$= 4\,s - 2\,\text{Arctan}\,s + \text{Log}\,(s-1) - \text{Log}\,(s+1)$$

Use the identity $-2\,\text{ArcTan}\,s = -2\,\frac{i}{2}\,\text{Log}\left(\frac{i+s}{i-s}\right) = i\,\text{Log}\left(\frac{i-s}{i+s}\right)$ and write the integral as

$$\int \frac{(z+1)^{\frac{1}{4}}}{z}\,dz = 4\,s + \text{Log}\left(\frac{s-1}{s+1}\right) + i\,\text{Log}\left(\frac{i-s}{i+s}\right)$$

Now use the substitution $s \to (z+1)^{\frac{1}{4}}$ and get

$$S(z) = B + A\int \frac{(z+1)^{\frac{1}{4}}}{z}\,dz$$
$$= B + A\left(4\,(z+1)^{1/4} + \text{Log}\,\frac{(z+1)^{1/4}-1}{(z+1)^{1/4}+1} + i\,\text{Log}\,\frac{i-(z+1)^{1/4}}{i+(z+1)^{1/4}}\right)$$

use the conditions $S(0) = 0$, $S(1) = i\,\pi$ and solve the resulting system for A and B. The desired result is

$$S(z) = 4\,(1+z)^{1/4} + \text{Log}\,\frac{(z+1)^{1/4}-1}{(z+1)^{1/4}+1} + i\,\text{Log}\,\frac{i-(z+1)^{1/4}}{i+(z+1)^{1/4}}.$$

Explore Extra Solution 2.

Extra Example 3. Use a Schwarz-Christoffel transformation to show that the conformal mapping

$$w = S(z) = \frac{1}{\pi}\,\text{Arcsin}\,z + \frac{i}{\pi}\,\text{Arcsin}\,\frac{1}{z} + \frac{1+i}{2}$$

will map the flow in the upper half-plane from a source at $z_0 = 0$ onto the flow in a right-angled channel with $w_0 = S(z_0) = -\infty$, as shown in Figure 11.107.

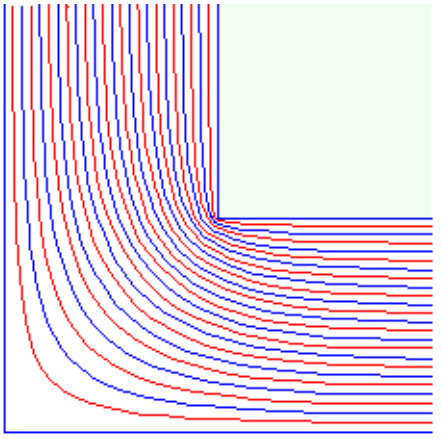

Figure 11.107 Flow in a right-angled channel.

Solution. Using the Schwarz-Christoffel transformation method, if we choose $x_1 = -1$, $x_2 = 0$, $x_3 = 1$ and $w_1 = 0$, $w_2 = d$, $w_3 = 1+i$, then the formula

$$g'(z) = A_1 (z - x_1)^{-\alpha_1/\pi} (z - x_2)^{-\alpha_2/\pi} (z - x_3)^{-\alpha_3/\pi}$$

$$= A_1 (z - (-1))^{-\alpha_1/\pi} (z - 0)^{-\alpha_2/\pi} (z - 1)^{-\alpha_3/\pi}$$

$$= A_1 (z + 1)^{-\alpha_1/\pi} (z)^{-\alpha_2/\pi} (z - 1)^{-\alpha_3/\pi}$$

will determine a mapping $w = g(z)$ of the upper half-plane onto the domain indicated in Figure 11.74(a). With $\alpha_1 = \frac{\pi}{2}$, we let $d \to \infty$, then $\alpha_2 \to \pi$ and $\alpha_3 \to -\frac{\pi}{2}$.

The limiting formula for the derivative $g'(z)$ becomes

$$S'(z) = A_1 (z + 1)^{-\alpha_1/\pi} (z)^{-\alpha_2/\pi} (z - 1)^{-\alpha_3/\pi}$$

$$= A_1 (z + 1)^{-\frac{\pi}{2}/\pi} (z)^{-\pi/\pi} (z - 1)^{-\left(-\frac{\pi}{2}\right)/\pi}$$

$$= A_1 (z + 1)^{-\frac{1}{2}} (z)^{-1} (z - 1)^{\frac{1}{2}}$$

$$= A_1 \frac{1}{z} \frac{(z-1)^{\frac{1}{2}}}{(z+1)^{\frac{1}{2}}}$$

$$= A_1 \frac{(z-1)^{\frac{1}{2}} (z-1)^{\frac{1}{2}}}{z (z+1)^{\frac{1}{2}} (z-1)^{\frac{1}{2}}}$$

$$= A_1 \frac{z-1}{z \left(z^2-1\right)^{\frac{1}{2}}}$$

$$= A \frac{z-1}{z \left(1-z^2\right)^{\frac{1}{2}}}$$

where $A = -i A_1$, which will determine a mapping $w = S(z)$ from the upper half plane onto the channel as indicated in Figure 11.74(b) and Figure 11.107.

Using integrals in Table 11.2, we obtain

$$S(z) = A \int \frac{z-1}{z \left(1-z^2\right)^{\frac{1}{2}}} dz + B$$

$$= A \int \frac{z}{z \left(1-z^2\right)^{\frac{1}{2}}} dz - A \int \frac{1}{z \left(1-z^2\right)^{\frac{1}{2}}} dz + B$$

$$= A \int \frac{1}{\left(1-z^2\right)^{\frac{1}{2}}} dz - A i \int \frac{1}{z \left(z^2-1\right)^{\frac{1}{2}}} dz + B$$

$$= A \left(\operatorname{Arcsin} z + i \operatorname{Arcsin}\left(\frac{1}{z}\right)\right) + B$$

If we use the principal branch of the inverse sine function, then the boundary values $S(-1) = 0$ and $S(1) = 1 + i$ lead to the system

$$A\left(-\frac{\pi}{2}-\frac{i\pi}{2}\right)+B=0 \quad \text{and} \quad A\left(\frac{\pi}{2}+\frac{i\pi}{2}\right)+B=1+i,$$

which we can solve to obtain $A=\frac{1}{\pi}$ and $B=\frac{1+i}{2}$. Hence the required solution is

$$w = S(z) = \frac{1}{\pi}\,\text{Arcsin}\,z + \frac{i}{\pi}\,\text{Arcsin}\,\frac{1}{z} + \frac{1+i}{2}$$

Explore Extra Solution 3.

Extra Example 4. Use a Schwarz-Christoffel transformation to show that the conformal mapping

$$w = S(z) = 2\,(z+1)^{1/2} - \text{Log}\left(\frac{1-(z+1)^{1/2}}{1+(z+1)^{1/2}}\right)$$

will map the flow in the upper half-plane onto the flow from a channel back into a quadrant, as shown in Figure 11.108,
where $z_1=0$, $z_2=1$ and $w_1=\infty$, $w_1=S(1)=2\sqrt{2}-i\pi-2\,\text{Log}\left(\sqrt{2}-1\right)$.

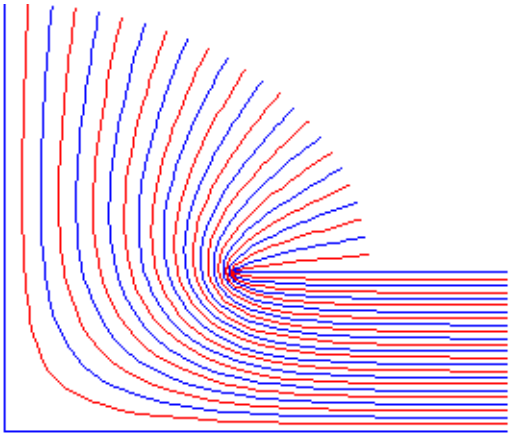

Figure 11.108 Flow from a channel back into the first quadrant.

Explore Extra Solution 4.

The "legacy version" being used to implement the commands "CartesianMap and PolarMap." The following Initialization Cell will load these subroutines.

Initialization Cell

Complex Analysis for Mathematics and Engineering

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Chapter 12 Fourier Series and the Laplace Transform

12.1 Fourier Series

Overview

In this chapter we show how Fourier Series, the Fourier Transform, and the Laplace Transform are related to the study of complex analysis. We develop the Fourier series representation of a real-valued function U(t) of the real variable t. We then discuss complex Fourier series and Fourier transforms. Finally, we develop the Laplace transform and the complex variable technique for finding its inverse. In this chapter we focus on applying these ideas to solving problems involving real-valued functions, so many of the theorems throughout are stated without proof.

Let U(t) be a real-valued function that is periodic with period 2π, that is

$$U(t + 2\pi) = U(t) \quad \text{for all } t.$$

One such function is $s = U(t) = \sin\left(t - \frac{\pi}{2}\right) + \frac{7}{10}\cos\left(2t - \pi - \frac{1}{4}\right) + \frac{17}{10}$. Its graph is obtained by repeating the portion of the graph in any interval of length 2π, as shown in Figure 12.1.

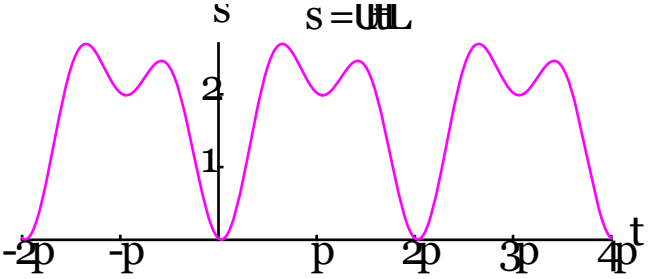

Figure 12.1. A function U(t) with period 2π.

Exploration.

Familiar examples of real functions that have period 2π, are cos(nt) and sin(nt), where n is an integer. These examples raise the question of whether any periodic function can be

represented by a sum of terms involving $a_n \cos(nt)$ and $b_n \sin(nt)$, where a_n and b_n are real constants. As we soon demonstrate, the answer to this question is often yes.

Definition 12.1 (Piecewise Continuous). The function $U(t)$ is piecewise continuous on the closed interval [a,b], if there exists values $t_0, t_1, \ldots, t_n$ with $a = t_0 < t_1 < \ldots < t_n = b$ such that U is continuous in each of the open intervals $t_{k-1} < t < t_k$, for $k = 1, 2, \ldots, n$ and has left-hand and right-hand limits at each of the values t_k, for $k = 0, 1, 2, \ldots, n$.

We use the symbols $U(a^-)$ and $U(a^+)$ for the left-hand and right-hand limit, respectively, of a function $U(a)$ as t approaches the point a. The graph of a piecewise continuous function is illustrated in Figure 12.2, where the function $U(a)$ is

$$U(t) = \begin{cases} \frac{2}{3}\left(t-\frac{1}{2}\right)^2+\frac{1}{4} & \text{, when } 1 \le t < 2, \\ \frac{5}{2}-(t-2)^2 & \text{, when } 2 < t < 3, \\ 1+\frac{t-3}{4} & \text{, when } 3 < t < 4, \\ \frac{6}{5}-(t-5)^3 & \text{, when } 4 < t \le 6. \end{cases}$$

The left-hand and right-hand limits at $t_0 = 2$, $t_1 = 3$, and $t_2 = 4$ are easily determined:

At $t_0 = 2$, the left-hand limit is $U(2^-) = \lim_{t\to 2^-}\left[\frac{2}{3}\left(t-\frac{1}{2}\right)^2+\frac{1}{4}\right] = \frac{2}{3}\left(2-\frac{1}{2}\right)^2+\frac{1}{4} = \frac{7}{4}$,

and the right-hand limit is $U(2^+) = \lim_{t\to 2^+}\left[\frac{5}{2}-(t-2)^2\right] = \frac{5}{2}-(2-2)^2 = \frac{5}{2}$.

At $t_1 = 3$, the left-hand limit is $U(3^-) = \lim_{t\to 3^-}\left[\frac{5}{2}-(t-2)^2\right] = \frac{5}{2}-(3-2)^2 = \frac{3}{2}$,

and the right-hand limit is $U(3^+) = \lim_{t\to 3^+}\left[1+\frac{t-3}{4}\right] = 1+\frac{3-3}{4} = 1$.

At $t_2 = 4$, the left-hand limit is $U(4^-) = \lim_{t\to 4^-}\left[1+\frac{t-3}{4}\right] = 1+\frac{4-3}{4} = \frac{5}{4}$,

and the right-hand limit is $U(4^+) = \lim_{t\to 4^+}\left[\frac{6}{5}-(t-5)^3\right] = \frac{6}{5}-(4-5)^3 = \frac{11}{5}$.

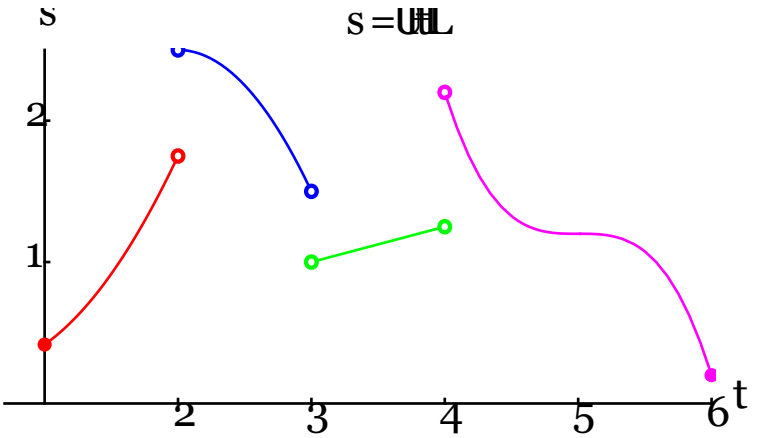

Figure 12.2. A piecewise continuous function $U(t)$ over the interval $[1, 6]$.

Exploration.

Definition 12.2 (Fourier Series). If $U(t)$ is periodic with period 2π and is piecewise continuous on $[-\pi, \pi]$, then the Fourier Series $S(t)$ for $U(t)$ is

(12-1) $$S(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty}\left(a_j \cos(jt) + b_j \sin(jt)\right),$$

where the coefficients a_j and b_j are given by the so-called **Euler's formulae**:

(12-2) $a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\cos(jt)\,dt \text{ for } j = 0, 1, \ldots,$

and

(12-3) $b_j = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\sin(jt)\,dt \text{ for } j = 1, 2, \ldots.$

We introduced the factor $\frac{1}{2}$ in the constant term $\frac{a_0}{2}$ on the right side of Equation (12-1) for convenience so that we can obtain a_0 from the general formula in Equation (12-1) by setting $j = 0$. We explain the reasons for this strategy shortly. Theorem 12.1 deals with convergence of the Fourier series.

Theorem 12.1 (Fourier Expansion). Assume that $S(t)$ is the Fourier Series for $U(t)$. If $U(t)$ and $U'(t)$ are piecewise continuous on $[-\pi, \pi]$, then $S(t)$ is convergent for all $t \in [-\pi, \pi]$.

The relation $U(t) = S(t)$ holds for all $t \in [-\pi, \pi]$ where U is continuous. If $t = a$ is a point of discontinuity of U, then

$$S(a) = \frac{U(a^-) + U(a^+)}{2},$$

where $U(a^-)$ and $U(a^+)$ denote the left-hand and right-hand limits, respectively. With this understanding, we have the Fourier Series expansion:

$$U(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos(jt) + b_j \sin(jt)).$$

Proof.

Method I. Fourier Series - Trigonometric Polynomials. Execute the following sells to set up the procedure S[n,t] for finding the Fourier Series - Trigonometric Polynomial of degree n.

```
Clear[a, b, j, n, t, u];
a[j_] = 1/π ∫_-π^π u[t] Cos[j t] dt;
b[j_] = 1/π ∫_-π^π u[t] Sin[j t] dt;
FS[n_, t_] := a[0]/2 + Sum[a[j] Cos[j t] + b[j] Sin[j t], {j, 1, n}];
Print["   a_n = ", a[j]];
Print["   b_n = ", b[j]];
Print["S_n[t] = a[0]/2 + Sum[a[j]Cos[j t]+b[j]Sin[j t], {j, 1, n}]"];
```

Example 12.1. The function $U(t) = \frac{t}{2}$ for $-\pi < t < \pi$, extended periodically by the equation $U(t + 2\pi) = U(t)$, has the Fourier series expansion

$$U(t) = \frac{t}{2} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}\sin(jt).$$

Solution.

Using Equation (12-2) and integrating by parts, we obtain

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t}{2}\,\text{Cos}[j\,t]\,dt = \left(\frac{t\,\text{Sin}[j\,t]}{2\,\pi\,j} + \frac{\text{Cos}[j\,t]}{2\,\pi\,j^2}\right)\Bigg|_{t=-\pi}^{t=\pi} = 0, \text{ for } j = 1, 2, \ldots.$$

Then using Equation (12-3) we get

$$b_j = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t}{2}\,\text{Sin}[j\,t]\,dt = \left(-\frac{t\,\text{Cos}[j\,t]}{2\,\pi\,j} + \frac{\text{Sin}[j\,t]}{2\,\pi\,j^2}\right)\Bigg|_{t=-\pi}^{t=\pi} = -\frac{\text{Cos}[j\,\pi]}{j} = \frac{(-1)^{j-1}}{j}.$$

We compute the coefficient a_0 by $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t}{2}\,dt = \left(\frac{t^2}{4\,\pi}\right)\Big|_{t=-\pi}^{t=\pi} = 0.$

Substituting the coefficients a_j and b_j into Equation (12.1) produces the required solution. The graphs of $U(t) = \frac{t}{2}$ and the first three partial sums $S_1(t) = \sin(t)$, $S_2(t) = \sin(t) - \frac{1}{2}\sin(2\,t)$, and $S_3(t) = \sin(t) - \frac{1}{2}\sin(2\,t) + \frac{1}{3}\sin(3\,t)$ are shown in Figure 11.3.

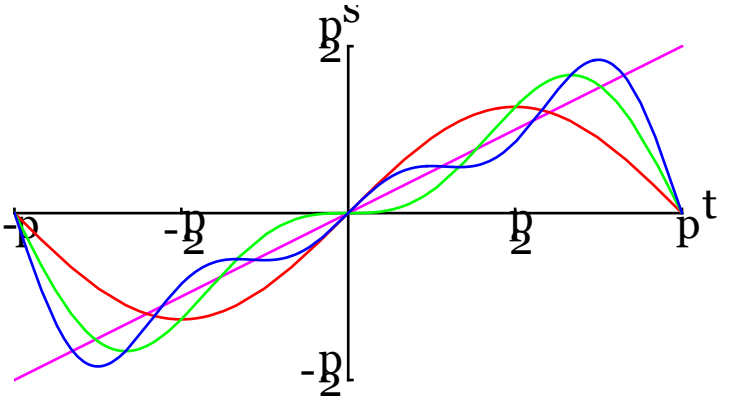

Figure 12.3. The function $U(t) = \frac{t}{2}$, and the approximations $S_1(t)$, $S_2(t)$, and $S_3(t)$.

Explore Solution 12.1.

Theorem 12.2. If $U(t)$ and $V(t)$ have Fourier series representations, then their sum $W(t) = U(t) + V(t)$ has a Fourier series representation, and the Fourier coefficients of W are obtained by adding the corresponding coefficients of U and V.

Proof.

Theorem 12.3 (Fourier Cosine Series). Assume that $U(t)$ is an even function and has period $2\,\pi$. If $U(t)$ and $U'(t)$ are piecewise continuous, the Fourier series for $U(t)$ involves only the cosine terms, (i.e. $b_j = 0$ for all j), and we write

$$U(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(j\,t),$$

where $a_j = \frac{2}{\pi}\int_0^{\pi} U(t)\cos(j\,t)\,dt$ for $j = 0, 1, \ldots$.

Proof.

Theorem 12.4 (Fourier Sine Series). Assume that U(t) is an odd function and has period 2π. If U(t) and U'(t) are piecewise continuous, the Fourier series for U(t) involves only the sine terms, (i.e. $a_j = 0$ for all j), and we write

$$U(t) = \sum_{j=1}^{\infty} b_j \sin(j\,t),$$

where $b_j = \frac{2}{\pi}\int_0^{\pi} U(t)\sin(j\,t)\,dt$ for $j = 1, 2, \ldots$.

Proof.

Theorem 12.5 (Termwise Integration). Assume that U(t) has the Fourier series representation

$$U(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j\cos(j\,t) + b_j\sin(j\,t)).$$

Then the integral of U(t) has a Fourier series representation which can be obtained by termwise integration of the Fourier series of U(t), that is

$$\int_0^t U(\tau)\,d\tau = \sum_{j=1}^{\infty}\left(\frac{\left(a_j + (-1)^{j+1}a_0\right)}{j}\sin(j\,t) - \frac{b_j}{j}\cos(j\,t)\right),$$

where we have used the expansion $\frac{a_0\,t}{2} = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}a_0}{j}\sin(j\,t)$.

Proof.

Theorem 12.6 (Termwise Differentiation). Assume that both U(t) and U'(t) have Fourier series representation and that

$$U(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j\cos(j\,t) + b_j\sin(j\,t)).$$

Then U'(t) can be obtained by termwise differentiation of U(t), that is

$$U'(t) = \sum_{j=1}^{\infty}(j\,b_j\cos(j\,t) - j\,a_j\sin(j\,t)).$$

Proof.

Method II. Fourier Series - Trigonometric Polynomials. Modify procedure S[n,t] for finding the Fourier Series - Trigonometric Polynomial of degree n to the case of a piecewise continuous function on the subintervals $[-\pi, 0)$ and $(0, \pi]$.

```
Clear[a, b, j, n, t, u1, u2];
a[j_] = 1/π (∫_-π^0 u2[t] Cos[j t] 𝕕t + ∫_0^π u1[t] Cos[j t] 𝕕t);
b[j_] = 1/π (∫_-π^0 u2[t] Sin[j t] 𝕕t + ∫_0^π u1[t] Sin[j t] 𝕕t);
TP[n_, t_] := a[0]/2 + ∑_j=1^n (a[j] Cos[j t] + b[j] Sin[j t]);
Print["   a_n = ", a[j]];
Print["   b_n = ", b[j]];
Print["T_n[t] = a[0]/2 + ∑_j=1^n (a[j]Cos[j t]+b[j]Sin[j t])"];
```

Example 12.2. The function $U(t) = |t|$ for $-\pi < t < \pi$, extended periodically by the equation $U(t + 2\pi) = U(t)$, has the Fourier series expansion

$$U(t) = |t| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\text{Cos}[(2j-1)t]}{(2j-1)^2}.$$

Solution.

The function $U(t)$ is an even function; hence we can use Theorem 11.3 to conclude that $b_j = 0$ for all j and that

$$a_j = \frac{2}{\pi}\int_0^{\pi} t\,\text{Cos}[j t]\,\mathbb{d}t = \left(\frac{2\,\text{Cos}[j t]}{\pi j^2} + \frac{2 t\,\text{Sin}[j t]}{\pi j}\right)\Bigg|_{t=0}^{t=\pi} = \frac{2\,\text{Cos}[j\pi] - 2}{\pi j^2} = \frac{2(-1)^j - 2}{\pi j^2},$$ for $j = 1, 2, \ldots$

We compute the coefficient a_0 by

$$a_0 = \frac{2}{\pi}\int_0^{\pi} t\,\mathbb{d}t = \left(\frac{t^2}{\pi}\right)\Bigg|_{t=0}^{t=\pi} = \pi.$$

Therefore, we have the found the Fourier series expansion

$$U(t) = |t| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^j - 1}{j^2}\,\text{Cos}[j t],$$

we leave it for the reader to show that this is equivalent to

$$U(t) = |t| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{\text{Cos}[(2j-1)t]}{(2j-1)^2}.$$

Using the a_j and Theorem 12.3 produces the required solution. The graphs of $U(t) = |t|$ and the first two partial sums $T_1(t) = \frac{\pi}{2} - \frac{4}{\pi}\cos(t)$,
and $T_3(t) = \frac{\pi}{2} - \frac{4}{\pi}\cos(t) - \frac{4}{9\pi}\cos(3t)$ are shown in figure below.

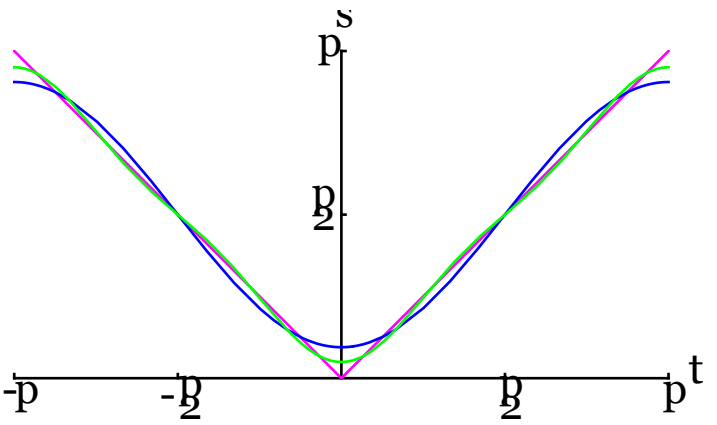

Figure. The function `U(t) = |t|`, and the approximations $T_1(t)$, and $T_3(t)$.

Explore Solution 12.2.

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12.2 The Dirichlet Problem for the Unit Disk

The Dirichlet problem for the closed unit disk $D: |z| \le 1$ is to find a real-valued function $u(x, y)$ that is harmonic in the unit disk D and that takes on the boundary values

(12-10) $$u(\cos\theta, \sin\theta) = U(\theta) \quad \text{for} \quad -\pi < \pi \le \pi,$$

at points $z = (\cos\theta, \sin\theta)$ on the unit circle $C: |z| = 1$.

Theorem 12.7 (The Dirichlet problem in D). If $U(t)$ has period 2π and has the Fourier series representation

$$U(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos(jt) + b_j \sin(jt)),$$

then the solution to the Dirichlet problem in the unit disk D is

(12-11) $$u(r\cos\theta, r\sin\theta) = \frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j r^j \cos(j\theta) + b_j r^j \sin(j\theta)),$$

where $x + iy = re^{i\theta}$ denotes a complex number in the closed disk $|z| \le 1$.

Proof.

The series representation in Equation (12-11) for u takes on the prescribed boundary values in Equation (12-10) at points on the unit circle $C: |z| = 1$. Each term, $a_j r^j \cos(j\theta)$ and $b_j r^j \sin(j\theta)$, in the series in Equation (12-11) is harmonic, so it is reasonable to conclude that the infinite series representing u will also be harmonic. The proof follows the proof of Theorem 12.8.

Theorem 12.8, which is attributed to the French mathematician Siméon Poisson, gives an integral representation for a function $u(x, y)$ that is harmonic in a domain containing the closed unit disk. The result is the analog to Poisson's integral formula for the upper half-plane.

Theorem 12.8 (Poisson Integral Formula for the Unit Disk). Let $u(x, y)$ be a function that is harmonic in a simply connected domain that contains the closed unit disk $D: |z| \le 1$. If

$u(x, y)$ takes on the boundary values

$$u(r\cos\theta, r\sin\theta) = U(\theta) \text{ for } -\pi < \theta \le \pi,$$

then $u(x, y)$ has the integral representation

$$u(r\cos\theta, r\sin\theta) = \int_{-\infty}^{\infty} \frac{(1 - r^2)\, U(t)}{1 + r^2 - 2r\cos(t - \theta)}\, dt,$$

which is valid for $|z| < 1$.

Proof.

Example 12.3. Find the function $u(x, y)$ that is harmonic in the unit disk $D: |z| < 1$ and takes on the boundary values $u(\cos\theta, \sin\theta) = U(\theta) = \frac{\theta}{2}$ for $-\pi < \theta < \pi$.

Solution.

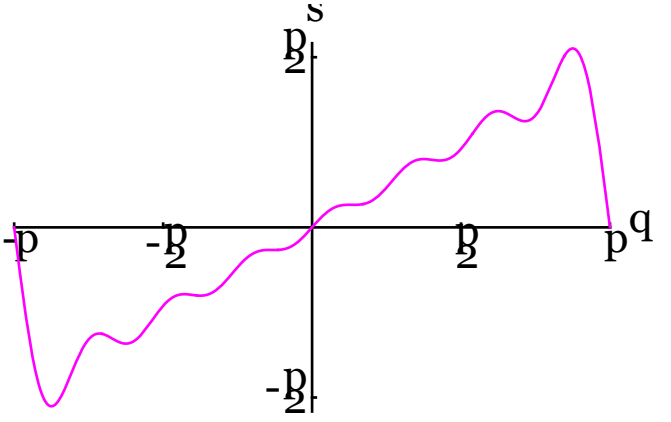
Using Example 12.1, we write the Fourier series for $U(t)$:

$$U(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nt).$$

Using Equation (12-11) for the solution of the Dirichlet problem, we obtain

$$u(r\cos\theta, r\sin\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} r^n \sin(nt).$$

This series representation of $u(r\cos\theta, r\sin\theta)$ takes on the prescribed boundary values at points where $U(t)$ is continuous.
The boundary function $U(t)$ is discontinuous at $z = -1$, which corresponds to $\theta = \pm\pi$; which are points where $U(t)$ was not prescribed.
Graphs of the approximations $u_7(t)$ and
$u_7(x, y) = u_7(r\cos\theta, r\sin\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} r^k \sin(n\theta)$, which involve the first seven terms in the preceding two equations, are shown in Figure 12.13.



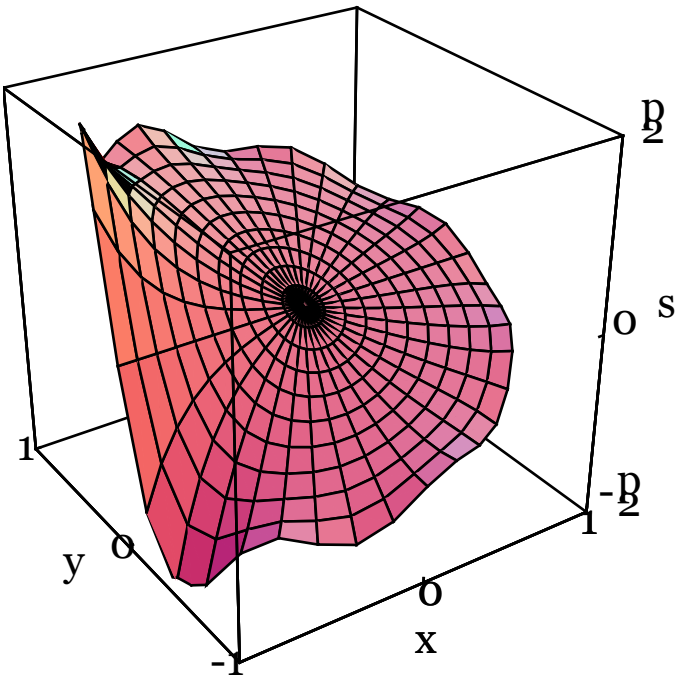


Figure 12.13. The functions $U_7(t)$ and $u_7(r\cos\theta, r\sin\theta) = \sum_{n=1}^{7} \frac{(-1)^{n+1}}{n} r^k \sin(n\theta)$.

Explore Solution 12.3.

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12.3 Vibrations in Mechanical Systems

Consider a spring that resists compression as well as extension, that is suspended vertically from a fixed support, and a body of mass m that is attached at the lower end of the spring. We make the assumption that the mass m is much larger than the mass of the spring so that we can neglect the mass of the spring. If there is no motion then the system is in static equilibrium, as illustrated in Figure 12.17(a). If the mass is pulled down further and released, then it will undergo an oscillatory motion.

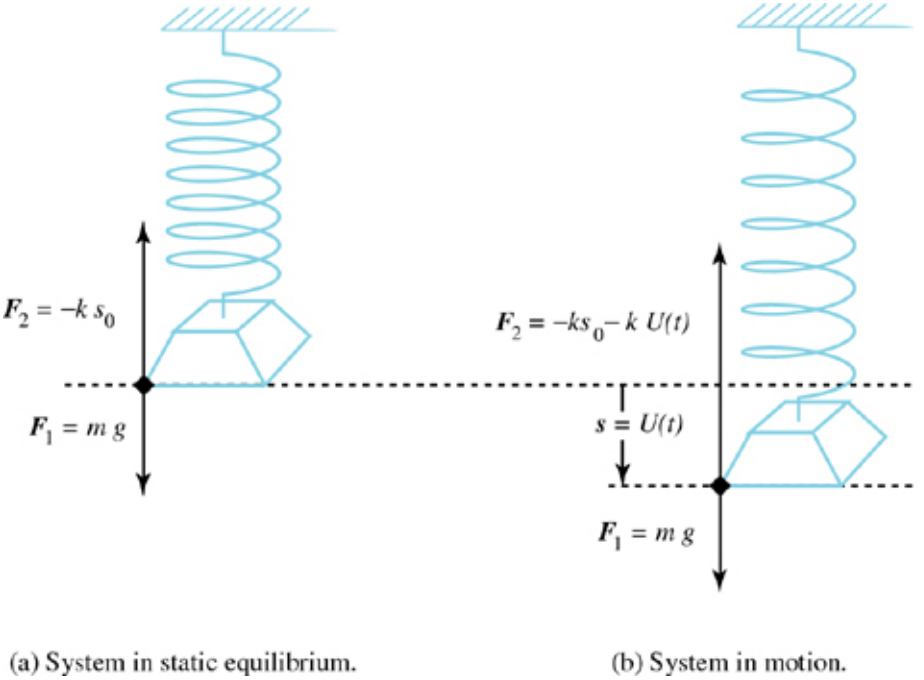

Figure 12.17. The spring-mass system.

We will determine the Simple Harmonic Motion of this mechanical system by considering the forces acting on the mass during the motion. For details see Section 12.3 in the textbook.

If there is no friction to slow the motion of the mass, then we say that the system is undamped. We determine the motion of this mechanical system by considering the forces acting on the mass during the motion. Doing so leads to a differential equation relating the displacement as a function of time. The most obvious force is that of gravitational attraction acting on the mass m and given by

$$F_1 = m\,g,$$

where g is the acceleration of gravity.

The next force to be considered is the spring force acting on the mass and directed upward if the spring is stretched and downward if it is compressed. It obeys Hooke's law,

$$F_2 = k\,s,$$

where s is the amount the spring is stretched when $s > 0$ and is the amount it is compressed when $s < 0$.

When the system is in static equilibrium and the spring is stretched by the amount s_0, the resultant of the spring force and the gravitational force is zero, which is expressed by the equation

$$m\,g - k\,s_0 = 0.$$

We let $s = U(t)$ denote the displacement from static equilibrium with the positive s direction pointed downward, as indicated in Figure 12.17(b), and write the spring force as

$$F_2 = -k[s_0 + U(t)] = -k\,s_0 - k\,U(t).$$

The resultant force F_R is

(12-16) $$F_R = F_1 + F_2 = m\,g - k\,s_0 - k\,U(t) = -k\,U(t).$$

We obtain the differential equation for motion by using Newton's second law, which states that the resultant of the forces acting on the mass at any instant satisfies

(12-17) $$F_R = m\,a.$$

The distance from equilibrium at time t is measured by $U(t)$, so the acceleration a is given by $a = U''(t)$. Applying Equations (12-16) and (12-17) yields

$$F_R = -k\,U''(t) = m\,U''(t).$$

Hence the undamped mechanical system is governed by the linear differential equation

$$m\,U''(t) + k\,U(t) = 0,$$

and the general solution to this D. E. is known to be

$U(t) = A\cos(\omega t) + B\sin(\omega t)$, where $\omega = \sqrt{\frac{k}{m}}$.

12.3.1 Damped Systems

If we consider frictional forces that slow the motion of the mass, then we say that the system is damped. To help visualize this situation, we connect a dashpot to the mass, as indicated in Figure 12.18. For small velocities we assume that the frictional force F_3 is proportional to the velocity; that is,

```
F3 = - c U' (t).
```

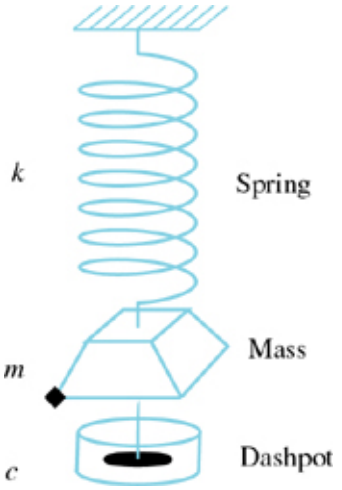

Figure 12.18. The spring-mass-dashpot system.

The damping constant c must be positive, for if `U' (t) > 0`, then the mass is moving downward and hence F_3 must point upward, which requires that F_3 be negative. The result of the three forces acting on the mass is given by

```
F1 + F2 + F3 = - k U (t) - c U' (t) = m U'' (t) = FR.
```

(For details see Section 12.3 in the textbook.)

Hence the Damped Mechanical Motion is governed by the linear differential equation

```
m U'' (t) + c U' (t) + k U (t) = 0.
```

12.3.2 Forced Vibrations

The vibrations discussed earlier are called free vibrations because all the forces that affect the motion of the system are internal to the system. We extend our analysis to cover the case in which an external force `F4 = F (t)` acts on the mass, as depicted in Figure 12.19. Such a force might occur from vibrations of the support to which the top of the spring is attached or from the effect of a magnetic field on a mass made of iron. As before, we sum the forces F_1, F_2, F_3, and F_4 and set this sum equal to the resultant force F_R, obtaining

```
F1 + F2 + F3 + F4 = FR = - K U (t) - c U' (t) + F (t) = m U'' (t).
```

Therefore the forced motion of the mechanical system satisfies the nonhomogenous linear differential equation

(12.18)

```
m U'' (t) + c U' (t) + k U (t) = F (t).
```

The function `F (t)` is called the input, or driving force, and the solution `U (t)` is called the output, or

response. Of particular interest are periodic inputs F(t) that can be represented by Fourier series.

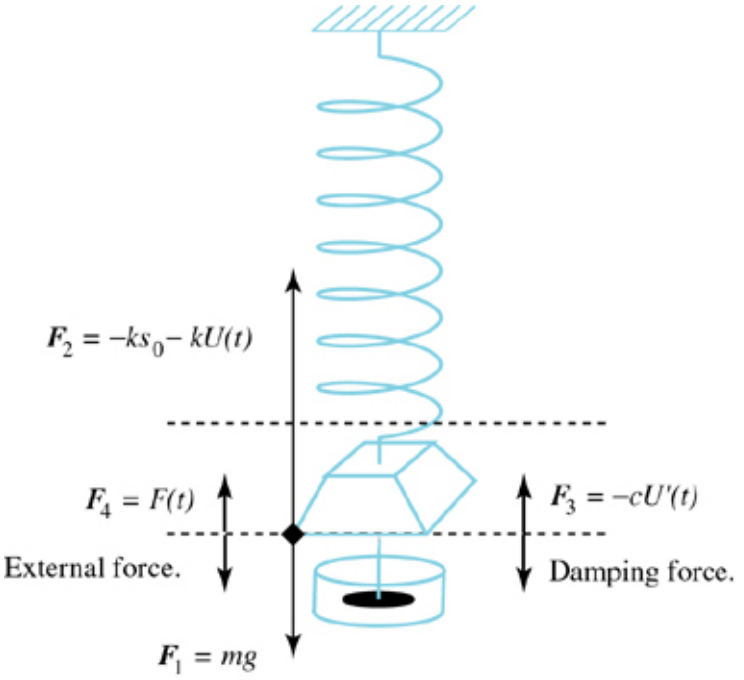

Figure 12.19. The spring-mass-dashpot system with an external force.

For damped mechanical systems that are driven by a periodic input F(t), the general solution involves a transient part that vanishes as $t \to +\infty$, and a steady state part that is periodic. We find the transient part of the solution $U_h(t)$ is found by solving the homogeneous differential equation

$$m\,U_h''(t) + c\,U_h'(t) + k\,U_h(t) = 0.$$

This homogeneous equation has the characteristic equation $m\lambda^2 + c\lambda + k = 0$, and it's roots are $\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$. The constants m, c, and k are all positive, and there are three cases to consider.

Case 1. If $c^2 - 4mk > 0$, then the roots are real and distinct, and since $\sqrt{c^2 - 4mk} < c$, it follows that the roots λ_1 and λ_2 are negative real numbers. Thus for this case we have

$$U_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}, \qquad \text{and } \lim_{t \to +\infty} U_h(t) = 0.$$

Case 2. If $c^2 - 4mk = 0$, then the roots are real and equal, $\lambda_1 = \lambda_2 = \lambda$, where λ is a negative real number, and for this case we have

$$U_h(t) = A_1 e^{\lambda t} + A_2\, t\, e^{\lambda t}, \qquad \text{and } \lim_{t \to +\infty} U_h(t) = 0.$$

Case 3. If $c^2 - 4mk < 0$, then the roots are complex conjugates, $\lambda = -\alpha \pm i\beta$, where α and β are positive real number, and it follows that

$$U_h(t) = A_1 e^{-\alpha t}\cos(\beta t) + A_2 e^{-\alpha t}\sin(\beta t), \quad \text{and}$$

$\lim_{t\to+\infty} U_h(t) = 0$.

In all three cases, the homogeneous solution $U_h(t)$ decays to 0 as $t \to +\infty$.

We obtain the steady state solution $U_p(t)$ by representing $U_p(t)$ by its Fourier series, substituting $U_p''(t)$, $U_p'(t)$, and $U_p(t)$ into the nonhomogeneous differential equation and solving the resulting system for the Fourier coefficients of $U_p(t)$. The general solution to the nonhomogeneous linear differential equation is

$$U(t) = U_h(t) + U_p(t).$$

Example 12.4 (a). Find the general solution to

$$U''(t) + 2U'(t) + U(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nt).$$

Solution 12.4 (a).

Example 12.4 (b). Find the general solution to

$$U''(t) + 2U'(t) + U(t) = \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos(nt).$$

Solution 12.4 (b).

Example 12.4 (c). Find the general solution to

$$U''(t) + 2U'(t) + U(t) = \sum_{n=1}^{\infty} \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi}\sin(nt).$$

Solution 12.4 (c).

Example 12.4 (d). Find the general solution to

$$U''(t) + 2U'(t) + U(t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nt).$$

Solution 12.4 (d).

Example 12.4 (e). Find the general solution to

$$U''(t) + 2U'(t) + U(t) = \sum_{n=1}^{\infty} \frac{1-(-1)^n}{2n^2}\cos(nt).$$

Solution 12.4 (e).

Example 12.4 (f). Find the general solution to

$$U''(t) + 2U'(t) + U(t) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1+(-1)^n}{1-n^2}\cos(nt).$$

Solution 12.4 (f).

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12.4 The Fourier Transform

In this section we develop the complex Fourier transform of the function $U(t)$, we will use ideas that were introduced in Section 12.1.

If we let $U(t)$ be a real-valued function with period 2π, which is piecewise continuous such that $U'(t)$ also exists and is piecewise continuous,

then $U(t)$ has the complex Fourier series representation

$$U(t) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

where

$$c_n = \frac{1}{\pi}\int_{-\pi}^{\pi} U(t)\, e^{-int}\, dt, \quad \text{for all } n.$$

The coefficients $\{c_n\}$ are complex numbers. Previously, we expressed $U(t)$ as the real trigonometric series

(12.19) $$U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)).$$

Hence a relationship between the coefficients is

$$a_n = c_n + c_{-n}, \quad \text{for } n = 0, 1, 2, \ldots,$$

and

$$b_n = i(c_n - c_{-n}), \quad \text{for } n = 1, 2, 3, \ldots.$$

We can easily establish these relations. We start by writing

(12.20)

$$U(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{i n t} + \sum_{n=1}^{\infty} c_{-n} e^{-i n t}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n (\cos(n t) + i \sin(n t)) + \sum_{n=1}^{\infty} c_{-n} (\cos(n t) - i \sin(n t))$$

$$= c_0 + \sum_{n=1}^{\infty} [(c_n + c_{-n}) \cos(n t) + i (c_n - c_{-n}) \sin(n t)]$$

Comparing Equations (12.20) and (12.19), we see that $a_0 = 2 c_0$, $a_n = c_n + c_{-n}$, and $b_n = i (c_n - c_{-n})$.

If $U(t)$ and $U'(t)$ are piecewise continuous and have period $2L$, then $U(t)$ has the complex Fourier series representation

(12.21) $$U(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \pi n t / L},$$

where

$$c_n = \frac{1}{2L} \int_{-L}^{L} U(t) e^{-i \pi n t / L} dt, \quad \text{for all } n.$$

We've shown how periodic functions are represented by trigonometric series, but many practical problems involve nonperiodic functions.

A representation analogous to a Fourier series for a nonperiodic function $U(t)$ is obtained by considering the Fourier series of $U(t)$ for $-L < t < L$

and then taking the limit as $L \to \infty$. The result is known as the Fourier transform of $U(t)$.

We start with the nonperiodic function $U(t)$ and consider the periodic function $U_L(t)$ with period $2L$, where

$$U_L(t) = U(t), \qquad \text{for } -L < t < L,$$

and

$$U_L(t) = U_L(t + 2L), \quad \text{for all } t.$$

Then $U_L(t)$ has the complex Fourier series representation

(12.23) $$U_L(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \pi n t / L}.$$

We need to introduce some terminology in order to discuss the terms in Equation (12.23). First

(12.24) $$\omega_n = \frac{\pi n}{L}$$

is called the frequency. If t denotes time, then the units for ω_n are radians per unit time. The set of all possible frequencies is called the frequency spectrum, that is,

$$\left\{ \ldots, -\frac{3\pi}{L}, -\frac{2\pi}{L}, -\frac{\pi}{L}, 0, \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots \right\}.$$

Note that, as L increases, the spectrum becomes finer and approaches a continuous spectrum of frequencies. It is reasonable to expect that the summation in the

Fourier series for $U_L(t)$ will give rise to an integral over $[-\infty, \infty]$. This result is stated in Theorem 12.9.

Theorem 12.9 (Fourier Transform) . Let $U(t)$ and $U'(t)$ be piecewise continuous, and

$$\int_{-\infty}^{\infty} |U(t)| \, dt < M,$$

for some positive constant M. The angular frequency Fourier transform $F(\omega)$ of $U(t)$ is defined as

(12.25) $$F(\omega) = \int_{-\infty}^{\infty} U(t) \, e^{-i\omega t} \, dt.$$

At points of continuity, $U(t)$ has the integral representation

$$U(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \, e^{i\omega t} \, d\omega.$$

At a point $t = a$ of discontinuity of $U(t)$, this integral converges to $\frac{U(a^-) + U(a^+)}{2}$.

The fact that $U(t)$ is transformed into $F(\omega)$ is commonly expressed by the operator notation

$$\mathcal{F}(U(t)) = F(\omega).$$

Proof.

Computer Software Remarks. Alternative definitions for the Fourier transform can be made in *Mathematica* by using subroutine call

```
FourierTransform[ f[t], t, ω, FourierParameters → {a, b} ],
```

where the Fourier transform with parameters **a** and **b** is computed using the formula $\sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{\infty} f(t) \, e^{ibωt} \, dt.$

Formula (12.25) in Theorem 12.9 is computed in *Mathematica* by setting **a = 1** and **b = −1** and using the option **FourierParameters → {1, −1}**.

Table 12.1 gives some important properties of the Fourier transform.

Linearity	$\mathcal{F}\left(a\,U_1(t) + b\,U_2(t)\right) = a\mathcal{F}\left(U_1(t)\right) + b\mathcal{F}\left(U_2(t)\right)$
Symmetry	If $\mathcal{F}\left(U(t)\right) = F(\omega)$, then $\mathcal{F}\left(F(t)\right) = 2\pi U(-\omega)$.
Time Scaling	$\mathcal{F}\left(U(a\,t)\right) = \frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
Time Shifting	$\mathcal{F}\left(U(t - t_0)\right) = e^{-i\,t_0\,\omega} F(\omega)$
Frequency Shifting	$\mathcal{F}\left(e^{i\,\omega_0\,t} U(t)\right) = F(\omega - \omega_0)$
Time Differentiation	$\mathcal{F}\left(U'(t)\right) = i\,\omega\,F(\omega)$
Frequency Differentiation	$\frac{d^n}{d\omega^n} F(\omega) = \mathcal{F}\left((-i\,t)^n U(t)\right)$
Moment Theorem	If $M_n = \int_{-\infty}^{\infty} t^n U(t)\,dt$, then $(-i)^n M_n = F^{(n)}(0)$.

Table 12.1 Properties of the Fourier transform.

Example 12.5. Show that $\mathcal{F}\left(e^{-|t|}\right) = \frac{2}{1+\omega^2}$.

Solution.

Using formula (12.25), we obtain

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} e^{-|t|}\, e^{-i\,\omega\,t}\,dt \\
&= \int_{-\infty}^{0} e^{-|t|}\, e^{-i\,\omega\,t}\,dt + \int_{0}^{\infty} e^{-|t|}\, e^{-i\,\omega\,t}\,dt \\
&= \int_{-\infty}^{0} e^{(1-i\,\omega)\,t}\,dt + \int_{0}^{\infty} e^{(-1-i\,\omega)\,t}\,dt \\
&= \frac{1}{1-i\,\omega}\, e^{(1-i\,\omega)\,t}\Big|_{t=-\infty}^{t=0} + \frac{-1}{1+i\,\omega}\, e^{(-1-i\,\omega)\,t}\Big|_{t=0}^{t=\infty} \\
&= \frac{1}{1-i\,\omega} + \frac{1}{1+i\,\omega} \\
&= \frac{2}{1+\omega^2}
\end{aligned}$$

establishing the result.

Explore Solution 12.5.

Example 12.6. Show that $\mathcal{F}\left(\frac{1}{1+t^2}\right) = \pi\, e^{-|\omega|}$.

Solution.

Using the result of Example 12.5 and the symmetry property, we obtain

$$\mathcal{F}\left(\frac{2}{1+t^2}\right) = 2\pi\, e^{-|-\omega|}$$

We use the linearity property and multiply each term by $\frac{1}{2}$ and get

$$\mathcal{F}\left(\frac{1}{2}\,\frac{2}{1+t^2}\right) = \left(\frac{1}{2}\right) 2\,\pi\, e^{-|-\omega|}.$$

Then rewrite this in the form

$$\mathcal{F}\left(\frac{1}{1+t^2}\right) = \pi\, e^{-|\omega|}$$

establishing the result.

Explore Solution 12.6.

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12.5 The Laplace Transform

In this section we investigate the Laplace transform, which is a very powerful tool for engineering applications. It's discovery is attributed to the French mathematician Pierre-Simon Laplace (1749-1827). The background we introduced in Section 12.4 regarding the Fourier transform in important for our approach to the theory of the Laplace transform.

12.5.1. From the Fourier Transform to the Laplace Transform

We have shown that certain real-valued functions $f(t)$ have a Fourier transform and that the integral

$$g(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt.$$

defines the complex function $g(\omega)$ of the real variable ω. If we multiply the integrand $f(t)\, e^{-i\omega t}$ by $e^{-\sigma t}$, then we create a complex function $G(\sigma + i\omega)$ of the complex variable $\sigma + i\omega$:

$$g(\sigma + i\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-\sigma t} e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} f(t)\, e^{-(\sigma + i\omega) t}\, dt.$$

The function $G(\sigma + i\omega)$ is called the two-sided Laplace transform of $f(t)$, (or bilateral Laplace transform of $f(t)$), and it exists when the Fourier transform of the function $f(t)\, e^{-\sigma t}$ exists. From Fourier transform theory, a sufficient condition for $G(\sigma + i\omega)$ to exist is that

$$\int_{-\infty}^{\infty} |f(t)|\, e^{-\sigma t}\, dt < \infty.$$

For a function $f(t)$, this integral is finite for values of σ that lie in some interval $a < \sigma < b$.

The two-sided Laplace transform has the lower limit of integration $t = -\infty$ and hence requires a knowledge of the past history of the function $f(t)$ (i.e., when $t < 0$). For most physical applications, we are interested in the behavior of a system only for $0 \le t$. The initial conditions $f(0), f'(0), f''(0), \ldots$ are a consequence of the past history of the system and are often all that we know. For this reason, it is useful to define the one-sided Laplace transform of $f(t)$,

which is commonly referred to simply as the Laplace transform of $f(t)$, which is also defined as an integral:

(12-28) $$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt,$$

where $s = \sigma + i\omega$. If the integral in Equation (12-28) for the Laplace transform exists for $s_0 = \sigma_0 + i\omega$, then values of σ with $\sigma > \sigma_0$ imply that $e^{-\sigma t} < e^{-\sigma_0 t}$ and so that

$$\int_{-\infty}^{\infty} |f(t)|\, e^{-\sigma t}\, dt < \int_{-\infty}^{\infty} |f(t)|\, e^{-\sigma_0 t}\, dt < \infty,$$

from which it follows that $F(s)$ exists for $s = \sigma + i\omega$. Therefore the Laplace transform $\mathcal{L}(f(t))$ is defined for all points s in the right half-plane $\mathrm{Re}(s) > \sigma_0$.

Another way to view the relationship between the Fourier transform and the Laplace transform is to consider the function $U(t)$ given by

$$U(t) = \begin{cases} f(t), & \text{for } 0 \le t; \\ 0, & \text{for } t < 0. \end{cases}$$

Then the Fourier transform theory, in Section 12.4, shows that

$$U(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} U(t)\, e^{-i\omega t}\, dt \right] e^{i\omega t}\, d\omega,$$

and, because the integrand $U(t)$ is zero for $t < 0$, we can write this equation as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_0^\infty f(t)\, e^{-i\omega t}\, dt \right] e^{i\omega t}\, d\omega.$$

Now use the change of variable $s = \sigma + i\omega$ and hold $\sigma > \sigma_0$ fixed. We have $ds = 0 + i\, d\omega$ and $d\omega = \frac{ds}{i}$. Then the new limits of integration are from $s = \sigma - i\omega$ to $s = \sigma + i\omega$. The resulting equation is

$$f(t) = \frac{1}{2\pi} \int_{\sigma - i\omega}^{\sigma + i\omega} \left[\int_0^\infty f(t)\, e^{-st}\, dt \right] e^{st}\, ds.$$

Definition (Laplace Transform). Therefore the Laplace transform is as the integral:

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt,$$

where $s = \sigma + i\omega$, and the inverse Laplace transform is given by:

(12.29) $$\mathcal{L}^{-1}(F(s)) = f(t) = \frac{1}{2\pi} \int_{\sigma - i\omega}^{\sigma + i\omega} F(s)\, e^{st}\, ds.$$

12.5.2 Properties of the Laplace Transform

Although a function $f(t)$ may be defined for all values of t, it's Laplace transform is not influenced by values of $f(t)$, when $t < 0$. The Laplace transform of $f(t)$ is actually defined for the function $U(t)$ given in the last section by

$$U(t) = \begin{cases} f(t) & \text{for } t \ge 0, \\ 0 & \text{for } t < 0. \end{cases}$$

A sufficient condition for the existence of the Laplace transform is that $|f(t)|$ does not grow too rapidly as $t \to +\infty$. We say that the function $f(t)$ is of exponential order if there exists real constants $M > 0$ and K such that

$$|f(t)| \le M e^{Kt} \text{ holds for all } t \ge 0.$$

All functions in this chapter are assumed to be of exponential order. Theorem 12.10 shows that the Laplace transform $F(s) = F(\sigma + i\tau)$ exists for values of $s = \sigma + i\tau$ in a domain that includes the right half-plane $\text{Re}(s) > K$.

Theorem 12.10 (Laplace Transform). If $f(t)$ is of exponential order, then its Laplace Transform $\mathcal{L}(f(t)) = F(s)$ exists and is given by

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt,$$

where $s = \sigma + i\tau$. The defining integral for $F(s)$ exists at points $s = \sigma + i\tau$ in the right half plane $\sigma > K$.

Proof.

Remark 12.1. The domain of definition of the defining integral for the Laplace transform $\mathcal{L}(f(t))$ seems to be restricted to a half plane. However, the resulting formula $F(s)$ might have a domain much larger than this half plane. Later we will show that $F(s)$ is an analytic function of the complex variable s. For most applications involving Laplace transforms that we present, the Laplace transforms are rational functions that take the form $\frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials; in other important applications, the functions take the form $\frac{e^{as}P(s)}{Q(s)}$.

Theorem 12.11 (Linearity of the Laplace Transform). Let $f(t)$ and $g(t)$ have Laplace transforms $F(s)$ and $G(s)$, respectively. If a and b are constants, then

$$\mathcal{L}(a f(t) + b g(t)) = a F(s) + b G(s).$$

Proof.

Theorem 12.12 (Uniqueness of the Laplace Transform). Let $f(t)$ and $g(t)$ have Laplace transforms $F(s)$ and $G(s)$, respectively.

$$\text{If } F(s) \equiv G(s), \text{ then } f(t) \equiv g(t).$$

Proof.

Table 12.2 gives the Laplace transforms of some well-known functions, and Table 12.3 highlights some important properties of Laplace transforms. (see below)

Example 12.7. Show that the Laplace transform of the step function given by

$$f(t) = \begin{cases} 1, & \text{for } 0 \le t < c; \\ 0, & \text{for } c < t. \end{cases}$$

$$\text{is } \mathcal{L}(f(t)) = F(s) = \frac{1 - e^{-cs}}{s}.$$

Solution.

Using the integral definition for $\mathcal{L}(f(t))$, we obtain

$$\mathcal{L}(f(t)) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^c 1 * e^{-st}\, dt + \int_c^\infty 0 * e^{-st}\, dt$$

$$= \left(-\frac{e^{-st}}{s}\right)\Bigg|_{t=0}^{t=c} = -\frac{e^{-cs}}{s} - \left(-\frac{1}{s}\right)$$

$$= \frac{1 - e^{-cs}}{s}$$

Explore Solution 12.7.

Example 12.8. Show that $\mathcal{L}(e^{at}) = \frac{1}{s-a}$, where a is a real constant.

Solution.

We actually show that the integral defining $\mathcal{L}(e^{at})$ equals the formula $F(s) = \frac{1}{s-a}$ for values of s with $\mathrm{Re}(s) > a$ and that the extension to other values of s is inferred by our knowledge about the domain of a rational function. Using straightforward integration techniques gives

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt$$

$$= \lim_{R\to\infty} \int_0^R e^{(a-s)t}\, dt$$

$$= \lim_{R\to\infty} \frac{e^{(a-s)R}}{a-s} - \frac{e^{(a-s)0}}{a-s}$$

$$= -\frac{1}{a-s} + \frac{1}{a-s} \lim_{R\to\infty} e^{(a-s)R}$$

Let $s = \sigma + i\tau$ be fixed, or where $\sigma > a$. Then, as $a - \sigma$ is a negative real number, we have $\lim_{R\to\infty} \left| e^{(a-s)R} \right| = \lim_{R\to\infty} e^{(a-\sigma)R} = 0$, which implies that $\lim_{R\to\infty} e^{(a-s)R} = 0$, and use this expression in the preceding equation to obtain $\mathcal{L}(e^{at}) = \frac{1}{s-a}$.

Explore Solution 12.8.

We can use the property of linearity to find new Laplace transforms from known transforms.

Example 12.9. Show that $\mathcal{L}(\sinh(at)) = \frac{a}{s^2 - a^2}$.

Solution.

Because $\sinh(at)$ can be written as the linear combination $\sinh(at) = \frac{1}{2} e^{at} - \frac{1}{2} e^{-at}$, we obtain

$$\mathcal{L}(\sinh(at)) = \mathcal{L}\left(\frac{1}{2}e^{at} - \frac{1}{2}e^{-at}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a}$$

$$= \frac{a}{s^2 - a^2}$$

Explore Solution 12.9.

Integration by parts is also helpful in finding new Laplace transforms.

Example 12.10. Show that $\mathcal{L}(t) = \frac{1}{s^2}$.

Solution.

Integration by parts yields

$$\mathcal{L}(t) = \int_0^\infty t\, e^{-st}\, dt$$

$$= \lim_{R\to\infty} \int_0^R t\, e^{-st}\, dt$$

$$= \lim_{R\to\infty} \left(\frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Big|_{t=0}^{t=R}$$

$$= \lim_{R\to\infty} \left(\frac{-R}{s}e^{-sR} - \frac{1}{s^2}e^{-sR}\right) + 0 + \frac{1}{s^2}$$

$$= (-0 - 0) + 0 + \frac{1}{s^2} = \frac{1}{s^2}$$

For values of s in the right half-plane $\mathrm{Re}(s) > 0$, an argument similar to that in Example 12.8 shows that the limit approaches zero, establishing the result.

Explore Solution 12.10.

Extra Example 1. Show that $\mathcal{L}(t^3) = \frac{3!}{s^4}$.

Explore Solution for Extra Example 1.

Example 12.11. Show that $\mathcal{L}(\cos(bt)) = \frac{s}{s^2 + b^2}$.

Solution.

A direct approach using the definition is tedious. Instead, let's assume that the complex constants $\pm ib$ are permitted and hence that the following Laplace transforms exist:

$$\mathcal{L}(e^{ibt}) = \frac{1}{s - ib} \text{ and } \mathcal{L}(e^{-ibt}) = \frac{1}{s + ib}.$$

Recall that $\cos(bt)$ can be written as the linear combination $\cos(bt) = \frac{1}{2}e^{ibt} + \frac{1}{2}e^{-ibt}$. Using the linearity of the Laplace transform, we have

$$\mathcal{L}(\cos(bt)) = \mathcal{L}\left(\frac{1}{2}e^{ibt} + \frac{1}{2}e^{-ibt}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{ibt}) + \frac{1}{2}\mathcal{L}(e^{-ibt})$$

$$= \frac{1}{2}\frac{1}{s-ib} + \frac{1}{2}\frac{1}{s+ib}$$

$$= \frac{s}{s^2+b^2}$$

Explore Solution 12.11.

Example 12.12. Find the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{3s+6}{s^2+9}\right)$.

Solution.

Using linearity and lines 6 and 7 of Table 11.2, we obtain

$$\mathcal{L}^{-1}\left(\frac{3s+6}{s^2+9}\right) = \mathcal{L}^{-1}\left(\frac{3s}{s^2+9} + \frac{6}{s^2+9}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{3s}{s^2+9}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^2+9}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{3s}{s^2+9}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^2+9}\right)$$

$$= 3\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + 2\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)$$

$$= 3\mathcal{L}^{-1}\left(\frac{s}{s^2+3^2}\right) + 2\mathcal{L}^{-1}\left(\frac{3}{s^2+3^2}\right)$$

$$= 3\cos(3t) + 2\sin(3t)$$

Explore Solution 12.12.

Inverting the Laplace transform is usually accomplished with the aid of a table of known Laplace transforms and the technique of partial fraction expansion. Table12.2 gives the Laplace transforms of some well-known functions, and Table 12.3 highlights some important properties of Laplace transforms.

	f(t)	$F(s) = \int_0^\infty f(t)e^{-st}dt$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	t^3	$\frac{3!}{s^4}$
5	t^n	$\frac{n!}{s^{n+1}}$
6	$U_0(t)$	$\frac{1}{s}$
7	$U_c(t)$	$\frac{e^{-cs}}{s}$
8	e^{at}	$\frac{1}{s-a}$
9	te^{at}	$\frac{1}{(s-a)^2}$

Table 12.2 Table of Laplace Transforms.

Definition	$\mathcal{L}(f(t))$
First Derivative	$\mathcal{L}(f'(t)) = s F(s) - f(0)$
Second Derivative	$\mathcal{L}(f''(t)) = s^2 F(s) - s f(0) - f'(0)$
Integral	$\mathcal{L}\left(\int_0^t f(\tau)\, d\tau\right) = \frac{F(s)}{s}$
Multiplication by t	$\mathcal{L}(t f(t)) = -F'(s)$
Division by t	$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\, d\sigma$
s - axis shifting	$\mathcal{L}(e^{at} f(t)) = F(s-a)$
t - axis shifting	$\mathcal{L}(U_a(t) f(t-a)) = e^{-as} F(s)$
Convolution	$\mathcal{L}(h(t)) = F(s) G(s)$, where $h(t) = (f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

Table 12.3 Properties of Laplace Transforms.

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12.6 Laplace Transforms of Derivatives and Integrals

This section is a continuation of our development of the Laplace Transform in Section 12.5.

Theorem 12.13 (Differentiation of f(t)). Let $f(t)$ and $f'(t)$ be continuous for $t \geq 0$, and of exponential order. Then

$$\mathcal{L}(f'(t)) = s\,F(s) - f(0),$$

where $F(s) = \mathcal{L}(f(t))$.

Proof.

Corollary 12.1 (Differentiation of f(t)). If $f(t)$, $f'(t)$ and $f''(t)$ are of exponential order, then

$$\mathcal{L}(f''(t)) = s^2 F(s) - s\,f(0) - f'(0).$$

Proof.

Example 12.13. Show that $\mathcal{L}(\cos^2(t)) = \frac{s^2+2}{s(s^2+4)}$.

Solution.

If we let $f(t) = \cos^2(t)$, then $f(0) = 1$ and $f'(t) = -2\sin(t)\cos(t) = -\sin(2t)$, and $F(s) = \mathcal{L}(\cos^2(t))$ is to be determined. Because $\mathcal{L}(\sin(2t)) = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$, we have $\mathcal{L}(-\sin(2t)) = -\frac{2}{s^2+4}$, and Theorem 12.13 implies that

$$-\frac{2}{s^2+4} = \mathcal{L}(-\sin(2t))$$

$$= s\,F(s) - f(0)$$

$$= s\,\mathcal{L}(\cos^2(t)) - 1$$

Thus, we have $s\,\mathcal{L}(\cos^2(t)) - 1 = -\frac{2}{s^2+4}$, and $s\,\mathcal{L}(\cos^2(t)) = -\frac{2}{s^2+4} + 1$ from which it follows that

$$\mathcal{L}(\cos^2(t)) = -\frac{2}{s(s^2+4)} + \frac{1}{s} = \frac{s^2+2}{s(s^2+4)}.$$

Explore Solution 12.13.

Theorem 12.14 (Integration of f(t)). Let $f(t)$ be continuous for $0 \le t$, and of exponential order and $\mathcal{L}(f(t)) = F(s)$, then

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s}.$$

Proof.

Example 12.14. (a) Show that $\mathcal{L}(t^2) = \frac{2!}{s^3}$, **(b)** Show that $\mathcal{L}(t^3) = \frac{3!}{s^4}$.

Solution.

Part (a). Use the fact that $\mathcal{L}(t) = \frac{1}{s^2}$, and apply Theorem 12.14, with $f(t) = 2t$ and $\mathcal{L}(f(t)) = F(s) = \frac{2}{s^2}$,

then obtain

$$\mathcal{L}(t^2) = \mathcal{L}\left(\int_0^t 2\tau\,d\tau\right) = \frac{1}{s}F(s) = \frac{1}{s}\frac{2}{s^2} = \frac{2!}{s^3}.$$

Part (b). Now we can use this first result as a fact, $\mathcal{L}(t^2) = \frac{2!}{s^3}$. This time we apply Theorem 12.14, with $f(t) = 3t^2$ and $\mathcal{L}(3t^2) = F(s) = \frac{6}{s^3}$,

and obtain

$$\mathcal{L}(t^2) = \mathcal{L}\left(\int_0^t 3\tau^2\,d\tau\right) = \frac{1}{s}F(s) = \frac{1}{s}\frac{6}{s^3} = \frac{3!}{s^4}.$$

Explore Solution 12.14 (a). Find $\mathcal{L}\left(t^2\right)$.

Explore Solution 12.14 (b). Find $\mathcal{L}\left(t^3\right)$.

One of the main uses of the Laplace transform is its role in the solution of differential equations. The utility of the Laplace transform lies in the fact that the transform of the derivative $f'(t)$ corresponds to multiplication of the transform $F(s)$ by s and then the subtraction of $f(0)$. This permits us to replace the calculus operation of differentiation with simple algebraic operations on transforms.

This idea is used to develop a method for solving linear differential equations with constant coefficients. Let's consider the initial value problem

$$y''(t) + a\,y'(t) + b\,y(t) = f(t),$$

with initial conditions $y(0) = y_0$ and $y'(0) = d_0$. We can use the linearity property of the Laplace transform to obtain

$$\mathcal{L}(y''(t)) + a\,\mathcal{L}(y'(t)) + b\,\mathcal{L}(y(t)) = \mathcal{L}(f(t)).$$

If we let $Y(s) = \mathcal{L}(y(t))$ and $F(s) = \mathcal{L}(f(t))$ and apply Theorem 12.13 and Corollary 12.1 then we have

$$(s^2 Y(s) - s\,y(0) - y'(0)) + a\,(s\,Y(s) - y(0)) + b\,Y(s) = F(s).$$

this in turn can be rearranged to obtain the form

(12-30) $$s^2 Y(s) + a\,s\,Y(s) + b\,Y(s) = F(s) + s\,y(0) + y'(0) + a\,y(0).$$

The Laplace transform $Y(s)$ of the solution $y(t)$ is easily found to be

(12-31) $$Y(s) = \frac{F(s) + s\,y(0) + y'(0) + a\,y(0)}{s^2 + a\,s + b}.$$

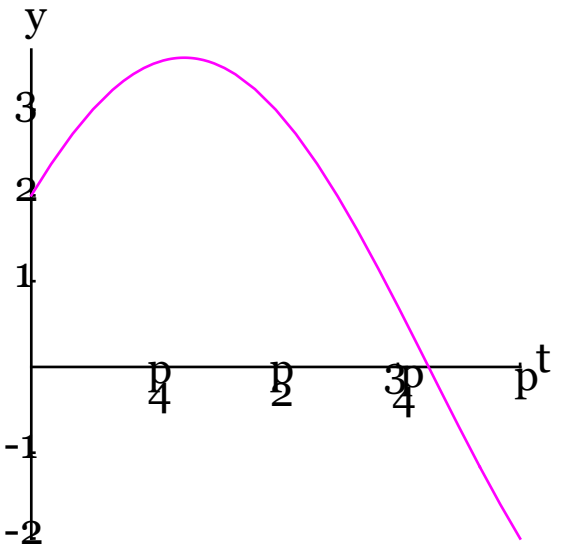
For many physical problems involving mechanical systems and electrical circuits, the transform $F(s)$ is known, and the inverse of $Y(s)$ can easily be computed. This process is referred to as operational calculus and has the advantage of changing problems in differential equations into problems in algebra. Then the solution obtained will satisfy the specific initial conditions.

Example 12.15. Solve the initial value problem

$$y''(t) + y(t) = 0$$

with

$$y(0) = 2, \text{ and } y'(0) = 3$$



A graph of the solution.

Solution.

The right side of the differential equation is $f(t) = 0$, so we have $F(s) = 0$. The initial conditions yield $\mathcal{L}(y''(t)) = s^2 Y(s) - 2\,s - 3$ and Equation (12-30)

becomes $s^2 Y(s) + 0 * s * Y(s) + Y(s) = F(s) + s * 2 + 3 + 0 * 2$ which simplifies, and we get

$$s^2 Y(s) + Y(s) = 2\,s + 3.$$

Solving we get $Y(s) = \frac{2s+3}{s^2+1}$.

We then solve $y(t)$ with the help of Table 12.2 to compute

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{2s+3}{s^2+1}\right) \\ &= \mathcal{L}^{-1}\left(\frac{2s}{s^2+1} + \frac{3}{s^2+1}\right) \\ &= \mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{3}{s^2+1}\right) \\ &= 2\,\mathcal{L}^{-1}\left(\frac{s}{s^2+1^2}\right) + 3\,\mathcal{L}^{-1}\left(\frac{1}{s^2+1^2}\right) \\ &= 2\cos(t) + 3\sin(t) \end{aligned}$$

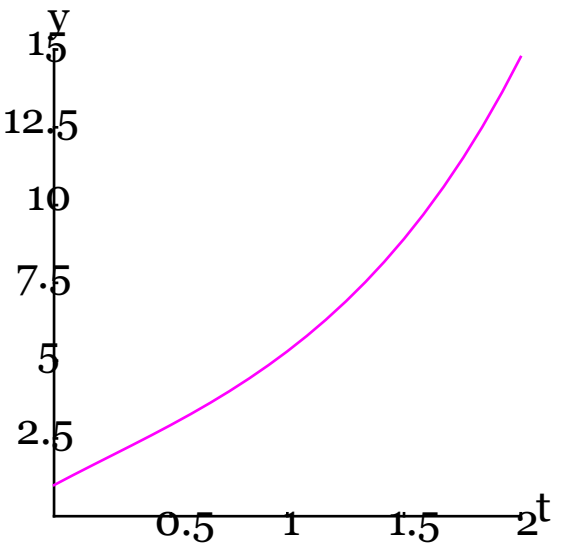
Explore Solution 12.15.

Example 12.16. Solve the initial value problem

$$y''(t) + y'(t) - 2y(t) = 0$$

with

$$y(0) = 1, \text{ and } y'(0) = 4$$



A graph of the solution.

Solution.

As in Example 12.15, the right side of the differential equation is $f(t) = 0$, so we have $F(s) = 0$. The initial conditions yield $\mathcal{L}(y''(t)) = s^2 Y(s) - 1 * s - 4$ and $\mathcal{L}(y'(t)) = s\,Y(s) - 1$, and Equation (12-30) becomes $(s^2 Y(s) - s - 4) + (s\,Y(s) - 1) - 2\,Y(s) = 0$, which can be rewritten as $s^2 Y(s) + s\,Y(s) - 2\,Y(s) = s + 4 + 1$ which simplifies, and we get $Y(s)(s^2 + s - 2) = s + 5$.

This time we use Equation (12-31) and obtain $Y(s) = \frac{F(s)+s\,y(0)+y'(0)+a\,y(0)}{s^2+a\,s+b} = \frac{0+s*1+4+1*1}{s^2+1*s-2}$, which simplifies, and we get $Y(s) = \frac{s+5}{s^2+s-2} = \frac{s+5}{(s-1)(s+2)}$. Now use the partial fraction expansion $Y(s) = \frac{s+5}{(s-1)(s+2)} = \frac{2}{s-1} - \frac{1}{s+2}$ to get the solution

$$y(t) = \mathcal{L}^{-1}\left(\frac{s+5}{(s-1)(s+2)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{2}{s-1} - \frac{1}{s+2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{2}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$$

$$= 2\,\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-(-2)}\right)$$

$$= 2\,e^{t} - e^{-2t}$$

Explore Solution 12.16.

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12.7 Laplace Transform Shifting Theorems and the Step Function

This section is a continuation of our development of the Laplace Transforms in Section 12.5 and Section 12.6.

We have shown how to use the Laplace transform to solve linear differential equations. Familiar functions that arise in solutions to differential equations are $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$. Theorem 12.15 (the first shifting theorem) shows how their transforms are related to those of $\cos(bt)$ and $\sin(bt)$ by shifting the variable s in $F(s)$. A companion result, called the second shifting theorem, Theorem 12.16, shows how the transform of $f(t-a)$ can be obtained by multiplying $F(s)$ by e^{-as}. Loosely speaking, these results show that multiplication of $f(t)$ by e^{at} corresponds to shifting $F(s-a)$, and that shifting $f(t-a)$ corresponds to multiplication of the transform $F(s)$ by e^{-as}.

Theorem 12.15 (Shifting the Variable s). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Proof.

Definition 12.3 (The Unit Step Function). Let $a \geq 0$. Then, the unit step function $U_a(t)$ is

$$U_a(t) = \begin{cases} 0, & \text{for } t < a, \\ 1, & \text{for } t > a. \end{cases}$$

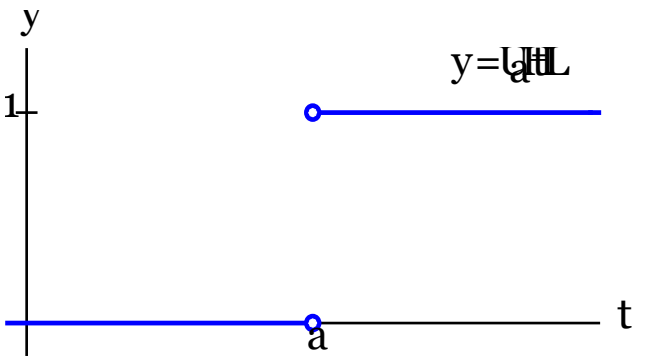

Figure 12.22. The graph of the unit step function $y = U_a(t)$.

Theorem 12.16 (Shifting the Variable t). If $F(s)$ is the Laplace transform of $f(t)$, and $a \geq 0$, then

$$\mathcal{L}(U_a(t)\, f(t-a)) = e^{-as} F(s),$$

where $f(t)$ and $U_a(t)\, f(t-a)$ are illustrated in Figure 12.23.

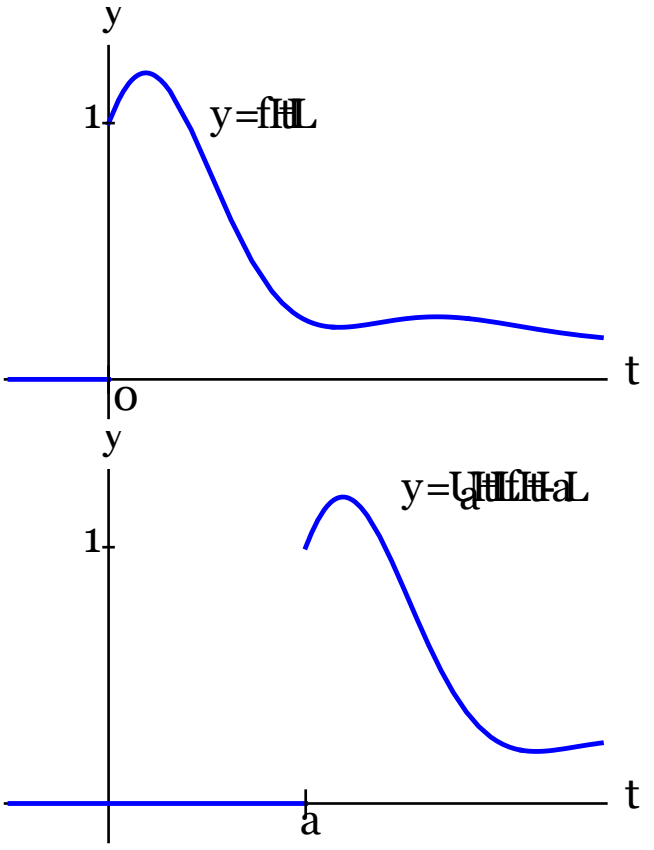

Figure 12.23. Comparison of the functions $y = f(t)$ and $y = U_a(t)\, f(t-a)$.

Proof.

Example 12.17. Show that $\mathcal{L}\left(t^n e^{at}\right) = \frac{n!}{(s-a)^{n+1}}$.

Solution.

If we let $f(t) = t^n$, then $F(s) = \mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$, and if we apply Theorem 12.15, we obtain the desired result:

$$\mathcal{L}\left(t^n e^{at}\right) = F(s-a) = \frac{n!}{(s-a)^{n+1}}.$$

Explore Solution 12.17.

Example 12.18. Show that $\mathcal{L}(U_c(t)) = \frac{e^{-cs}}{s}$.

Solution.

If set $f(t) = 1$, and then set $F(s) = \frac{1}{s}$. We apply Theorem 12.16 to get

$$\begin{aligned}\mathcal{L}(U_c(t)) &= \mathcal{L}(U_c(t) * 1)\\ &= \mathcal{L}(U_c(t) f(t))\\ &= e^{-cs} F(s)\\ &= e^{-cs}\frac{1}{s}\\ &= \frac{e^{-cs}}{s}\end{aligned}$$

Explore Solution 12.18.

Extra Example 1. Use Theorem 12.16 and find $\mathcal{L}(U_c(t)\cos(t-c))$.

Explore Solution for Extra Example 1.

Example 12.19. Find $\mathcal{L}(f(t))$ if $f(t)$ is as given in Figure 12.24.

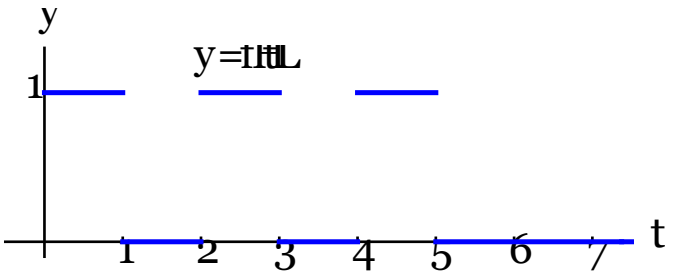

Figure 12.24. The function $y = f(t)$.

Solution.

We represent $f(t)$ in terms of step functions
$f(t) = 1 - U_1(t) + U_2(t) - U_3(t) + U_4(t) - U_5(t)$. Using the result of Example 12.18 and linearity, we obtain

$$\mathcal{L}(f(t)) = \mathcal{L}(1 - U_1(t) + U_2(t) - U_3(t) + U_4(t) - U_5(t))$$

$$= \mathcal{L}(1) - \mathcal{L}(U_1(t)) + \mathcal{L}(U_2(t)) - \mathcal{L}(U_3(t)) + \mathcal{L}(U_4(t)) - \mathcal{L}(U_5(t))$$

$$= \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-5s}}{s}$$

Explore Solution 12.19.

Example 12.20. Solve the initial value problem

$$y''(t) + y(t) = U_{\pi}(t)$$

with

$$y(0) = 0, \text{ and } y'(0) = 0$$

A graph of the solution.

Solution.

As usual, we let $Y(s)$ denote the Laplace transform of $y(t)$. The right hand side of the D.E. is $f(t) = U_{\pi}(t)$ and $F(s) = \frac{e^{-\pi s}}{s}$.

Taking Laplace transforms we write $s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = F(s)$. Using the initial conditions $y(0) = 0$, $y'(0) = 0$ and $F(s) = \frac{e^{-\pi s}}{s}$ we get

$$s^2 Y(s) + Y(s) = \frac{e^{-\pi s}}{s}.$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{e^{-\pi s}}{s\left(s^2 + 1\right)}.$$

Use the facts that $\mathcal{L}(1) = \frac{1}{s}$, $\mathcal{L}(\cos(t)) = \frac{s}{s^2+1}$ and get $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$, $\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos(t)$, respectively. Then we will apply Theorem 12.16. We compute the solution, $y(t)$, as

$$y(t) = \mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s\left(s^2+1\right)}\right) = \mathcal{L}^{-1}\left(e^{-\pi s}\,\frac{1}{s\left(s^2+1\right)}\right)$$

$$= \mathcal{L}^{-1}\left(e^{-\pi s}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)\right)$$

$$= \mathcal{L}^{-1}\left(e^{-\pi s}\,\frac{1}{s}-e^{-\pi s}\,\frac{s}{s^2+1}\right)$$

$$= \mathcal{L}^{-1}\left(e^{-\pi s}\,\frac{1}{s}\right)-\mathcal{L}^{-1}\left(e^{-\pi s}\,\frac{s}{s^2+1}\right)$$

$$= U_\pi(t)\,\mathcal{L}^{-1}\left(\frac{1}{s}\right)-U_\pi(t)\,\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= U_\pi(t)-U_\pi(t)\cos(t-\pi)$$

$$= U_\pi(t)+U_\pi(t)\cos(t-\pi)$$

Then using the trigonometric identity $\cos(t-\pi) = -\cos(t)$ we can write this in a more familiar form

$$y(t) = \begin{cases} 0, & \text{for } t \le \pi; \\ 1+\cos(t), & \text{for } t > \pi. \end{cases}$$

Explore Solution 12.20.

Complex Analysis for Mathematics and Engineering

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12.8 Laplace Transforms: Multiplication and Division by t

This section is a continuation of our development of the Laplace Transforms in Sections 12.5-7.

Sometimes the solutions to nonhomogeneous linear differential equations with constant coefficients involve the functions $t\cos(bt)$, $t\sin(bt)$, or $t^n e^{at}$ as part of the solution. We now show how the Laplace transforms of $tf(t)$ and $\frac{f(t)}{t}$ are related to the Laplace transform of $f(t)$. The transform of $tf(t)$ will be obtained via differentiation and the transform of $\frac{f(t)}{t}$ will be obtained via integration. To be precise, we present Theorems 12.17 and 12.18.

Theorem 12.17 (Multiplication by t). If $F(s)$ is the Laplace transform of $f(t)$, then

$$\mathcal{L}(tf(t)) = -F'(s).$$

Proof.

Theorem 12.18 (Division by t). Let both $f(t)$ and $\frac{f(t)}{t}$ have Laplace transforms and let $F(s)$ denote the Laplace transform of $f(t)$. If $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists then

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\sigma)\,d\sigma.$$

Proof.

Example 12.21. Show that $\mathcal{L}(t\cos(bt)) = \frac{s^2-b^2}{(b^2+s^2)^2}$.

Solution.

If we let $f(t) = \cos(bt)$, then $F(s) = \mathcal{L}(\cos(bt)) = \frac{s}{s^2+b^2}$. Hence we can differentiate $F(s)$ to obtain the desired result:

$$\mathcal{L}(t\,\mathrm{Cos}(b\,t)) = -F'(s)$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)$$

$$= -\frac{1\left(s^2+b^2\right)-(2\,s+0)\,s}{\left(b^2+s^2\right)^2}$$

$$= \frac{s^2-b^2}{\left(b^2+s^2\right)^2}$$

Explore Solution 12.21.

Example 12.22. Show that $\mathcal{L}\left(\frac{\sin(t)}{t}\right) = \mathrm{Arctan}\left(\frac{1}{s}\right)$.

Solution.

We let $f(t) = \sin(t)$, then $F(s) = \mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$. Because $\lim_{t\to 0}\frac{\sin(t)}{t} = 1$, we can integrate $F(s)$ to obtain the desired result:

$$\mathcal{L}\left(\frac{\sin(t)}{t}\right) = \int_s^\infty \frac{1}{\sigma^2+1}\,d\sigma$$

$$= -\mathrm{Arctan}\left(\frac{1}{\sigma}\right)\Big|_{\sigma=s}^{\sigma=\infty}$$

$$= -\mathrm{Arctan}(0) + \mathrm{Arctan}\left(\frac{1}{s}\right)$$

$$= \mathrm{Arctan}\left(\frac{1}{s}\right)$$

Explore Solution 12.22.

Some types of differential equations involve the terms $t\,y'(t)$ or $t\,y''(t)$. We can use Laplace transforms to find the solution if we use the additional substitutions

(12-32) $$\mathcal{L}(t\,y'(t)) = -s\,Y'(s) - Y(s),$$

and

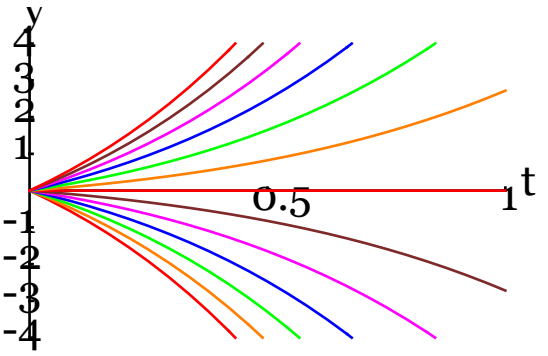
(12-33) $$\mathcal{L}(t\,y''(t)) = -s^2\,Y'(s) - 2\,s\,Y(s) + y(0).$$

Example 12.23. Use Laplace transforms to solve the initial value problem

$$t\,y''(t) - t\,y'(t) - y(t) = 0$$

with

$$y(0) = 0$$



Some of the functions in the family of solutions.

Solution.

If we let $Y(s)$ denote the Laplace transform of $y(t)$ and substitute Equations (12-32) and (12-33) into the preceding equation, we get

$-s^2 Y'(s) - 2sY(s) + 0 + sY'(s) + Y(s) - Y(s) = 0$, which can be simplifies as
$-s^2 Y'(s) - 2sY(s) + sY'(s) = 0$ and then rewritten in the form

(12-34) $$(s-1)Y'(s) + 2Y(s) = 0$$

Equation (12-34) involves $Y'(s)$ and can be written as a first-order linear differential equation

(12-35) $$Y'(s) + \frac{2}{s-1} Y(s) = 0.$$

The integrating factor for the differential equation is

$$\rho = \text{Exp}\left(\int \frac{2}{s-1}\, ds\right) = e^{2\ln(s-1)} = e^{\ln((s-1)^2)} = (s-1)^2.$$

Multiplying Equation (12-35) by produces

$$(s-1)^2 Y'(s) + 2(s-1)Y(s) = 0,$$

which in turn can be written as

$$\frac{d}{ds}\left[(s-1)^2 Y(s)\right] = 0.$$

Now integrate both sides and obtain $\int \frac{d}{ds}\left[(s-1)^2 Y(s)\right] ds = \int 0\, ds$ which yields

$$(s-1)^2 Y(s) = C,$$

where C is the constant of integration. Hence the solution to Equation (12-35) is

$$Y(s) = \frac{C}{(s-1)^2}.$$

Thus $y(t) = \mathcal{L}^{-1}(Y(s)) = C\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)$ in this equation is the desired solution:

$$y(t) = Cte^t.$$

Remark. For this differential equation there is a family of solutions.

Explore Solution 12.23.

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12.9 Inverting the Laplace Transform

This section is a continuation of our development of the Laplace Transforms in Sections 12.5-8.

So far, most of the applications involving the Laplace transform involves a transform (or part of a transform) that is expressed by

(12.36) $$Y(s) = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are polynomials that have no common factors. The inverse of $Y(s)$ is found by using its partial fraction representation and referring to Table 11.2. We now show how the theory of complex variables can be used systematically to find the partial fraction representation. Theorem 11.19 is an extension of Example 8.7 to n linear factors. We leave the proof to you.

Theorem 12.19 (Nonrepeated Linear Factors). Let $P(s)$ be a polynomial of degree at most $n-1$. If $Q(s)$ has degree n, and has distinct complex roots $a_1, a_2, \ldots, a_n$, then equation (12.36) becomes

(12.37) $$Y(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-a_1)(s-a_2)\ldots(s-a_n)} = \sum_{k=1}^{n} \frac{Res[Y, a_k]}{(s-a_k)}.$$

Proof.

Theorem 12.20 (A Repeated Linear Factors). If $P(s)$ and $Q(s)$ are polynomials of degree μ and ν, respectively and $\mu < \nu + n$ and $Q(a) \neq 0$, then equation (12.36) becomes

(12.38) $$Y(s) = \frac{P(s)}{(s-a)^n Q(s)} = \sum_{k=1}^{n} \frac{A_k}{(s-a)^k} + R(s),$$

where $R(s)$ is the sum of all partial fractions that do not involve factors of the form $(s-a)^j$. Furthermore, the coefficients A_k can be computed with the formula

$$A_k = \frac{1}{(n-k)!} \lim_{s \to a} \frac{d^{n-k}}{ds^{n-k}} \frac{P(s)}{Q(s)} \quad \text{for } k = 1, 2, \ldots, n.$$

Proof.

Example 12.24. Let $Y(s) = \frac{s^3 - 4s + 1}{s(s-1)^3}$. Find $\mathcal{L}^{-1}(Y(s))$.

Solution.

From Equations (12-37) and (12-37) we write

$$\frac{s^3 - 4s + 1}{s(s-1)^3} = \frac{A_3}{(s-1)^3} + \frac{A_2}{(s-1)^2} + \frac{A_1}{s-1} + \frac{B_1}{s}$$

We calculate the coefficient $B_1 = -1$ by

$$B_1 = \text{Res}[Y, 0] = \lim_{s\to 0} s\,\frac{s^3 - 4s + 1}{s(s-1)^3}$$

$$= \lim_{s\to 0} \frac{s^3 - 4s + 1}{(s-1)^3} = \frac{0 - 0 + 1}{(-1)^3} = -1$$

We find the coefficients $A_1 = 2$, $A_2 = 1$, and $A_3 = -2$ by using Theorem 12.20.

In this case $a = 1$ and $\frac{P(s)}{Q(s)} = \frac{s^3 - 4s + 1}{s}$, and we get

$$A_1 = \lim_{s\to 1} \frac{P(s)}{Q(s)} = \lim_{s\to 1} \frac{s^3 - 4s + 1}{s}$$

$$= \frac{1 - 4 + 1}{1} = -2$$

$$A_2 = \frac{1}{1!} \lim_{s\to 1} \frac{d}{ds} \frac{P(s)}{Q(s)} = \lim_{s\to 1} \frac{d}{ds}\left(\frac{s^3 - 4s + 1}{s}\right)$$

$$= \lim_{s\to 1} \frac{d}{ds}\left(s^2 - 4 + \frac{1}{s}\right) = \lim_{s\to 1}\left(2s - \frac{1}{s^2}\right)$$

$$= (2 - 1) = 1$$

$$A_3 = \frac{1}{2!} \lim_{s\to 1} \frac{d^2}{ds^2} \frac{P(s)}{Q(s)} = \frac{1}{2} \lim_{s\to 1} \frac{d^2}{ds^2}\left(\frac{s^3 - 4s + 1}{s}\right)$$

$$= \frac{1}{2} \lim_{s\to 1} \frac{d^2}{ds^2}\left(s^2 - 4 + \frac{1}{s}\right) = \frac{1}{2} \lim_{s\to 1} \frac{d}{ds}\left(2s - \frac{1}{s^2}\right)$$

$$= \frac{1}{2} \lim_{s\to 1}\left(2 + \frac{2}{s^3}\right) = \frac{1}{2}(2 + 2) = 2$$

Now substitute these values into the formula $Y(s) = \frac{A_3}{(s-1)^3} + \frac{A_2}{(s-1)^2} + \frac{A_1}{s-1} + \frac{B_1}{s}$, and get the partial fraction representation for $Y(s)$:

$$Y(s) = \frac{s^3 - 4s + 1}{s(s-1)^3} = -\frac{2}{(s-1)^3} + \frac{1}{(s-1)^2} + \frac{2}{s-1} - \frac{1}{s},$$

The solution $y(t) = \mathcal{L}^{-1}(Y(s))$ is found with the computation

$$y(t) = \mathcal{L}^{-1}\left(\frac{s^3 - 4s + 1}{s(s-1)^3}\right)$$

$$= \mathcal{L}^{-1}\left(-\frac{2}{(s-1)^3} + \frac{1}{(s-1)^2} + \frac{2}{s-1} - \frac{1}{s}\right)$$

$$= -\mathcal{L}^{-1}\left(\frac{2}{(s-1)^3}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) + \mathcal{L}^{-1}\left(\frac{2}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$= -t^2 e^t + t e^t + 2 e^t - 1$$

Explore Solution 12.24.

Theorem 12.21 (Irreducible Quadratic Factors). Let $P(s)$ and $Q(s)$ be polynomials with real coefficients such that the degree of $P(s)$ is at most 1 larger than the degree of $Q(s)$. If $T(s)$ does not have a factor of the form $(s-a)^2 + b^2$, then

$$Y(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{\left[(s-a)^2 + b^2\right] T(s)} = \frac{2A(s-a) - 2Bb}{(s-a)^2 + b^2} + R(s),$$

where

(12.21) $$A + i B = \frac{P(a + i b)}{Q'(a + i b)}.$$

Proof.

Example 12.25. Let $Y(s) = \frac{5s}{(s^2+4)(s^2+9)}$. Find $\mathcal{L}^{-1}(Y(s))$.

Solution.

This solution is a little tedious and illustrates how Theorem 12.21 is applied to obtain the partial fraction expansion of $Y(s)$ when two irreducible quadratic factors are involved.

Here we have $P(s) = 5s$ and $Q(s) = (s^2 + 4)(s^2 + 9) = s^4 + 13s^2 + 36$, and the roots of $Q(s)$ occur at $0 \pm 2i$ and $0 \pm 3i$. The derivative of $Q(s)$ is $Q'(s) = 4s^3 + 26s$. Computing the residues at at $2i$ and $3i$ yields

$$\text{Res}[Y, 2i] = \left(\frac{P(2i)}{Q'(2i)}\right) = \frac{5(2i)}{4(2i)^3 + 26(2i)} = \frac{10i}{-32i + 52i} = \frac{1}{2},$$

and

$$\text{Res}[Y, 3i] = \left(\frac{P(3i)}{Q'(3i)}\right) = \frac{5(3i)}{4(3i)^3 + 26(3i)} = \frac{15i}{-108i + 78i} = -\frac{1}{2}.$$

The above residues are used to determine the values in Equation (12-21), and we get

$$A_1 + i B_1 = \frac{P(a_1 + i b_1)}{Q'(a_1 + i b_1)} = \frac{P(2i)}{Q'(2i)} = \frac{1}{2} + 0i$$

and

$$A_2 + i B_2 = \frac{P(a_2 + i b_2)}{Q'(a_2 + i b_2)} = \frac{P(3i)}{Q'(3i)} = -\frac{1}{2} + 0i$$

We find that $A_1 + i B_1 = \frac{1}{2} + 0i$ and $A_2 + i B_2 = -\frac{1}{2} + 0i$, which correspond to $a_1 + i b_1 = 0 + 2i$ and $a_2 + i b_2 = 0 + 3i$, respectively. Applying the formula in Theorem 12.21 we obtain

$$Y(s) = \frac{2(A_1)(s - a_1) - 2(B_1)(b_1)}{(s - a_1)^2 + (b_1)^2} + \frac{2A_2(s - a_2) - 2(B_2)(b_2)}{(s - a_2)^2 + (b_2)^2}$$

which becomes

$$Y(s) = \frac{2\left(\frac{1}{2}\right)(s-0) - 2(0)(2)}{(s-0)^2 + (2)^2} + \frac{2\left(-\frac{1}{2}\right)(s-0) - 2(0)(3)}{(s-0)^2 + (3)^2}$$

and is simplified as

$$Y(s) = \frac{s}{s^2+4} - \frac{s}{s^2+9}.$$

Finally $\mathcal{L}^{-1}(Y(s))$ is found with the calculation

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{5s}{(s^2+4)(s^2+9)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+3^2}\right)$$

$$= \cos(2t) - \cos(3t)$$

Explore Solution 12.25.

Example 12.26. Let $Y(s) = \frac{s^3+3s^2-s+1}{s(s+1)^2(s^2+1)}$. Find $\mathcal{L}^{-1}(Y(s))$.

Solution.

This solution is a little tedious and illustrates how Theorems 12.20 and 12.21 are applied to obtain the partial fraction expansion of $Y(s)$.

The partial fraction expression for $Y(s)$ has the form

$$Y(s) = \frac{D}{s} + \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2} + \frac{2A(s-0)-2B}{(s-0)^2+1^2}.$$

First, we calculate the coefficient $D = 1$. In the denominator of $Y(s)$ the linear factor (s) is nonrepeated, so we have

$$D = \text{Res}[Y(s), 0] = \lim_{s\to 0}\left(s\,\frac{s^3+3s^2-s+1}{s(s+1)^2(s^2+1)}\right)$$

$$= \lim_{s\to 0}\left(\frac{s^3+3s^2-s+1}{(s+1)^2(s^2+1)}\right) = \frac{0+3*0-0+1}{(0+1)^2(0+1)} = 1$$

Second, we calculate the coefficients $C_1 = -2$, and $C_2 = -2$ by using Theorem 12.20. In this case $a = -1$ and $\frac{P(s)}{Q(s)} = \frac{s^3+3s^2-s+1}{s(s^2+1)}$. In the denominator of $Y(s)$ the term $(s+1)^2$ is a repeated factor, so we have

$$C_1 = \text{Res}[Y(s), -1] = \frac{1}{1!}\lim_{s\to -1}\frac{d}{ds}\frac{P(s)}{Q(s)}$$

$$= \lim_{s\to -1}\frac{d}{ds}\left(\frac{s^3+3s^2-s+1}{s(s^2+1)}\right)$$

$$= \lim_{s\to -1}\frac{-3s^4+4s^3-1}{s^2(s^2+1)^2}$$

$$= \frac{-3-4-1}{1(1+1)^2} = -2$$

$$C_2 = \text{Res}[(s+1)Y(s), -1] = \lim_{s\to -1}\frac{P(s)}{Q(s)}$$

$$= \lim_{s\to -1}\frac{s^3+3s^2-s+1}{s(s^2+1)}$$

$$= \frac{-1+3+1+1}{-1(1+1)} = -2$$

Third, we calculate the coefficients $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ by using Theorem 12.21. Here we have $P(s) = s^3+3s^2-s+1$, $Q(s) = s(s+1)^2(s^2+1)$, and $Q'(s) = (s+1)(5s^3+3s^2+3s+1)$. In the denominator of $Y(s)$ the factor (s^2+1) is an irreducible quadratic, with roots $0 \pm i$, so that

$$\text{Res}[Y, i] = \left(\frac{P(i)}{Q'(i)}\right)$$

$$= \frac{i^3+3i^2-i+1}{(i+1)(5i^3+3i^2+3i+1)}$$

$$= \frac{-i-3i-i+1}{(i+1)(-5i-3i+3i+1)}$$

$$= \frac{1-5i}{6-4i} = \frac{1-i}{2}$$

Hence we have $A + iB = \frac{1-i}{2}$. Or if you prefer taking limits then we can use the calculation

$$A + iB = \text{Res}[Y, i] = \lim_{s\to i}\frac{s^3+3s^2-s+1}{s(s+1)^2(s+i)}$$

$$= \frac{i^3+3i^2-i+1}{i(i+1)^2(i+i)} = \frac{-i-3-i+1}{i(i+1)^2(i+i)}$$

$$= \frac{-2-2i}{-4i} = \frac{1-i}{2}$$

Either way, we obtain $A = \frac{1}{2}$ and $B = -\frac{1}{2}$.

Now substitute these values into the formula $Y(s) = \frac{D}{s} + \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2} + \frac{2A(s-0)-2B}{(s-0)^2+1^2}$, and get the partial fraction representation for $Y(s)$:

$$Y(s) = \frac{1}{s} + \frac{-2}{s+1} + \frac{-2}{(s+1)^2} + \frac{2\left(\frac{1}{2}\right)(s-0) - 2\left(-\frac{1}{2}\right)}{(s-0)^2+1^2}$$

$$= \frac{1}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2} + \frac{s}{s^2+1^2} + \frac{1}{s^2+1^2}$$

Now we use Table 12.2 to get

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{2}{s+1} - \frac{2}{(s+1)^2} + \frac{s}{s^2+1^2} + \frac{1}{s^2+1^2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{2}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{2}{(s+1)^2}\right) + \mathcal{L}^{-1}\left(\frac{s}{s^2+1^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2+1^2}\right)$$

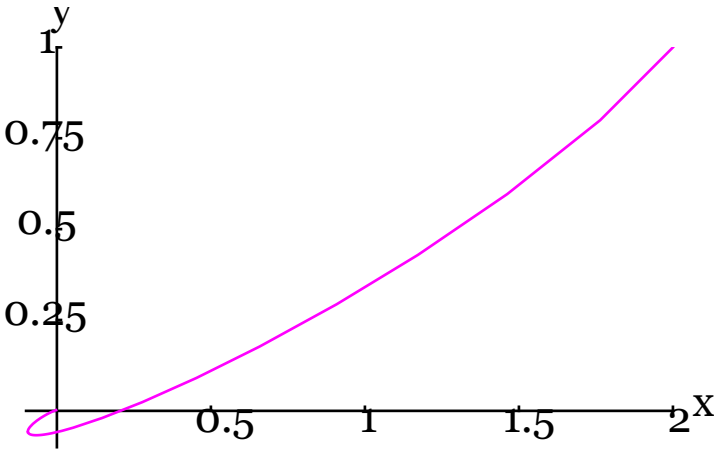
$$= 1 - 2e^{-t} - 2te^{-t} + \cos(t) + \sin(t)$$

Explore Solution 12.26.

Example 12.27. Use Laplace transforms to solve the system

$$y'(t) = y(t) - x(t) \quad \text{with} \quad y(0) = 1;$$

$$x'(t) = 5y(t) - 3x(t) \quad \text{and} \quad x(0) = 2.$$



A graph of the solution.

Solution.

We let $Y(s)$ and $X(s)$ denote the Laplace transforms of $y(t)$ and $x(t)$, respectively. Taking the transforms of the two differential equations gives

$$sY(s) - 1 = Y(s) - X(s);$$

and

$$sX(s) - 2 = 5Y(s) - 3X(s).$$

which can be written as

$$(s-1)Y(s) + X(s) = 1;$$

and

$$5Y(s) - (s+3)X(s) = -2.$$

We use Cramer's rule to solve for $Y(s)$ and $X(s)$:

$$Y(s) = \frac{\begin{vmatrix} 1 & 1 \\ -2 & -(s+3) \end{vmatrix}}{\begin{vmatrix} (s-1) & 1 \\ 5 & -(s+3) \end{vmatrix}} = \frac{-(s+3)-(-2)}{-(s-1)(s+3)-5} = \frac{s+1}{(s+1)^2+1}$$

$$X(s) = \frac{\begin{vmatrix} (s-1) & 1 \\ 5 & -2 \end{vmatrix}}{\begin{vmatrix} (s-1) & 1 \\ 5 & -(s+3) \end{vmatrix}} = \frac{-2(s-1)-5}{-(s-1)(s+3)-5} = \frac{2s+3}{(s+1)^2+1}$$

We obtain the desired solution by computing the inverse transforms:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+1^2}\right) \\ &= e^{-t}\cos(t) \end{aligned}$$

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left(\frac{2(s+1)+1}{(s+1)^2+1^2}\right) \\ &= 2\mathcal{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+1^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1^2}\right) \\ &= 2e^{-t}\cos(t) + e^{-t}\sin(t) \end{aligned}$$

Explore Solution 12.27.

According to Equation (12-29), the inverse Laplace transform is given by the integral formula

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} F(s)\, e^{st}\, ds,$$

where σ_0 is any suitably chosen large positive constant. This improper integral is a contour integral taken along the vertical line $s = \sigma_0 + i\tau$ in the complex $s = \sigma + i\tau$ plane. We use the residue theory in Chapter 8 to evaluate it. We leave the cases in which the integrand has either infinitely many poles or branch points for you to research in advanced texts. We state the following more elementary theorem.

Theorem 12.22 (Inverse Laplace Transform). Let $F(s) = \frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials of degree m and n, respectively, and $n > m$. The inverse Laplace transform of $F(s)$ is given by

$$f(t) = \mathcal{L}^{-1}(F(s)) = \sum \text{Res}[F(s)\, e^{st}, s_k],$$

where the sum is taken over all the residues of the complex function $F(s)\, e^{st}$.

Proof.

Using the residue calculus, the English mathematician Oliver Heaviside discovered a method for inverting the Laplace transform.

Theorem 12.23 (Heaviside Expansion Theorem). Let $P(s)$ and $Q(s)$ be polynomials of degree m and n, respectively, where $n > m$. If $Q(s)$ has n distinct simple zeros at the points $s_1, s_2, \ldots, s_n$, then $\frac{P(s)}{Q(s)}$ is the Laplace transform of the function $f(t)$ given by

(12.44) $f(t) = \mathcal{L}^{-1}\left(\frac{P(s)}{Q(s)}\right) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} e^{s_k t}.$

Proof.

Example 12.28. Let $Y(s) = \frac{4s+3}{s^3+2s^2+s+2}$. Find $\mathcal{L}^{-1}(Y(s))$.

Solution.

Here we have $P(s) = 4s+3$ and $Q(s) = s^3+2s^2+s+2 = (s+2)(s^2+1)$ so that $Q(s)$ has simple zeros located at the points $s_1 = -2$, $s_2 = i$, and $s_3 = -i$. When we use $Q'(s) = 3s^2+4s+1$, calculation reveals that

$$\frac{P(s_1)}{Q'(s_1)} = \frac{P(-2)}{Q'(-2)} = \frac{4(-2)+3}{3(-2)^2+4(-2)+1} = \frac{-5}{5} = -1$$

$$\frac{P(s_2)}{Q'(s_2)} = \frac{P(i)}{Q'(i)} = \frac{4(i)+3}{3(i)^2+4(i)+1}$$

$$= \frac{3+4i}{-2+4i} = \frac{1}{2} - i$$

$$\frac{P(s_3)}{Q'(s_3)} = \frac{P(-i)}{Q'(-i)} = \frac{4(-i)+3}{3(-i)^2+4(-i)+1}$$

$$= \frac{3-4i}{-2-4i} = \frac{1}{2} + i$$

Applying Equation (12-44) in Theorem 12.23 gives $f(t) = \sum_{k=1}^{n} \frac{P(s_k)}{Q'(s_k)} e^{s_k t}$ as

$$f(t) = \frac{P(s_1)}{Q'(s_1)} e^{s_1 t} + \frac{P(s_2)}{Q'(s_2)} e^{s_2 t} + \frac{P(s_3)}{Q'(s_3)} e^{s_3 t}$$

$$= (-1) e^{-2t} + \left(\frac{1}{2} - i\right) e^{it} + \left(\frac{1}{2} + i\right) e^{-it}$$

$$= -e^{-2t} + \left(\frac{e^{it}+e^{-it}}{2}\right) + 2\frac{e^{it}-e^{-it}}{2i}$$

$$= -e^{-2t} + \cos(t) + 2\sin(t)$$

Solution 12.28.

Complex Analysis for Mathematics and Engineering

by

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12.10 Convolution for the Laplace Transform

This section is a continuation of our development of the Laplace Transforms in Sections 12.5-9.

If we let $F(s)$ and $G(s)$ denote the transforms of $f(t)$ and $g(t)$, respectively, then the inverse of the product $F(s)\,G(s)$ is given by the function $h(t) = (f * g)(t)$. It is called the convolution of $f(t)$ and $g(t)$ and can be regarded as s generalized product of $f(t)$ and $g(t)$. Convolution will assist us in solving integral equations.

Theorem 12.24 (Convolution Theorem). Let $F(s)$ and $G(s)$ denote the Laplace transforms of $f(t)$ and $g(t)$, respectively. Then the product $H(s) = F(s)\,G(s)$ is the Laplace transform of the convolution of $f(t)$ and $g(t)$, and is denoted by $h(t) = (f * g)(t)$, and has the integral representation

$$h(t) = (f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

or

$$h(t) = (g * f)(t) = \int_0^t g(\tau)\, f(t-\tau)\, d\tau$$

Proof.

Example 12.29. Show that $\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = t \sin(t)$.

Solution.

If we let $F(s) = \frac{1}{s^2+1}$, and $G(s) = \frac{2s}{s^2+1}$, then $f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$, and $g(t) = \mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) = 2\cos(t)$, respectively. Now, applying the convolution theorem, we get

$$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\,\frac{2s}{s^2+1}\right)$$

$$= \mathcal{L}^{-1}(F(s)\,G(s))$$

$$= \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$$= \int_0^t \sin(t-\tau)\,2\cos(\tau)\,d\tau$$

$$= \int_0^t (\cos(\tau)\sin(t) - \cos(t)\sin(\tau))\,2\cos(\tau)\,d\tau$$

$$= \int_0^t (2\cos^2(\tau)\sin(t) - 2\cos(t)\cos(\tau)\sin(\tau))\,d\tau$$

$$= \left(\sin(t)\,(\tau + \cos(\tau)\sin(\tau)) - \cos(t)\sin^2(\tau)\right)\Big|_{\tau=0}^{\tau=t}$$

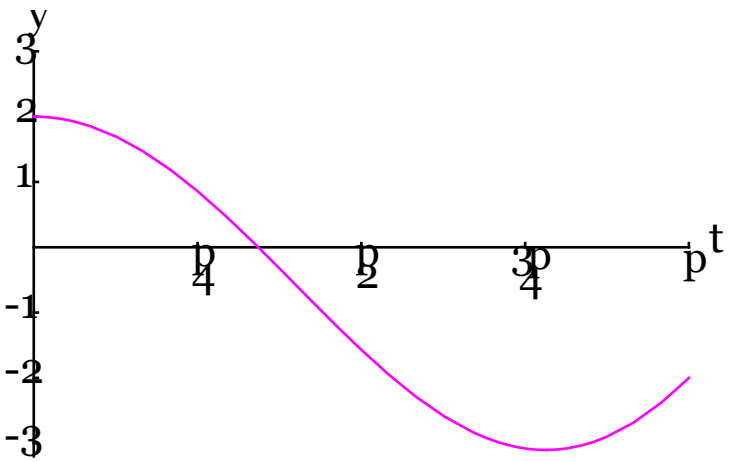
$$= \sin(t)\,(t + \cos(t)\sin(t)) - \cos(t)\sin^2(t)$$

$$= t\sin(t) + \cos(t)\sin^2(t) - \cos(t)\sin^2(t)$$

$$= t\sin(t)$$

Explore Solution 12.29.

Example 12.30. Use the convolution theorem to solve the integral equation
$f(t) = 2\cos(t) - \int_0^t (t-\tau)\,f(\tau)\,d\tau$.



A graph of the solution.

Solution.

Letting $F(s) = \mathcal{L}(f(t))$ and using $G(s) = \mathcal{L}(t) = \frac{1}{s^2}$ in the convolution theorem, we obtain

$$\mathcal{L}(f(t)) = \mathcal{L}(2\cos(t)) - \mathcal{L}\left(\int_0^t (t-\tau)\,f(\tau)\,d\tau\right)$$

$$F(s) = \frac{2s}{s^2+1} - \frac{1}{s^2}F(s).$$

Solving for $F(s)$, we have $F(s) + \frac{1}{s^2}F(s) = \frac{2s}{s^2+1}$, and then $F(s)\left(1 + \frac{1}{s^2}\right) = \frac{2s}{s^2+1}$, and then $F(s)\frac{s^2+1}{s^2} = \frac{2s}{s^2+1}$, and then we get

$$F(s) = \frac{2s^3}{(s^2+1)^2} = \frac{2s}{s^2+1} - \frac{2s}{(s^2+1)^2}.$$

Finally, the solution $f(t)$ is obtained using facts from Table 12.2: $\mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) = 2\cos(t)$ and $\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = t\sin(t)$, and the computation

$$f(t) = \mathcal{L}^{-1}\left(\frac{2s^3}{(s^2+1)^2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) - \mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right)$$

$$= 2\cos(t) - t\sin(t)$$

Explore Solution 12.30.

Engineers and physicists sometimes consider forces that produce large effects but that are applied over a very short time interval. The force acting at the time an earthquake starts is an example. This phenomenon leads to the idea of a unit impulse function $\delta(t)$. Let's consider the small positive constan a. The function $\delta_a(t)$ is defined by

$$\delta_a(t) = \begin{cases} \frac{1}{a} & \text{for } 0 < t < a \\ 0 & \text{otherwise} \end{cases}.$$

The unit impulse function is obtained by letting the interval width go to zero, or

$$\delta(t) = \lim_{a \to 0} \delta_a(t).$$

Figure 11.29 shows the graph $y = \delta_a(t)$ for $a = 10, 40, 100$. Although $\delta(t)$ is called the Dirac Delta function, it is not an ordinary function. To be precise it is a distribution, and the theory of distributions permits manipulations of $\delta(t)$ as though it were a function. Here, we will treat $\delta(t)$ as a function and investigate its properties.

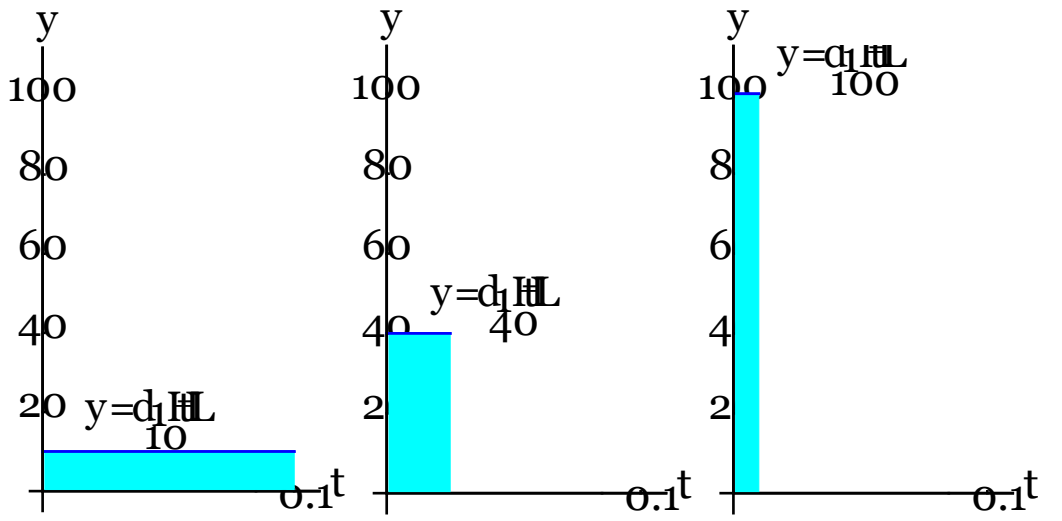


Figure 12.29. Graphs of $y = \delta_a(t)$ for $a = 10, 40, 100$.

Example 12.31. Show that $\mathcal{L}(\delta(t)) = 1$.

Solution.

By definition, the Laplace transform of $\delta_a(t)$ is

$$\mathcal{L}(\delta_a(t)) = \int_0^\infty \delta_a(t)\, e^{-st}\, dt$$

$$= \int_0^a \frac{1}{a} e^{-st}\, dt + \int_a^\infty 0 * e^{-st}\, dt$$

$$= \left(-\frac{e^{-st}}{as}\right)\Bigg|_{t=0}^{t=a}$$

$$= \left(-\frac{e^{-as}}{as}\right) - \left(-\frac{1}{as}\right)$$

$$= \frac{1 - e^{-as}}{as}$$

Letting $a \to 0$ in the above equation and using L'Hôpital's rule, we obtain

$$\mathcal{L}(\delta(t)) = \lim_{a\to 0} \mathcal{L}(\delta_a(t))$$

$$= \lim_{a\to 0} \frac{1 - e^{-as}}{as}$$

$$= \lim_{a\to 0} \frac{0 + se^{-as}}{s}$$

$$= \lim_{a\to 0} e^{-as}$$

$$= e^0 = 1$$

Explore Solution 12.31.

We now turn to the unit impulse function. First, we consider the function $f_a(t)$ obtained by integrating $\delta_a(t)$:

$$f_a(t) = \int_0^t \delta_a(\tau)\, d\tau = \begin{cases} 0, & \text{for } t < 0; \\ \frac{1}{a}, & \text{for } 0 \le t \le a; \\ 1, & \text{for } a < t. \end{cases}$$

Taking the limit as $a \to 0$ results in the important fact that

$$U_0(t) = \lim_{a \to 0} f_a(t),$$

where $U_0(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t > 0 \end{cases}$ is the unit step function that was introduced in Section 12.7. The situation is illustrated in Figure 12.30.

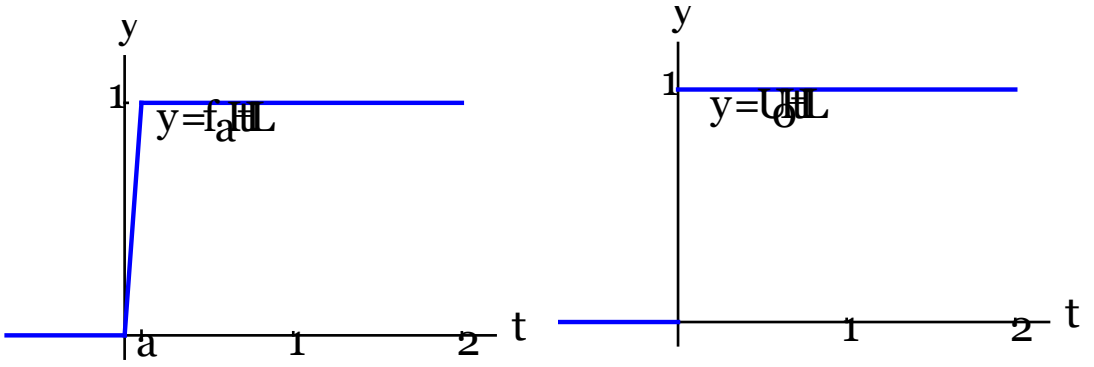


Figure 12.30. The integral of $\delta_a(t)$ is $f_a(t)$, which becomes $U_0(t)$ when for $a \to 0$.

We demonstrate the response of a system to the unit impulse function in Example 12.32.

Example 12.32. Solve the initial value problem

$$y''(t) + 4y'(t) + 13y(t) = 3\delta(t)$$

with

$$y(0) = 0 \text{ and } y'(0) = 0.$$

Solution.

Taking transforms results in $(s^2 + 4s + 13)Y(s) = 3\mathcal{L}(\delta(t)) = 3$ so that

$$Y(s) = \frac{3}{s^2 + 4s + 13} = \frac{3}{(s+2)^2 + 3^2},$$

and the solution is

$$y(t) = \mathcal{L}^{-1}\left(\frac{3}{(s-(-2))^2 + 3^2}\right)$$

$$= e^{-2t}\sin(3t)$$

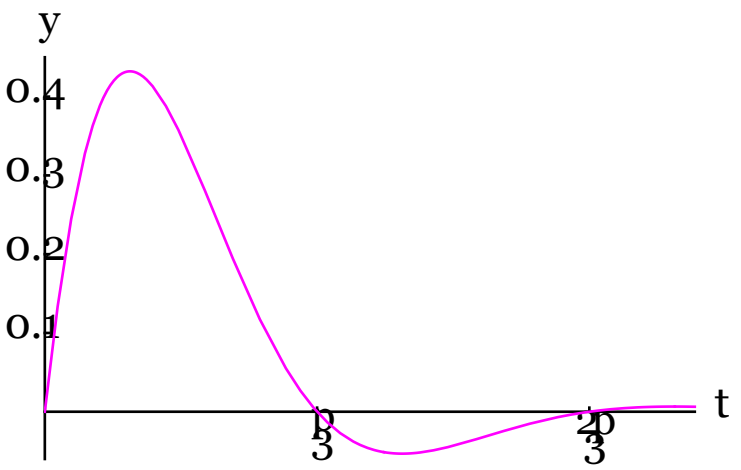


Figure 12.31. A graph of the solution $y = y(t)$.

Remark. The condition $y'(0^-) = 0$ is not satisfied by the "solution" $y(t)$. Recall that all solutions involving the use of the Laplace transform are to be considered zero for values of $t < 0$, hence the graph of $y(t)$ is given above in Figure 12.31. Note that $y'(t)$ has a jump discontinuity of magnitude $+3$ at the origin. This discontinuity occurs because either $y(t)$ or $y'(t)$ must have a jump discontinuity at the origin whenever the Dirac delta function, $\delta(t)$, occurs as part of the input or driving function.

Explore Solution 12.32.

The convolution method can be used to solve initial value problems. The tedious mechanical details of problem solving can be facilitated with computer software such as Maple™, Matlab™, or Mathematica™.

Theorem 12.25 (Initial value Problem - IVP convolution method). The unique solution to the initial value problem

$$a\,y''(t) + b\,y'(t) + c\,y(t) = g(t),$$

with $y(0) = y_0$ and $y'(0) = y_1$, is given by

$$y(t) = u(t) + (h * g)(t),$$

where $u(t)$ is the solution to the homogeneous equation

$$a\,u''(t) + b\,u'(t) + c\,u(t) = 0,$$

with $u(0) = y_0$, and $u'(0) = y_1$, and

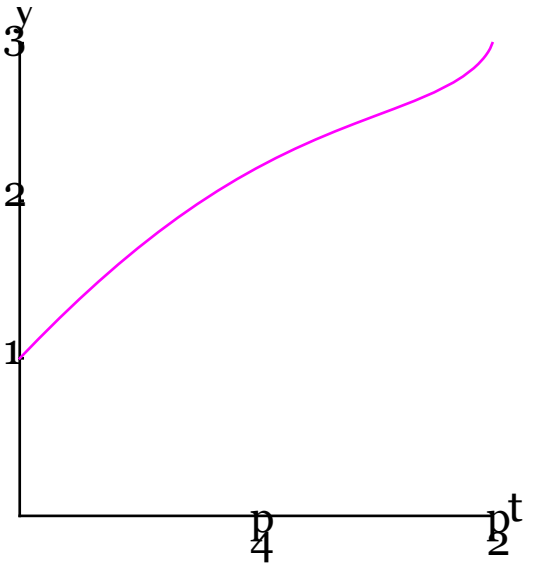
$h(t)$ has the Laplace transform given by $H(s) = \frac{1}{a\,s^2 + b\,s + c}$.

Proof.

Proof of Theorem 12.25 is in the book.

Example 12.33. Use convolution to solve the initial value problem

$y''(t) + y(t) = \tan(t)$
with
$y(0) = 1$ and $y'(0) = 2$.



A graph of the solution.

Solution.

First, we obtain the portion $u(t)$ of the solution by solving $u''(t) + u(t) = 0$ with $u(0) = 1$ and $u'(0) = 2$. Taking the Laplace transform yields $s^2 U(s) - s - 2 + U(s) = 0$, and we can rearrange the terms to obtain $U(s)(s^2 + 1) = s + 2$. Solving for $U(s)$ gives $U(s) = \frac{s+2}{s^2+1} = \frac{s}{s^2+1} + 2\frac{1}{s^2+1}$, and it follows that

$$u(t) = \mathcal{L}^{-1}\left(\frac{s+2}{s^2+1}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + 2\,\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \cos(t) + 2\sin(t)$$

Second, we observe that $H(s) = \frac{1}{s^2+1}$ and $h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$, and also that $g(t) = \tan(t)$. We compute the portion $v(t)$ of the solution with a convolution:

$$v(t) = (h * g)(t) = \int_0^t \sin(t-\tau)\tan(\tau)\,d\tau$$

$$= \left[\cos(t)\ln\left[\frac{\cos(\tau)}{1+\sin(\tau)}\right] - \sin(t-\tau)\right]_{\tau=0}^{\tau=t}$$

$$= \cos(t)\ln\left[\frac{\cos(t)}{1+\sin(t)}\right] - (-\sin(t))$$

$$= \cos(t)\ln\left[\frac{\cos(t)}{1+\sin(t)}\right] + \sin(t)$$

Now we can compute the solution using the convolution method in Theorem 12.25:

$$y(t) = u(t) + v(t) = u(t) + (h * g)(t)$$

$$= \cos(t) + 2\sin(t) + \cos(t)\ln\left[\frac{\cos(t)}{1+\sin(t)}\right] + \sin(t)$$

$$= \cos(t) + 3\sin(t) + \cos(t)\ln\left[\frac{\cos(t)}{1+\sin(t)}\right]$$

Explore Solution 12.33.